

Fekete-Szegő Problems and Toeplitz Determinants for Meromorphic Functions involving q -Calculus in a Cardioid Domain

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ABSTRACT

In this paper, we introduce and investigate a new subclass of meromorphic p -valent functions, denoted by $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$, defined in the punctured unit disk \mathbb{U}^* . The definition of this class is characterized by a subordination condition involving the q -derivative operator and a specific subordinating function, $\mathcal{C}(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, which maps the unit disk onto a Cardioid-shaped domain. We first establish a necessary and sufficient condition for functions to belong to this class using the Hadamard product (convolution). Subsequently, we derive sharp bounds for the first two initial Taylor-Laurent coefficients $|a_{1-p}|$ and $|a_{2-p}|$. In addition, we also deal with the Fekete-Szegő functional $|a_{2-p} - \mu a_{1-p}^2|$ for both real and complex parameters μ . The results obtained in this work generalize several existing findings in the literature and highlight the geometric impact of the Cardioid-shaped domain on meromorphic p -valent starlike functions.

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INTRODUCTION AND PRELIMINARIES

The study of univalent and multivalent functions has been a central theme in geometric function theory for decades, with basic results and detailed treatments found in the classical works of Pommerenke [16] and Duren [2]. In recent years, geometric function theory has grown a lot through the use of q -calculus, also known as quantum calculus. The q -difference operator, introduced by Jackson ([6],[7]) provides a natural generalization of the classical derivative. It has been extensively used to define various new subclasses of analytic and meromorphic functions (see,[1],[2],[8],[11],[13],[14],[22],[23] and reference therein) The study of meromorphic p -valent functions is important due to their applications in various branches of mathematics and physics.

Let Σ_p denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p}, \quad (z \in \mathbb{U}^*), \quad (1.1)$$

which are p -valent in the punctured unit disk $\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

Subordination plays an important role in the characterization of function classes in Geometric Function Theory[15]. While traditional classes are often subordinate to half-planes or disks, modern research has shifted toward domains with specific geometric shapes, such as the Nephroid, the Cardioid, and the Lemniscate of Bernoulli. In this article, we consider the subordinating function:

$$L(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2, \quad (1.2)$$

As illustrated in Figure 1, the boundary of the domain is a cardioid-shaped region with a cusp at the origin. This function takes the unit disk \mathbb{U} onto the **Cardioid domain** $\mathcal{C} = L(\mathbb{U})$ [13]. The function $L(z)$ is univalent in \mathbb{U} and produces a region that is symmetric with respect to the real axis.

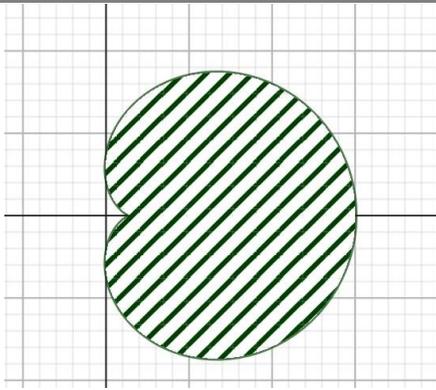


Figure 1: Image of the unit disk under $f(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$.

The study of coefficient functionals, such as the Fekete-Szegő functional[5] and second order Hermitian Toeplitz determinants([11],[24]) provides deep insights into the structural properties of starlike functions. These results are well known for analytic functions, but their study for meromorphic p -valent functions involving the q -derivative operator is still open for research.

Motivated by the works of [12] and recent developments in q -calculus (see [3], [10], [13], [18], [20], [25] and references therein), we introduce and define the class $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$.

