



The Utilization of the Emad-Sara Integral Transform in Solving the Heat and Wave Equations

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ABSTRACT

Heat and wave equations are widely recognized partial differential equations that find applications in fundamental sciences and engineering disciplines. Integral transform techniques offer efficient approaches to address a range of issues encountered in the basic sciences and engineering fields. This chapter introduces the Emad-Sara Integral Transform for solving heat and wave equations expressed in terms of partial differential equations.

Keywords: Emad-Sara Integral Transform, Heat equation, Wave equation, Differential Equations.

INTRODUCTION

Partial differential equations are used extensively in chemistry, physics, and a variety of other disciplines. As a result, the literature includes a variety of approaches for solving partial differential equations with the goal of establishing symmetry. These equations are used to analyze practical challenges that people face, such as biological growth, tumour size expansion, heat transmission, carbon dating, compound interest calculations, chemical reactions, mixing problems, compartment problems, electric circuits, and trajectory problems [1]. Partial differential equations in mathematics are commonly employed to address a variety of problems, such as boundary value problems, initial value problems, heat and wave equations, and other differential equations. Various methods exist for solving these equations, including numerical approaches, decomposition techniques, separation methods, integral transforms, and more. Nowadays, integral transforms are considered the most convenient and straightforward mathematical tools for solving advanced problems related to initial-value problems, boundary-value problems, differential equations, and integral equations that are encountered in various fields such as technology, science, social sciences, commerce, economics, and engineering. One important feature of integral transformations is that they provide an exact solution to problems without requiring large amounts of computation. Owing to this significant characteristic, a large number of scholars work in this area and become acquainted with different integral transforms. Yang [2] used a Fourier-like integral transformation to find accurate solutions for the steady heat transfer problem. Su *et al.* [3] applied the Fourier-like integral transform to wave and heat-transfer problems. Vaidya *et al.* [4] used various approaches to solve the partial differential heat equation. Peker *et al.* [5] solved heat transfer problems using the Kashuri Fundo transform. Using the Rohit Transform, Gupta *et al.* [6] solved the wave and heat equations. Gupta Transform was utilized by Gupta *et al.* [7] to solve the one-dimensional heat and wave equation. Mulugeta *et al.* [8] used Anduaalem and Khan Transform to achieve exact solutions for wave and heat equations. Iman Ahmed Almaryd [9] employed Iman transform to solve Ordinary Differential Equations. Dinesh and Prakash [10] solved the linear second-kind Volterra integral problem by applying the Upadhyaya transform. Gavit *et al.* [11] solving the system

of differential equations by using Emad-Sara Transform. Kuffi *et al.* [12-13], obtain the solution of partial differential equations by employing Emad-Sara Transform.

MATERIAL AND METHOD: EMAD-SARA INTEGRAL TRANSFORM

1.1. Definition: For an exponential order function, the Emad-Sara Integral Transform is defined as [12]

$$A = \left\{ f(t) : \exists K, \lambda_1, \lambda_2 > 0, |f(t)| < Ke^{\lambda_j t}; \quad \text{if } t \in (-1)^j X[0, \infty) \right\} \quad (1)$$

where, K be the finite constant number
 $f(t)$ be the function in the set A and
 λ_1, λ_2 may be finite or infinite number
 ES – Emad - Sara Operator,

2.2. Definition: The kernel function of Emad-Sara Integral Transform symbolized by $I(\cdot)$, written in the integral form as [12]

$$ES[f(t)] = \frac{1}{\eta^2} \int_0^\infty \exp(-\eta t) f(t) dt = P(\eta); \quad (2)$$

Where, $t \geq 0, \lambda_1 < \infty < \lambda_2; \eta \geq 0$ and ∞ be a variable that is used as a factor to the variable t in the function $f(t)$. In Emad-Sara Integral Transform, t is an argument of the function $f(t)$, which is factorized by the transform variable. Inverse of Emad-Sara Integral Transform is denoted by ES^{-1} and η be the factor of t variable.

