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Pythagorean Fuzzy Level Subgroup Cut Set Structures

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ABSTRACT

In this paper, we study a new concept, (α, β)-level of pythagorean fuzzy subgroup which different from fuzzy groups, intuitionistic fuzzy group, $(2, 1)$-fuzzy subgroups. Also we define a new kind of pythagorean fuzzy subgroup and its level cut sets. Finally, some properties of pythagorean fuzzy subgroups are studied.

Key words: fuzzy set, intuitionistic fuzzy set, pythagorean fuzzy set, (α, β)-level cut, fuzzy group, fuzzy level subgroup, intuitionistic fuzzy subgroup, pythagorean fuzzy subgroup.

INTRODUCTION

Due to the failure of the classical set theory to address vagueness and uncertainties Zadeh introduced the notion of fuzzy set. These sets have become acceptable and very useful among many researchers in Mathematics, Computer Science, Engineering and other areas.

Fuzzy set is able to accommodate the element in a set to some degree values between [0, 1].

Many works have been done in this area by many scientists. In order to put algebraic structure on these sets as we have in the case of classical group, Rosenfeld developed the idea of fuzzy subgroups and fuzzy ideals. Mukhuger and Bhattacharya introduced the notion of fuzzy normal subgroups and fuzzy sets. However, Atanassov demonstrated that fuzzy sets do not completely capture the imprecision in vagueness. A situation where the degree of indeterminacy is involved cannot be fully represented by Zadeh's fuzzy sets.

In fuzzy sets, for a non-empty set X, the degree of existence of an element $x \in X$ is represented by $\alpha(x) \in [0, 1]$. In this case $\alpha(x) + \beta(x) = 1$. But Atanassov demonstrated the possibility of having $\alpha(x) + \beta(x) \leq 1$. In this case, $\pi(x) = 1 - \left(\alpha(x) + \beta(x)\right)$ is referred to as the degree of indeterminacy of $x \in X$. Hence the triple $\left(x,\ \alpha(x),\ \beta(x)\right)$ is referred to as the intuitionistic fuzzy sets. Furthermore Biswas, like Rosenfeld developed some group structures on the IFS's of Atanassov. Meanwhile, Yager has demonstrated the possibility of $\alpha(x) + \beta(x) \geq 1$, in which case, the inequality was replaced with $\alpha^2(x) + \beta^2(x) \leq 1$ which is referred to as pythagorean fuzzy sets.

The measure of indeterminacy of $x \in X$ is $\pi(x) = \sqrt{1 - \left(\alpha^2(x) + \beta^2(x)\right)}$. It turned out that every IFS is a PFS but it is not true conversely. Recently, Bhunia and Ghoria put group structure on the PFSs and developed pythagorean fuzzy subgroups.

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As always the case, some limitations of PFSs were found and demonstrated by Senapati and Yager. Thus, the condition $\alpha^2(x) + \beta^2(x) \leq 1$ was replaced by $\alpha^3(x) + \beta^3(x) \leq 1$, in which case the extend of indeterminacy is $\pi(x) = \sqrt{1 - \left(\alpha^3(x) + \beta^3(x)\right)}$ is referred to as fermatean fuzzy sets.

This is a special case of the over of Yager.

Preliminaries:

Definition-2.1: A fuzzy set 'A' in X is a set of ordered pairs $A = \{(x,\ J_A(x))/x \in X\}$, where $J_A(x)$ is the grade of membership of $x \in A$ and $J_A: X \to [0, 1]$ is the membership function.

Definition-2.2: Let $A = \{(x,\ J_A(x))/x \in X\}$ be a fuzzy set. The complement of A is defined as

$A' = \{x \in X, K_A(x) = 1 - J_A(x)\}$.

Definition-2.3: Let X be a group and 'J' a fuzzy subset of X. Then J is called a fuzzy subgroup of X if for any $x, y \in X$, $J(xy) \geq T\{J(x), J(y)\}$ and

$$J(x^{-1}) \geq J(x)$$

Remark-2.4: Let J and K be two fuzzy subgroups of a group X. Their intersection $J \cap K$ =min$\{J, K\}$ be a fuzzy subgroup. But their union $J \cup K$ =max$\{J, K\}$ is not usually a fuzzy subgroup.

Furthermore, let J and K be two anti-fuzzy subgroups of a group X. It can easily be shown that their intersection $J \cap K$ =max$\{J, K\}$ is an anti-fuzzy subgroup. Also their union $J \cup K$ =min$\{J, K\}$ is an anti-fuzzy subgroup.

Theorem-2.5: Let X be a group and 'J'be a fuzzy subset of X. Then J is a fuzzy subgroup of X if and only if $J(xy^{-1}) \geq T\{J(x), J(y)\}$, for any $x, y \in X$.