Definition 1. For $b \in \mathbb{C} \setminus \{0\}$, the class $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ is defined by:

$$\mathcal{MS}_{p,q}^*[b; \mathcal{C}] = \left\{ f \in \Sigma_p : 1 - \frac{1}{b} \left[\frac{zD_q f(z)}{f(z)} + \frac{[p]_q}{q^p} \right] < 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\}, \quad (1.3)$$

The class $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ represents the q -analog of meromorphic p -valent starlike functions of complex order b . This notation means that the functions are defined using the q -derivative operator and are characterized by subordination to the Cardioid domain \mathcal{C} , providing a broad framework for studying their geometric properties. By choosing appropriate values of the parameters p, q , and b , the class $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ reduces to several well-known and new subclasses. For instance:

- Taking $q \rightarrow 1^-$ and $b = 1$, we obtain the class of meromorphic p -valent starlike functions associated with the Cardioid domain:

$$\mathcal{MS}_p^*[1; \mathcal{C}] = \left\{ f \in \Sigma_p : 1 - \left[\frac{zf'(z)}{f(z)} + p \right] < 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\}.$$

- Taking $p = 1$, the class reduces to the meromorphic q -starlike functions of complex order b associated with the Cardioid domain:

$$\mathcal{MS}_q^*[b; \mathcal{C}] = \left\{ f \in \Sigma_1 : 1 - \frac{1}{b} \left[\frac{zD_q f(z)}{f(z)} + \frac{1}{q} \right] < 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\}.$$

- If we let $q \rightarrow 1^-$, $b = 1$, and replace the Cardioid domain $L(z)$ with the Janowski function $\frac{1+Az}{1-Bz}$, the class reduces to the classical meromorphic p -valent Janowski starlike functions.

Next, we recall some essential definitions and lemmas from q -calculus and subordination theory which are required to prove our main results.

Definition 2. For $0 < q < 1$, the q -number $[n]_q$ is defined by:

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \quad (n \in \mathbb{N}). \quad (1.4)$$

As $q \rightarrow 1^-$, $[n]_q \rightarrow n$.

Definition 3. The q -derivative operator D_q is defined by:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (0 < q < 1). \quad (1.5)$$

Moreover, as $q \rightarrow 1^-$, the operator $D_q f(z)$ converges to the classical derivative $f'(z)$.

For $f \in \Sigma_p$, the q -derivative is given by:

$$D_q f(z) = \frac{-[p]_q}{q^p z^{p+1}} + \sum_{k=1}^{\infty} [k-p]_q a_k z^{k-p-1}, \quad (1.6)$$

Definition 4. Let f, g be analytic functions in \mathbb{U} . We say that f is subordinate to g , written $f < g$, if there exists a Schwarz function ω , analytic in \mathbb{U} with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$. Specifically, if g is univalent in \mathbb{U} , then $f < g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Definition 5. For $f, g \in \Sigma_p$, the Hadamard product (or convolution) $(f * g)(z)$ is defined by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} \quad (1.7)$$

By the definition, it is easy to verify that for any function $f \in \Sigma_p$

$$f(z) * \frac{1}{z^p(1-z)} = f(z) \quad (1.8)$$

Lemma 1. For any meromorphic function $f \in \Sigma_p$, the q -derivative operator D_q satisfies the following convolution identity:

$$f(z) * \frac{1 - \left(q + \frac{1}{[p]_q}\right)z}{z^p(1-z)(1-qz)} = -\frac{q^p}{[p]_q} z D_q f(z), \quad (z \in \mathbb{U}^*). \quad (1.9)$$

Proof. Let $f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}$. The q -derivative of $f(z)$ is given by:

$$D_q f(z) = \frac{-[p]_q}{q^p z^{p+1}} + \sum_{k=1}^{\infty} [k-p]_q a_{k-p} z^{k-p-1}. \quad (1.10)$$

Multiplying both sides by the factor $-\frac{q^p}{[p]_q} z$, we obtain the power series expansion of the right-hand side (RHS):

$$\text{RHS} = z^{-p} - \sum_{k=1}^{\infty} \frac{q^p [k-p]_q}{[p]_q} a_{k-p} z^{k-p}. \quad (1.11)$$

Now, consider the left-hand side (LHS). Let $g(z)$ be defined as:

$$g(z) = \frac{1 - \alpha z}{z^p(1-z)(1-qz)} = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p}, \quad (1.12)$$

where $\alpha = q + \frac{1}{[p]_q}$.