$$f(t) = ES^{-1}[P(\eta)], \quad t \geq 0 \quad (3)$$

2.3. Linearity Property of Emad-Sara Integral Transform [12]:

If $B_1(\eta)$ and $B_2(\eta)$ respectively, are the Emad-Sara Integral Transform of functions $f_1(t)$ and $f_2(t)$

Therefore, Emad-Sara Integral Transform of

$$ES[pf_1(t) + qf_2(t)] = pB_1(\eta) + qB_2(\eta) \quad (4)$$

2.4. Derived Properties Property of Emad-Sara Integral Transform [13]:

$$\left. \begin{aligned} \text{First derivative : } ES[f'(t)] &= \eta ES(\eta) - \frac{1}{\eta^2} f(0) = I\left[\frac{df(t)}{dt}\right] \\ \text{Second derivative : } ES[f''(t)] &= \eta ES[f'(t)] - \frac{1}{\eta^2} f'(0) = I\left[\frac{d^2 f(t)}{dt^2}\right] \\ \text{nth derivative : } ES[f^n(t)] &= \eta ES[f^{n-1}(t)] - \frac{1}{\eta^2} f^{n-1}(0) = I\left[\frac{d^n f(t)}{dt^n}\right] \end{aligned} \right\} \quad (5)$$

Property 2.5. Partial derivative properties of Emad-Sara Integral Transform [12]:

$$\left. \begin{aligned}
 \text{First derivative : } & ES \left[\frac{\partial f(x,t)}{\partial t} \right] = \eta ES(x,\eta) - \frac{1}{\eta^2} f(x,0) \\
 \text{Second derivative : } & ES \left[\frac{\partial^2 f(x,t)}{\partial t^2} \right] = \eta^2 ES(x,\eta) - \frac{f(x,0)}{\eta} - \frac{1}{\eta^2} \left(\frac{\partial f(x,0)}{\partial t} \right) \\
 \text{nth derivative : } & ES \left[\frac{\partial^n f(x,t)}{\partial t^n} \right] = - \frac{\partial^{(n-1)} f(x,0)}{\eta^2 \partial t} - \frac{\partial^{(n-2)} f(x,0)}{\eta \partial t} - \dots - \frac{f(x,0)}{\eta^{3-n}} + \eta^n ES(x,\eta)
 \end{aligned} \right\} \quad (6)$$

TABULATED VALUES

Emad-Sara Integral Transform and Inverse of Emad-Sara Integral Transform of some function are as below by [13].

3(a). Emad-Sara Integral Transform

3(b). Inverse of Emad-Sara Transform

$f(t)$	$ES[f(t)] = P(\eta)$	$f(t)$	$ES[f(t)] = p(\eta)$	$ES^{-1}[P(\eta)]$	$f(t)$	$ES^{-1}[P(\eta)]$	$f(t)$
$ES(1) = \frac{1}{\eta^3},$	$ES(\exp(at)) = \frac{1}{\eta^2(\eta-a)}$	$ES(\exp(-at)) = \frac{1}{\eta^2(\eta+a)}$	$ES(\sinh(at)) = \frac{a}{\eta^2(\eta^2-a^2)}$	$ES^{-1}\left(\frac{1}{\eta^3}\right) = 1;$	$ES^{-1}\left(\frac{1}{\eta^2(\eta-a)}\right) = \exp(at);$	$ES^{-1}\left(\frac{1}{\eta^2(\eta+a)}\right) = \exp(-at);$	$ES^{-1}\left(\frac{a}{\eta^2(\eta^2-a^2)}\right) = \sinh(at);$
$ES(t) = \frac{1}{\eta^4},$	$ES(t^n) = \frac{n!}{\eta^{n+3}}, n \in N,$	$ES(\cosh(at)) = \frac{1}{\eta(\eta^2-a^2)}$	$ES(\sin(at)) = \frac{a}{\eta^2(\eta^2+a^2)}$	$ES^{-1}\left(\frac{1}{\eta^4}\right) = t;$	$ES^{-1}\left(\frac{1}{\eta^2(\eta^2+a^2)}\right) = \sin(at);$	$ES^{-1}\left(\frac{1}{\eta^2(\eta^2-a^2)}\right) = \cosh(at);$	$ES^{-1}\left(\frac{a}{\eta^2(\eta^2+a^2)}\right) = \sin(at);$
				$ES^{-1}\left(\frac{N!}{\eta^{N+3}}\right) = t^N, m \in N,$			