Definition-2.6: An intuitionistic fuzzy set A in X is defined as $A = \{(x,\ J_A(x),\ K_A(x)/\ x \in X)\}$ where $J_A: X \to [0, 1]$ and $K_A: X \to [0, 1]$ are respectively degree of membership and degree of non-membership for every $x \in X$ with $0 \leq J_A(x) + K_A(x) \leq 1$ and $\pi_A(x) = 1 - \left(J(x) + K(x)\right)$ is the degree of indeterminacy of $x \in X$.

Definition-2.7: Let X be a group and $A = \{(x,\ J_A(x),\ K_A(x)/\ x \in X)\}$ an intuitionistic fuzzy set. Then A is an intuitionistic fuzzy group (IFG) if for any $x, y \in X$, the following are satisfied.

(IFG1): $J_A(xy) \geq T\{J_A(x), J_A(y)\}$

(IFG2): $J_A(x^{-1}) \geq J_A(x)$

(IFG3): $K_A(xy) \leq S\{\ K_A(x),\ K_A(y)\}$

(IFG4): $K_A(x^{-1}) \leq\ K_A(x)$

Proposition-2.8: Let X be a group and $A = \{(x,\ J_A(x),\ K_A(x)/\ x \in X)\}$ an intuitionistic fuzzy set.

Then A is an intuitionistic fuzzy group (IFG) if and only if for any $x, y \in X$, the following are satisfied.

(i) $J_A(xy^{-1}) \geq T\{J_A(x), J_A(y)\}$

(ii) $K_A(xy^{-1}) \leq S\{\ K_A(x),\ K_A(y)\}$

Definition-2.9: A pythagorean fuzzy set A in X is defined as $A = \{(x,\ J_A(x),\ K_A(x) / \ x \in X)\}$ where $J_A: X \to [0, 1]$ and $K_A: X \to [0, 1]$ are respectively degree of membership and degree of non-membership for every $x \in X$ with $0 \leq J^2{}_A(x) + K^2{}_A(x) \leq 1$ and degree of indeterminacy of $x \in X$, $\pi_A(x) = \sqrt{1 - \left(J^2{}_A(x) + K^2{}_A(x)\right)}$.

Definition-2.10: Let X be a group and $A = \{(x,\ J_A(x),\ K_A(x) / \ x \in X)\}$ a pythagorean fuzzy set. Then A is a pythagorean fuzzy subgroup (PFSG) if for any $x, y \in X$, the following are satisfied.

(PFSG 1): $J^2{}_A(xy) \geq T\{J^2{}_A(x), J^2{}_A(y)\}$

(PFSG 2): $J^2{}_A(x^{-1}) \geq J^2{}_A(x)$

(PFSG 3): $K^2{}_A(xy) \leq S\{\ K^2{}_A(x),\ K^2{}_A(y)\}$

(PFSG 4): $K^2{}_A(x^{-1}) \leq\ K^2{}_A(x)$

Proposition-2.11: Let X be a group and $A = \{(x,\ J_A(x),\ K_A(x) / \ x \in X)\}$ a pythagorean fuzzy set. Then A is a pythagorean fuzzy subgroup (PFSG) if and only if for any $x, y \in X$,

(i) $J^2{}_A(xy^{-1}) \geq T\{J^2{}_A(x), J^2{}_A(y)\}$

(ii) $K^2{}_A(xy^{-1}) \leq S\{\ K^2{}_A(x),\ K^2{}_A(y)\}$

Proof: Let A be a pythagorean fuzzy set.

(i) By definition -2.9 and definition- 2.7,

$$J^2{}_A(xy^{-1}) \geq T\{J^2{}_A(x), J^2{}_A(y^{-1})\} = T\{J^2{}_A(x), J^2{}_A(y)\}$$

(ii) By definition -2.9 and definition- 2.7,

$$K^2{}_A(xy^{-1}) \leq S\{\ K^2{}_A(x),\ K^2{}_A(y^{-1})\} = S\{\ K^2{}_A(x),\ K^2{}_A(y)\}$$

Conversely,

(i) Assume that $J^2{}_A(xy^{-1}) \geq T\{J^2{}_A(x), J^2{}_A(y)\}$, then

$$\begin{aligned} J^2{}_A(e) &= T\{J^2{}_A(x), J^2{}_A(x^{-1})\} \\ &\geq T\{J^2{}_A(x), J^2{}_A(x)\} \\ &= J^2{}_A(x) \end{aligned}$$

This implies that $J^2{}_A(e) \geq J^2{}_A(x)$, for all $x \in X$.