By expanding the rational function into a power series, we find that the coefficients b_{k-p} are given by:

$$b_{k-p} = [k+1]_q - \alpha [k]_q. \quad (1.13)$$

Substituting the value of α and using the q -number identity $[k+1]_q = 1 + q[k]_q$, we have

$$b_{k-p} = (1 + q[k]_q) - \left(q + \frac{1}{[p]_q}\right) [k]_q = 1 - \frac{[k]_q}{[p]_q}. \quad (1.14)$$

Thus, the coefficients of the convolution $(f * g)(z)$ match the coefficients of the modified q -derivative exactly for all $k \geq 1$. This completes the proof. \square

And

$$f(z) * \frac{1 - \left(q + \frac{1}{[p]_q}\right)z}{z^p(1-z)(1-qz)} = -\frac{q^p}{[p]_q} zD_q f(z). \quad (1.15)$$

Definition 6. The Fekete-Szegő functional for a function $f \in \Sigma_p$ is defined as the absolute value of the combination $|a_{2-p} - \mu a_{1-p}^2|$, where μ is a real or complex parameter. The problem of finding sharp upper bounds for this functional is known as the Fekete-Szegő inequality, originally introduced by Fekete and Szegő [5] for univalent functions.

Definition 7. For a function $f \in \Sigma_p$ with coefficients a_{k-p} , the second order Hermitian Toeplitz determinants $T_m(n)$ are defined by the $m \times m$ symmetric matrices as follows:

$$T_m(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+m-1} \\ a_{n+1} & a_n & \dots & a_{n+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+m-1} & a_{n+m-2} & \dots & a_n \end{vmatrix}. \quad (1.16)$$

In the present investigation, we focus on the estimate for $|T_2(1-p)| = |a_{1-p}^2 - a_{2-p}^2|$.

Lemma 2. [24] If $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ is a Schwarz function, then for any $n \in \mathbb{N}$, $|c_n| \leq 1$. Furthermore, for any $n \geq 2$, the following inequality holds:

$$|c_2| \leq 1 - |c_1|^2. \quad (1.17)$$

The next lemma plays a key role in solving the Fekete-Szegő problem.

Lemma 3. [25] Let $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ be a Schwarz function. then for any $\lambda \in \mathbb{C}$,

$$|c_2 - \lambda c_1^2| \leq \max\{1, |\lambda|\}. \quad (1.18)$$

The result is sharp for the functions $\omega(z) = z^2$ and $\omega(z) = z$.

Hadamard Product Properties

Theorem 1. A function $f \in \Sigma_p$ belongs to the class $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ if and only if for all $\theta \in [0, 2\pi)$ and $z \in \mathbb{U}^*$:

$$z^p \left[f(z) * \frac{1 + [\Lambda(\theta) - q]z}{z^p(1-z)(1-qz)} \right] \neq 0. \quad (2.1)$$

Where

$$\Lambda(\theta) = \frac{3}{bq^p(4e^{i\theta} + 2e^{2i\theta})} \quad (2.2)$$

Proof. First, let $f \in \mathcal{MS}_{p,q}^*[b; \mathcal{C}]$, then in order to prove (2.1), we will write (1.13) by using the definition of the subordination, that is

$$-\frac{q^p}{[p]_q} \frac{zD_q f(z)}{f(z)} = 1 + \frac{bq^p(4w(z) + 2w^2(z))}{3[p]_q} \quad (z \in \mathbb{U}^*) \quad (2.3)$$

where w is a Schwarz function, hence

$$z^p \left[-q^p z D_q f(z) - \left\{ [p]_q + \frac{bq^p}{3} (4e^{i\theta} + 2e^{2i\theta}) \right\} f(z) \right] \neq 0 \quad (z \in \mathbb{U}^*). \quad (2.4)$$