3. Methodology:

Methodology for Solving the Applications of Heat and Wave Equations Via Emad-Sara Integral Transform

In this chapter, the Emad-Sara Integral Transform is employing for solving the heat and wave equations which are expressed in terms of differential equations, approaching to light in the field of basic sciences, material sciences and engineering.

Application-4.1. [HEAT EQUATION]:

Consider the heat equation [6]

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2} \quad (7)$$

with the condition

$$f(x,0) = 3 \sin 2\pi x, f(0,t) = 0, f(2,t) = 0, 0 < x < 2 \text{ and } t > 0. \quad (8)$$

Relating the Emad-Sara Integral Transform to the equation (7), both the sides, we have

$$ES\left(\frac{\partial f(x,t)}{\partial t}\right) = ES\left(\frac{\partial^2 f(x,t)}{\partial x^2}\right) \tag{9}$$

Using the results of partial derivative properties of Emad-Sara Integral Transform from equation (6), then equation (9), becomes

$$\eta P(\eta) - \frac{f(x,0)}{\eta^2} = \frac{\partial^2}{\partial x^2} ES(f(x,t)) \tag{10}$$

Using given condition $f(x,0) = 3 \sin 2\pi x$, in equation (10), we get

$$\eta P(\eta) - \frac{3 \sin 2\pi x}{\eta^2} = \frac{\partial^2}{\partial x^2} P(\eta) \tag{11}$$

$$\frac{\partial^2}{\partial x^2} P(\eta) - \eta P(\eta) = -\frac{3 \sin 2\pi x}{\eta^2}$$

$$(D^2 - v^2)P(\eta) = -\frac{3 \sin 2\pi x}{v^2} \tag{12}$$

Equating to zero left side of equation (12) as $(D^2 - v^2)P(\eta) = 0$, which is a homogeneous equation. Therefore, the solution of this homogeneous equation written as

$$P(\eta) = C_1 e^{\sqrt{\eta} x} + C_2 e^{-\sqrt{\eta} x} \tag{13}$$

where, C_1 and C_2 are constants and as per the initial condition from (13), as

$$f(0,t) = 0, f(2,t) = 0, 0 < x < 2 \text{ and } t > 0.$$

By using $P(0,\eta) = 0$, equation (13) becomes as

$$0 = C_1 e^{2\sqrt{\eta}} + C_2 e^{-2\sqrt{\eta}}$$

$$C_1 = -C_2$$

Again, using $P(2,\eta) = 0$, in equation (13), we get

$$0 = -C_2 e^{2\sqrt{\eta}} + C_2 e^{-2\sqrt{\eta}}$$

$$0 = C_2 (e^{-2\sqrt{\eta}} + e^{2\sqrt{\eta}})$$

But $e^{-2\sqrt{\eta}} + e^{2\sqrt{\eta}} \neq 0$, therefore, we can write $C_2 = 0$

Therefore, complementary solution of equation (12) is Zero.