Also note that, $J^2{}_A(x^{-1}) = J^2{}_A(ex^{-1})$

$$\begin{aligned} &\geq\ T\{J^2{}_A(e), J^2{}_A(x)\} \\ &= J^2{}_A(x). \end{aligned}$$

This implies that $J^2{}_A(x^{-1}) \geq J^2{}_A(x)$, for all $x \in X$.

Furthermore, let $J^2{}_A(xy) = J^2{}_A(x(y^{-1})^{-1})$

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$$\geq T\{J^2{}_A(x), J^2{}_A(y^{-1})\}$$

There are two possible cases, namely:

$K^2{}_A(x) \geq K^2{}_A(x^{-1})$ and

$K^2{}_A(x) \leq K^2{}_A(x^{-1})$.

Case(i) If $J^2{}_A(x) \geq J^2{}_A(x^{-1})$, then

$$J^2{}_A(x) \geq T\{J^2{}_A(x), J^2{}_A(y^{-1})\} = J^2{}_A(y^{-1}) \geq J^2{}_A(y).$$

In this case, $J^2{}_A(x) \geq J^2{}_A(y)$, then

$$J^2{}_A(xy) = J^2{}_A(x(y^{-1})^{-1}) \geq T\{J^2{}_A(x), J^2{}_A(y^{-1})\}$$

$$= J^2{}_A(y^{-1}) \geq J^2{}_A(y)$$

$$= T\{J^2{}_A(x), J^2{}_A(y)\}.$$

Hence, $J^2{}_A(xy) \geq T\{J^2{}_A(x), J^2{}_A(y)\}$

Case(ii) On the other hand, let $J^2{}_A(x) < J^2{}_A(x^{-1})$

$\Rightarrow J^2{}_A(y^{-1}) < J^2{}_A(y)$.

We know that $J^2{}_A(y^{-1}) \geq J^2{}_A(y)$.

Hence, $J^2{}_A(y^{-1}) \geq T\{J^2{}_A(x), J^2{}_A(y)\}$

Then $J^2{}_A(xy) = J^2{}_A(x(y^{-1})^{-1})$

$$\geq T\{J^2{}_A(x), J^2{}_A(y^{-1})\}$$

$$\geq T\{J^2{}_A(x), J^2{}_A(y)\}$$

So, in the both cases, $J^2{}_A(xy) \geq T\{J^2{}_A(x), J^2{}_A(y)\}$.

(ii) Assume that $K^2{}_A(xy^{-1}) \leq S\{K^2{}_A(x), K^2{}_A(y)\}$.

But $K^2{}_A(e) = S\{K^2{}_A(x), K^2{}_A(x^{-1})\}$

$$\leq S\{K^2{}_A(x), K^2{}_A(x)\}$$

$$= K^2{}_A(x)$$

This implies that $K^2{}_A(e) \leq K^2{}_A(x)$, for all $x \in X$.

Also note that, $K^2{}_A(x^{-1}) = K^2{}_A(ex^{-1})$

$$\leq S\{K^2{}_A(e), K^2{}_A(x)\}$$

$$= K^2{}_A(x).$$

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This implies that ${K^2}_A(x^{-1}) \leq {K^2}_A(x)$, for all $x \in X$.

Furthermore, let ${K^2}_A(xy) = {K^2}_A(x(y^{-1})^{-1})$

$$\leq S\{{K^2}_A(x), {K^2}_A(y^{-1})\}$$

There are two possible cases, namely:

${J^2}_A(x) \geq {J^2}_A(y^{-1})$ and

${J^2}_A(x) < {J^2}_A(y^{-1})$.

Case(i) Let ${K^2}_A(y^{-1}) \geq {K^2}_A(x)$. We know that ${K^2}_A(y^{-1}) \geq {K^2}_A(y)$.

Hence ${K^2}_A(y^{-1}) \leq S\{{K^2}_A(x), {K^2}_A(y)\}$

Then,

$${K^2}_A(xy) = {K^2}_A(x(y^{-1})^{-1}) \leq S\{{K^2}_A(x), {K^2}_A(y^{-1})\}$$

$$\leq S\{{K^2}_A(x), {K^2}_A(y)\}.$$

Hence, ${K^2}_A(xy) \leq S\{{K^2}_A(x), {K^2}_A(y)\}$

Case(ii) On the other hand, let ${K^2}_A(x) < {K^2}_A(y^{-1}) \Rightarrow {K^2}_A(y^{-1}) > {K^2}_A(y)$.

Then ${K^2}_A(x) < {K^2}_A(x^{-1}) \leq {K^2}_A(x)$,

which implies that ${K^2}_A(y) = S\{{K^2}_A(x), {K^2}_A(y)\}$.