Now from (1.8) and (1.15), we may write (2.4) as

$$z^p [f(z) * \frac{\left\{ 1 - \left(q + \frac{1}{[p]_q} \right) z \right\} [p]_q}{z^p (1-z)(1-qz)} - \left\{ [p]_q + \frac{bq^p}{3} (4e^{i\theta} + 2e^{2i\theta}) \right\} \left(f(z) * \frac{1}{z^p (1-z)} \right)] \neq 0 \quad (z \in \mathbb{U}^*),$$

which is equivalent to

$$z^p \left[f(z) * \frac{1 + \left(-q + \frac{3}{bq^p (4e^{i\theta} + 2e^{2i\theta})} \right) z}{z^p (1-z)(1-qz)} \left[-\frac{bq^p}{3} (4e^{i\theta} + 2e^{2i\theta}) \right] \right] \neq 0$$

or

$$z^p \left[f(z) * \frac{1 + \left(\frac{3}{bq^p (4e^{i\theta} + 2e^{2i\theta})} - q \right) z}{z^p (1-z)(1-qz)} \right] \neq 0 \quad (z \in \mathbb{U}^*) \quad (2.5)$$

which is the required condition (2.1). This proves the necessary part of Theorem 1. Conversely, suppose that $f \in \Sigma_p$ satisfy the condition (2.1). Since it was shown in the first part of the proof that assumption (2.3) is equivalent to (2.5), we obtain that

$$-\frac{q^p z D_q f(z)}{[p]_q f(z)} \neq 1 + \frac{bq^p (4e^{i\theta} + 2e^{2i\theta})}{3[p]_q} \quad (z \in \mathbb{U}^*) \quad (2.6)$$

Let us assume that

$$\varphi(z) = -\frac{q^p z D_q f(z)}{[p]_q f(z)} \quad \text{and} \quad \psi(z) = 1 + \frac{bq^p (4z + 2z^2)}{3[p]_q}.$$

The relation (2.1) means that

$$\varphi(\mathbb{U}^*) \cap \psi(\partial\mathbb{U}^*) = \emptyset.$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C} \setminus \psi(\partial\mathbb{U}^*)$. Therefore, using the fact that $\varphi(0) = \psi(0)$ and the univalence of the function ψ , it follows that $\varphi(z) < \psi(z)$, which implies that $f \in \mathcal{MS}_{p,q}^*[b; \mathbb{C}]$. Thus, the proof of Theorem 2.1 is completed. \square

In particular, from Theorem 2.1, we obtain the following corollaries for $p = 1$ and $q \rightarrow 1$:

Corollary 2.1. *If $f \in \Sigma$, then $f \in \mathcal{MS}^*[b; \mathbb{C}]$ if and only if $z \left[f(z) * \frac{1 + [\Lambda_1(\theta) - 1]z}{z(1-z)^2} \right] \neq 0 \quad (z \in \mathbb{U}^*),$*

where $\Lambda_1(\theta) = \frac{3}{b(4e^{i\theta} + 2e^{2i\theta})}$.

Theorem 2.2. *A necessary and sufficient condition for a function $f \in \Sigma_p$ to be in the class $\mathcal{MS}_{p,q}^*[b; \mathbb{C}]$ is that*

$$1 + \sum_{k=1}^{\infty} \left[\frac{3[k]q + bq^p (4e^{i\theta} + 2e^{2i\theta})}{bq^p (4e^{i\theta} + 2e^{2i\theta})} \right] a_{k-p} z^k \neq 0, \quad (2.9)$$

for all $z \in \mathbb{U}$ and $\theta \in [0, 2\pi)$.

Proof. From Theorem 2.1, we find that $f \in \mathcal{MS}^*[b; \mathcal{C}]$ if and only if condition [eq:21] holds. Since

$$\frac{1}{z^p(1-z)(1-qz)} = \frac{1}{z^p} + (1+q)z^{1-p} + (1+q+q^2)z^{2-p} + (1+q+q^2+q^3)z^{3-p} + \dots, \quad (z \in \mathbb{U}^*),$$

hence

$$\frac{1 + [\Lambda(\theta) - q]z}{z^p(1-z)(1-qz)} = \frac{1}{z^p} + \sum_{k=1}^{\infty} (1 + \Lambda(\theta)[k]_q) z^{k-p},$$

where $\Lambda(\theta)$ is given by (2.2).