Also, For determining the particular solution of equation (12)

$$\text{Particular Integral, P.I.} = \frac{1}{D^2 - \eta^2} \left(\frac{-3 \sin 2\pi x}{\eta^2} \right) \quad (14)$$

After, simplification from equation (14), we get

$$P(x, \eta) = P \cos 2\pi x + Q \sin 2\pi x \quad (15)$$

On the way to determine the value of P and Q

By using the Technique of undetermined coefficients, we get

$$\frac{\partial}{\partial x} (P(x, v)) = -2\pi P \sin 2\pi x + 2\pi Q \cos 2\pi x \quad (16)$$

$$\frac{\partial^2}{\partial x^2} (P(x, v)) = -4\pi P \cos 2\pi x - 4\pi Q \sin 2\pi x \quad (17)$$

Substituting the value of $P(x, v)$ and $\frac{\partial^2}{\partial x^2} (P(x, v))$ from equation (16) and (17), in equation (12),

we get

$$-(4\pi^2 P \cos 2\pi x) - (4\pi^2 Q \sin 2\pi x) - \eta(P \cos 2\pi x + Q \sin 2\pi x) = -\frac{3 \sin 2\pi x}{\eta^2}$$

$$(4\pi^2 + \eta)P \cos 2\pi x + (4\pi^2 + \eta)Q \sin 2\pi x = \frac{3 \sin 2\pi x}{\eta^2} \quad (18)$$

Equating the coefficients of like term to both the sides of equation (18)

$$(4\pi^2 + \eta)P = 0 \quad ; \quad (4\pi^2 + \eta)Q = \frac{3}{\eta^2}.$$

$$P = 0. \quad \text{and} \quad Q = \frac{3}{\eta^2(4\pi^2 + \eta)}$$

Therefore, the solution of equation (12), obtain

$$P(x, v) = 0(\cos 2\pi x) + \frac{3}{\eta^2(4\pi^2 + \eta)} \sin 2\pi x$$

$$P(x, v) = \frac{3}{\eta^2(4\pi^2 + \eta)} \sin 2\pi x \quad (19)$$

Relating the inverse Emad-Sara Integral Transform to equation (19), we have

$$ES^{-1}(P(x, \eta)) = \sin 2\pi x ES^{-1}\left(\frac{3}{\eta^2(4\pi^2 + \eta)}\right)$$

$$f(x, t) = 3e^{-4\pi^2 t}(\sin 2\pi x) = P.I \tag{20}$$

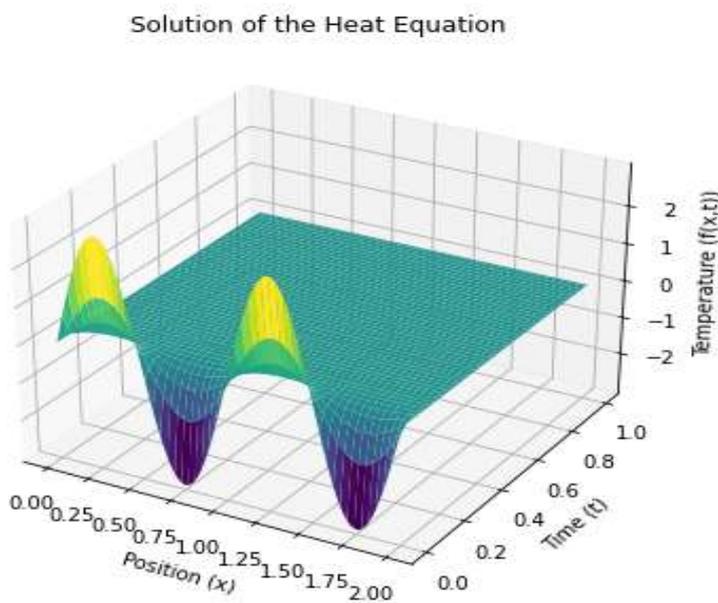
This is the particular solution of (12) and the solution of heat equation (7), we get

$$f(x, t) = 3e^{-4\pi^2 t}(\sin 2\pi x)$$

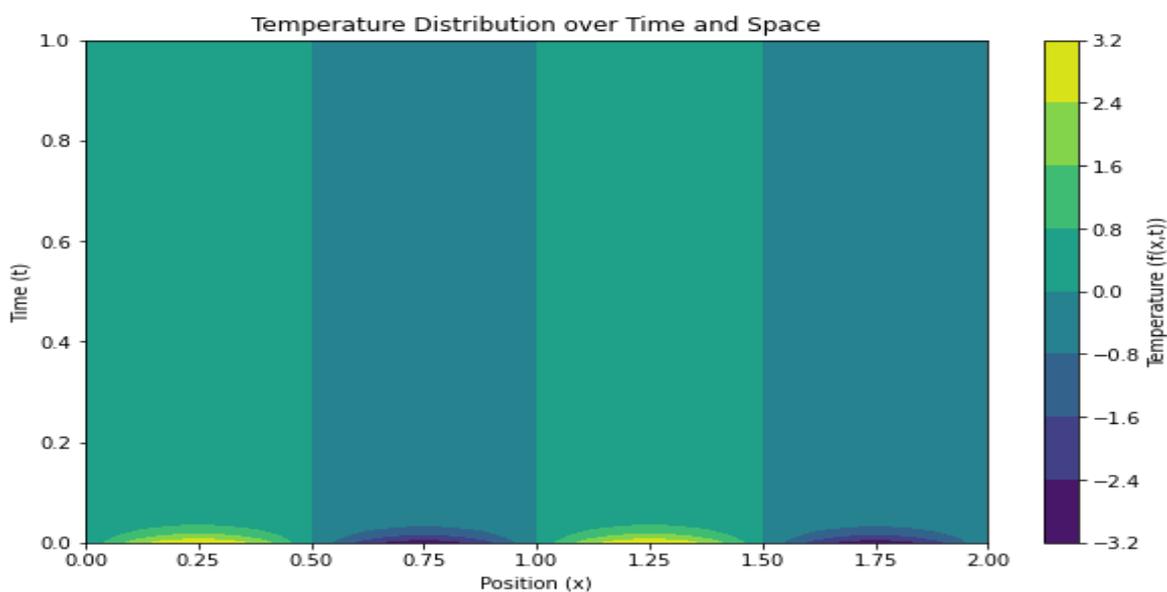
Hence, is the solution of the heat equation, as the given condition $t = 0$, $f(x, 0) = 3 \sin 2\pi x$.

Heat Equation Figures:

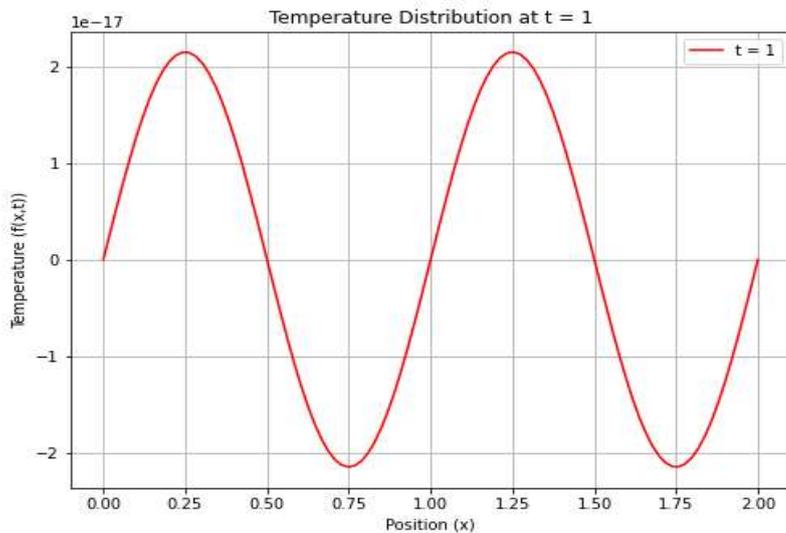
The provided graphs depict the temperature distribution over time and space according to the heat equation and given initial conditions.