Now a simple computation shows that (2.1) is identical to (2.9). Thus, the proof of Theorem 2.2 is completed. \square

Theorem 2.3. *If $f \in \Sigma_p$ satisfies the inequality:*

$$\sum_{k=1}^{\infty} [3[k]_q + 2|b|q^p] |a_{k-p}| < 2|b|q^p, \quad (2.10)$$

then $f \in \mathcal{MS}^*_{p,q}[b; \mathcal{C}]$.

Proof. Since

$$\begin{aligned} & \left| 1 + \sum_{k=1}^{\infty} \frac{3[k]_q + (4e^{i\theta} + 2e^{2i\theta})bq^p}{(4e^{i\theta} + 2e^{2i\theta})bq^p} a_{k-p} z^k \right| \\ & \geq 1 - \left| \sum_{k=1}^{\infty} \frac{3[k]_q + (4e^{i\theta} + 2e^{2i\theta})bq^p}{(4e^{i\theta} + 2e^{2i\theta})bq^p} a_{k-p} z^k \right| \\ & \geq 1 - \sum_{k=1}^{\infty} \frac{3[k]_q + 2|b|q^p}{2|b|q^p} |a_{k-p}| > 0. \end{aligned}$$

Thus, the inequality (2.10) holds and our result follows from Theorem 2.2. \square

Coefficient Inequalities And Determinants

In this section, we apply the subordination principles and the properties of Schwarz functions to derive coefficient estimates and the second-order Hermitian Toeplitz determinant for the class $\mathcal{MS}^*_{p,q}[b; \mathcal{C}]$.

Lemma 3.1. *Let $f \in \Sigma_p$ of the form (1.1) belong to the class $\mathcal{MS}^*_{p,q}[b; \mathcal{C}]$.*

Then the coefficients a_{1-p} and a_{2-p} satisfy:

$$a_{1-p} = -\frac{4bq^p}{3} c_1 \quad (3.1)$$

and

$$a_{2-p} = -\frac{4bq^p}{3[2]_q} \left[c_2 - \left(\frac{4bq^p}{3} - \frac{1}{2} \right) c_1^2 \right], \quad (3.2)$$

where c_1, c_2 are the coefficients of a Schwarz function $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$. See [9]

Proof. Let $f \in \mathcal{MS}^*_{p,q}[b; \mathcal{C}]$. By the definition of subordination, there exists a Schwarz function $\omega(z)$ such that:

$$1 - \frac{1}{b} \left[\frac{z^D q^p f(z)}{f(z)} + \frac{[p]_q}{q^p} \right] = L(\omega(z)), \quad (3.3)$$

where $L(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$.

Expanding the right side of (3.3) using $\omega(z) = c_1z + c_2z^2 + \dots$, we obtain:

$$L(\omega(z)) = 1 + \frac{4}{3}c_1z + \left(\frac{4}{3}c_2 + \frac{2}{3}c_1^2\right)z^2 + \dots \quad (3.4)$$

Now, we expand the left side of (3.3). The q -derivative expansion for meromorphic functions:

$$zD_q f(z) = -\frac{[p]_q}{q^p}z^{-p} + [1-p]_q a_{1-p}z^{1-p} + [2-p]_q a_{2-p}z^{2-p} + \dots \quad (3.5)$$

Dividing by $f(z) = z^{-p} + a_{1-p}z^{1-p} + a_{2-p}z^{2-p} + \dots$ and adding the normalization constant $\frac{[p]_q}{q^p}$, we get :

$$1 - \frac{1}{b} \left[\frac{1}{q^p} a_{1-p}z + \left(\frac{[2]_q}{q^p} a_{2-p} + \frac{1}{q^{2p}} a_{1-p}^2 \right) z^2 + \dots \right] \quad (3.6)$$