Solution of the Heat Equation (Application 4.1)



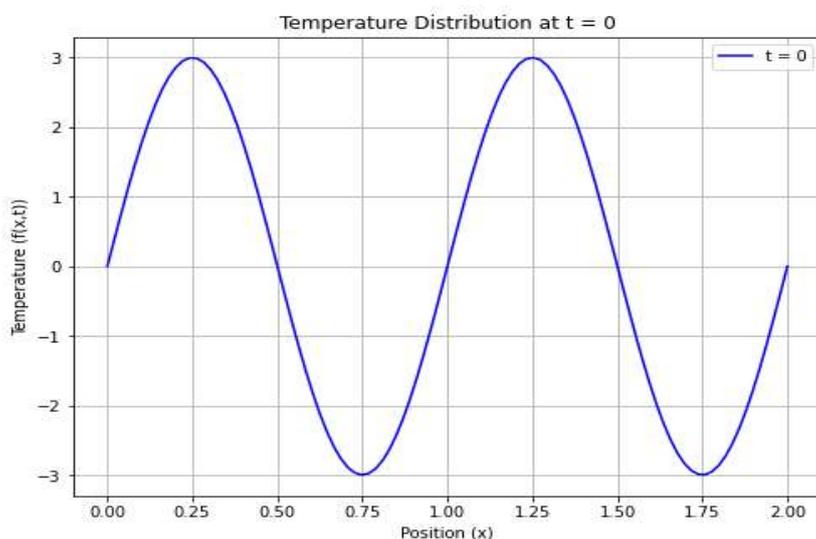
Temperature Distribution Over Time and Space



The **3D surface plot** illustrates how the temperature $f(x, t)$ evolves across both the spatial domain x and time t . In this plot, the x-axis represents the position x , the y-axis represents time t , and the z-axis represents the temperature $f(x, t)$. The color gradient on the surface indicates the temperature at each point in space and time. Cooler temperatures are depicted in blue, while warmer temperatures are shown in yellow. This visualization provides a comprehensive view of how the temperature changes over both space and time.

The contour plot offers a different perspective on the temperature distribution, representing it with contour lines. Similar to the 3D plot, the x-axis represents the position x , the y-axis represents time t , and the color indicates the temperature $f(x, t)$. The contour lines connect points of equal temperature, allowing for a clear visualization of temperature variations over time and space. Cooler temperatures are represented by blue contours, while warmer temperatures are shown in yellow. This plot provides a more detailed view of temperature changes over time and space compared to the 3D surface plot. To conclude, that the line plots at $t = 0$ and $t = 1$ focus on the temperature distribution along the spatial domain x at specific time points. At $t = 0$, the temperature distribution is sinusoidal, reflecting the initial condition $f(x, 0) = 3\sin(2\pi x)$. As time progresses to $t = 1$, the temperature distribution decays due to the heat equation's diffusion process, resulting in a smoother curve compared to $t = 0$. These line plots provide insight into how the temperature evolves at discrete time points along the spatial domain.

Overall, these visualizations offer a comprehensive understanding of how the temperature evolves over time and space according to the heat equation and given initial conditions. They highlight the process of heat diffusion and how it influences the temperature distribution within the domain.



Application-4.2. [WAVE EQUATION]:

Consider the wave equation [6]

$$\frac{\partial^2 f(x, t)}{\partial t^2} = \frac{\partial^2 f(x, t)}{\partial x^2} \tag{21}$$

with the condition $f(x, 0) = \sin x$, $f(0, t) = 0$, $f(\pi, t) = 0$, $\frac{\partial f(x, 0)}{\partial t} = 0$ $0 < x < \pi$ and $t > 0$. (22)

Relating the Emad-Sara Integral Transform to both the side of equation (21), we get

$$ES\left(\frac{\partial^2 f(x, t)}{\partial t^2}\right) = ES\left(\frac{\partial^2 f(x, t)}{\partial x^2}\right) \tag{23}$$

$$\eta^2 P(x, \eta) - \frac{f(x, 0)}{\eta} - \frac{f'(x, 0)}{\eta^2} = \frac{\partial^2}{\partial x^2} ES(f(x, t))$$

$$\frac{\partial^2}{\partial x^2} P(x, \eta) - \eta^2 P(x, \eta) = -\frac{\sin x}{\eta} \tag{24}$$

The complementary solution of equation (24), is the solution of homogeneous equation

$$\frac{\partial^2}{\partial x^2} P(x, \eta) - \eta^2 P(x, \eta) = 0$$
 , which is given by

$$P(x, \eta) = k_1 e^{\eta x} + k_2 e^{-\eta x} \tag{25}$$

where, k_1 and k_2 are constants

Calculating these constants by using the given initial condition as below:

$$f(0, \eta) = 0, \quad f(\pi, \eta) = 0 .$$

Applying the above condition and the condition $P(0, \eta) = 0$, in equation (25), we get

$$0 = k_1 e^{\eta(0)} + k_2 e^{-\eta(0)}$$

$$k_1 = -k_2$$

Again, applying the boundary condition $P(\pi, \eta) = 0$, in equation (25), we get

$$0 = k_2 \left(e^{-\eta(\pi)} - e^{\eta(\pi)} \right)$$

But $\left(e^{-\eta(\pi)} - e^{\eta(\pi)} \right) \neq 0$, therefore, $k_2 = 0$.

Therefore, the solution of equation (24) can be written as

$$P(x, \eta) = M \cos x + N \sin x \tag{26}$$

By using the technique of undetermined coefficients to order determine the constants M and N .

$$\frac{\partial}{\partial x}(P(x, \eta)) = -M \sin x + N \cos x \tag{27}$$

$$\frac{\partial^2}{\partial x^2}(P(x, \eta)) = -M \cos x - N \sin x \tag{28}$$

Substituting equations (26) and (28) in equation (24), we get

$$-M \cos x - N \sin x - \eta^2(M \cos x + N \sin x) = -\frac{\sin x}{\eta}$$

$$(1 + \eta^2)M \cos x + (1 + \eta^2)N \sin x = \frac{\sin x}{\eta} \tag{29}$$

Equating like term to both the sides in equation (29), we get

$$(1 + \eta^2)M = 0; \quad \text{and} \quad (1 + \eta^2)N = \frac{1}{\eta};$$

$$M = 0. \quad N = \frac{1}{\eta(1 + \eta^2)}$$

Hence, the solution of equation (24) becomes

$$P(x, \eta) = \frac{1}{\eta(1 + \eta^2)} \sin x \tag{30}$$

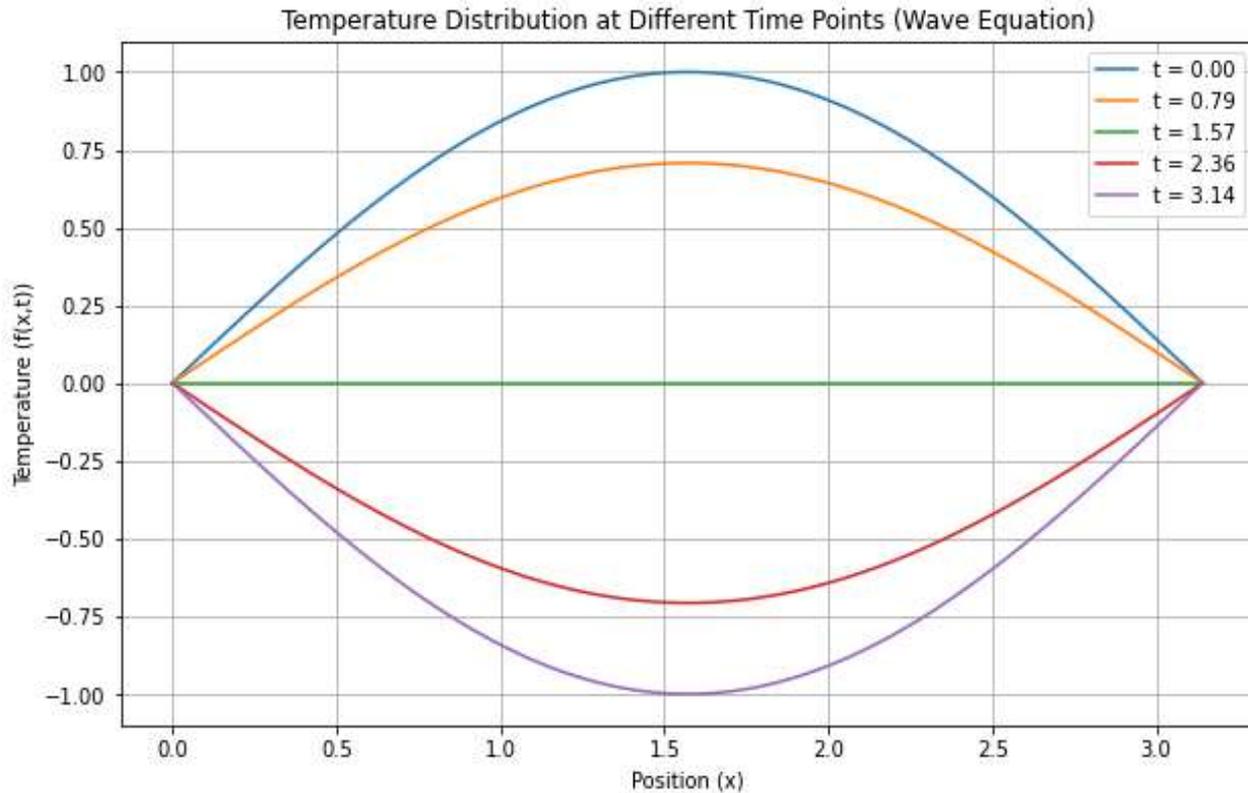
Relating inverse Emad-Sara Integral Transform of equation (30) to both the sides, we get

$$ES^{-1}(B(x, v)) = \sin x ES^{-1}\left(\frac{1}{\eta(1 + \eta^2)}\right)$$

$$f(x, t) = \sin x \cos t, \text{ which is the required solution of (21).}$$

It can be checked that this is certainly the solution of the wave equation, as $t = 0, f(x, 0) = \sin x$.