By comparing the coefficients of z and z^2 from (3.4) and (3.6): For z^1 :

$$-\frac{1}{bq^p} a_{1-p} = \frac{4}{3}c_1 \implies a_{1-p} = -\frac{4bq^p}{3}c_1. \quad (3.7)$$

For z^2 :

$$-\frac{1}{b} \left(\frac{[2]_q}{q^p} a_{2-p} + \frac{1}{q^{2p}} a_{1-p}^2 \right) = \frac{4}{3}c_2 + \frac{2}{3}c_1^2. \quad (3.8)$$

On solving for a_{2-p} gives us:

$$\frac{[2]_q}{q^p} a_{2-p} = -\frac{2bq^p}{3} (2c_2 + c_1^2) - \frac{1}{q^p} a_{1-p}^2. \quad (3.9)$$

Rearranging gives the expression in (3.2). This completes the proof. \square

Theorem 3.1.

Let $f \in \mathcal{MS}_{p,q}^*[b; \mathbb{C}]$ be of the form (1.1). Then for any $\mu \in \mathbb{C}$:

$$|a_{2-p} - \mu a_{1-p}^2| \leq \frac{4|b|q^p}{3[2]_q} \max \left\{ 1, \left| \frac{1}{2} + \frac{4bq^p}{3} (1 - \mu[2]_q) \right| \right\}. \quad (3.10)$$

Proof. From Lemma 3.1, we have

$$a_{1-p} = -\frac{4bq^p}{3}c_1, \quad (3.11)$$

and

$$a_{2-p} = -\frac{4bq^p}{3[2]_q}c_2 + \frac{q^p}{[2]_q} \left(\frac{16b^2q^p - 6b}{9} \right) c_1^2. \quad (3.12)$$

For any $\mu \in \mathbb{C}$, the Fekete-Szegő functional is expressed as:

$$a_{2-p} - \mu a_{1-p}^2 = -\frac{4bq^p}{3[2]_q}c_2 + \left[\frac{q^p(16b^2q^p - 6b)}{9[2]_q} - \frac{16\mu b^2q^{2p}}{9} \right] c_1^2,$$

which simplifies to

$$a_{2-p} - \mu a_{1-p}^2 = -\frac{4bq^p}{3[2]_q} \left[c_2 - \left(\frac{4bq^p}{3} (1 - \mu[2]_q) - \frac{1}{2} \right) c_1^2 \right].$$

Taking the absolute value and using the Keogh-Merkes Lemma, $|c_2 - vc_1^2| \leq \max\{1, |v|\}$,

we obtain

$$|a_{2-p} - \mu a_{1-p}^2| \leq \frac{4|b|q^p}{3[2]_q} \max\left\{1, \left|\frac{4bq^p}{3} (1 - \mu[2]_q) - \frac{1}{2}\right|\right\}.$$

The proof is complete. \square

Theorem 3.2. Let $f \in \mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ be of the form (1.1). Then, the second-order Hermitian Toeplitz determinant $T_2(1)$ satisfies:

$$|T_2(1 - p)| \leq \left(\frac{4|b|q^p}{3}\right)^2 \left[1 + \frac{1}{[2]_q^2} \left(\max\left\{1, \left|\frac{4bq^p}{3} - \frac{1}{2}\right|\right\}\right)^2\right]. \quad (3.13)$$

Proof. By the definition of the Toeplitz determinant for $m = 2$ and $n = 1 - p$, we have:

$$T_2(1 - p) = \begin{vmatrix} a_{1-p} & a_{2-p} \\ a_{2-p} & a_{1-p} \end{vmatrix} = a_{1-p}^2 - a_{2-p}^2. \quad (3.14)$$

so:

$$|a_{1-p}^2 - a_{2-p}^2| \leq |a_{1-p}|^2 + |a_{2-p}|^2. \quad (3.15)$$

From Lemma 3.1, the first coefficient satisfies:

$$|a_{1-p}| = \left| -\frac{4bq^p}{3} c_1 \right| = \frac{4|b|q^p}{3} |c_1|. \quad (3.16)$$