Wave Equation:



Temperature Distribution at Different Time Points (Application 4.2)

The provided graph depicts the temperature distribution across a one-dimensional spatial domain (x) at various time points, as described by the wave equation $f(x, t) = \sin(x)\cos(t)$. At $t = 0$, the temperature distribution $f(x, 0) = \sin(x)\cos(0) = \sin(x)$ is sinusoidal, exhibiting heights and troughs along the spatial domain x . As time progresses, the cosine term ($\cos(t)$) modulates the amplitude of this sine wave. Consequently, the temperature distribution undergoes sinusoidal oscillations in both space and time. At $t = \pi/2$, the cosine term reaches its maximum value of 1, leading to maximal modulation of the sine wave's amplitude. This modulation significantly alters the temperature distribution compared to its initial state at $t = 0$, resulting in observable changes in the peaks and troughs along x . The graph effectively illustrates the wave-like behavior of the temperature distribution over time, showcasing how it propagates and evolves according to the wave equation. It demonstrates the sinusoidal nature of the temperature variations and emphasizes their dependence on both spatial position and time. Overall, the graph provides a clear visual representation of the dynamic nature of wave propagation and how it influences the temperature distribution within the medium.

DELIBERATION AND OUTCOME

In this chapter, the Iman Transform is effectively demonstrated as a reliable method for solving well-known differential equations that represent heat and wave equations. By applying the Iman Transform, the exact solutions for these equations are derived, providing validation for the effectiveness of this approach. The Iman Transform simplifies the crucial computations involved in solving differential equations that represent the heat and wave equations, resulting in accurate solutions.

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