Since $|c_1| \leq 1$ for any Schwarz function, it follows that $|a_{1-p}| \leq \frac{4|b|q^p}{3}$

Next, for the second coefficient a_{2-p} , putting $\mu = 0$ in (3.10) we obtain:

$$|a_{2-p}| \leq \frac{4|b|q^p}{3[2]_q} \max\left\{1, \left|\frac{4bq^p}{3} - \frac{1}{2}\right|\right\}. \quad (3.17)$$

Substituting the bounds from (3.16) and (3.17) into (3.14), we get:

$$|T_2(1 - p)| \leq \left(\frac{4|b|q^p}{3}\right)^2 + \left[\frac{4|b|q^p}{3[2]_q} \max\left\{1, \left|\frac{4bq^p}{3} - \frac{1}{2}\right|\right\}\right]^2. \quad (3.18)$$

Factoring out the term $\left(\frac{4|b|q^p}{3}\right)^2$ gives the desired inequality. This completes the proof. \square

Remark 1. In view of the special cases of the class $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ discussed in Section 1, Theorem 3.3 yields several interesting consequences:

1. For $q \rightarrow 1^-$ and $b = 1$, we obtain the second-order Hermitian Toeplitz determinant for the class of meromorphic p -valent starlike functions associated with the Cardioid domain.

$$\mathcal{MS}_p^*[1; \mathcal{C}]: |T_2(1 - p)| \leq \frac{16}{9} \left[1 + \frac{1}{4} \left(\max\left\{1, \left|\frac{4}{3} - \frac{1}{2}\right|\right\}\right)^2\right] = \frac{16}{9} \left(1 + \frac{25}{144}\right) = \frac{169}{81}.$$

2. For $p = 1$, the result provides the bound for the class $\mathcal{MS}_q^*[b; \mathcal{C}]$, which characterizes the q -starlike functions of complex order b . In this case, the index $n = 1 - p$ becomes $n = 0$, and the bound is given by:

$$|T_2(0)| \leq \left(\frac{4|b|q}{3}\right)^2 \left[1 + \frac{1}{[2]_q^2} \left(\max\left\{1, \left|\frac{4bq}{3} - \frac{1}{2}\right|\right\}\right)^2\right].$$

- By further specializing $q \rightarrow 1^-$, $b = 1$, and $p = 1$, the bound reduces to that of the classical meromorphic starlike functions subordinated to the Cardioid mapping.

While Theorem 3.2 is derived for the Cardioid domain \mathcal{C} , analogous bounds for the Janowski cases mentioned in Section 1 can be obtained by taking note of the coefficients of z and z^2 as $A+B$ and $B(A+B)$ respectively.

CONCLUSIONS

In the present investigation, we introduced and studied the properties of the subclass $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ of meromorphic multivalent functions by employing the q -derivative operator. This class is defined through subordination to the analytic function $L(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, which maps the open unit disc onto a cardioid-shaped bounded symmetric region. The main contributions of this research are summarized as follows: We established necessary and sufficient conditions for membership in the class $\mathcal{MS}_{p,q}^*[b; \mathcal{C}]$ using the convolution (Hadamard product) technique. We derived sharp estimates for the Fekete-Szegő inequality $|a_{2-p} - \mu a_{1-p}^2|$ and the second-order Hermitian Toeplitz determinant $|T_2(1-p)|$. These results illustrate how the geometric parameters of the cardioid domain influence the coefficient bounds of p -valent meromorphic functions. We demonstrated the consistency of our results by showing that they reduce to several well-known and new subclasses as $q \rightarrow 1^-$, $p = 1$, and $b = 1$. By establishing these bounds, we provide a mathematical foundation for future studies into higher-order determinants, such as the third-order Hankel determinant $H_3(1-p)$, or the investigation of other leaf-shaped domains within the framework of q -calculus.

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