Then ${K^2}_A(xy) = {K^2}_A(x(y^{-1})^{-1})$

$$\leq S\{{K^2}_A(x), {K^2}_A(y^{-1})\}$$

$$= {K^2}_A(y^{-1}) = {K^2}_A(y)$$

$$\leq S\{{K^2}_A(x), {K^2}_A(y)\}$$

Hence, A is a pythagorean fuzzy subgroup of X.

Some Standard Results

Proposition-3.1: Let X be a group and $A = \{(x,\ J(x),\ K(x)/\ x \in X)\}$ a pythagorean fuzzy set. Then, for all $x \in X$,

(i) $\{(x,\ J^2(x)/\ x \in X)\}$ is a fuzzy subgroup of X.

(ii) $\{(x,\ K^2(x)/\ x \in X)\}$ is an anti-fuzzy subgroup of X.

Proof: The proof is straight forward by definition-2.1.

Example-3.2: Let the group $X = \{1, -1, i, -i\}$ such that $J(1) = 0.71, J(-1) = 0.59, J(i) = 0.42$ and

$J(-i) = 0$ and $K(1) = 0.10$, $K(-1) = 0.27$, $K(i) = 0.6$, $K(-i)$ =0.6. Obviously, $\big(x,\ J(x),\ K(x)\big)$ is a pythagorean fuzzy subgroup of X.

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Definition-3.3: Let X be a group such that $B_1 = (x,\ J_1,\ K_1)$ and $B_2 = (x,\ J_2,\ K_2)$ are pythagorean fuzzy subgroups from X. For all $x \in X$, the intersection of B_1 and B_2 is $B = \big(x,\ J(x),\ K(x)\big)$, where

$J^2(x) = T\{J^2{}_1(x), J^2{}_2(y)\}$ and $K^2(x) = S\{K^2{}_1(x), K^2{}_2(x)\}$.

Also, the union of B_1 and B_2 is $B = \big(x,\ J(x),\ K(x)\big)$, where $J^2(x) = S\{J^2{}_1(x), J^2{}_2(y)\}$ and $K^2(x) = T\{K^2{}_1(x), K^2{}_2(x)\}$.

Theorem-3.4: Let X be a group such that $B_1 = (x,\ J_1,\ K_1)$ and $B_2 = (x,\ J_2,\ K_2)$ are pythagorean fuzzy subgroups from X. For all $x \in X$, the intersection of B_1 and B_2 is $B = (x,\ J,\ K)$ is also

pythagorean fuzzy subgroup.

Proof: $J^2(xy^{-1}) = T\{J^2{}_1(xy^{-1}), J^2{}_2(xy^{-1})\}$

$$\geq T\left\{T\{J^2{}_1(x), J^2{}_1(y)\},\ T\{J^2{}_2(x), J^2{}_2(y)\}\right\}$$

$$= T\left\{T\{J^2{}_1(x), J^2{}_2(x)\},\ T\{J^2{}_1(y), J^2{}_2(y)\}\right\}$$

$$= T\{J^2(x),\ J^2(y)\}.$$

$$K^2(xy^{-1}) = S\{K^2{}_1(xy^{-1}), K^2{}_2(xy^{-1})\}$$

$$\leq S\{S\{K^2{}_1(x), K^2{}_1(y)\},\ S\{K^2{}_2(x), K^2{}_2(y)\}\}$$

$$= S\{S\{K^2{}_1(x), K^2{}_2(x)\},\ S\{K^2{}_1(y), K^2{}_2(y)\}\}$$

$$= S\{K^2(x),\ K^2(y)\}.$$

Corollary-3.5: Let X be a group such that $\{B_i = (x,\ J_i,\ K_i)\}$ is an arbitrary collection of pythagorean fuzzy subgroups from X. For all $x \in X$, the arbitrary intersection of B_i is also a pythagorean fuzzy subgroup.

Proof:

This is same step in the above theorem-3.4 will proof the corollary.

Remark-3.6: The following example shows that the union of pythagorean fuzzy subgroups is not a pythagorean fuzzy subgroup.

Example-3.7: Consider the group $(Z, +)$.

Let $B_1 = (x,\ J_1,\ K_1)$ and $B_2 = (x,\ J_2,\ K_2)$ be two pythagorean fuzzy subgroups such that

$B = B_1 \cup B_2$=$(x,\ J,\ K)$ and $J,\ K, J_1,\ K_1, J_2,\ K_2$ are defined as follows.

$J(x) = \{0.5, x \in 5Z\ 0.3, x \in (2z - 5z)\ 0.2, elsewhere$ $K(x) = \{0.2, x \in 5Z\ 0.4, x \in (2z - 5z)\ 0.6, elsewhere$

$J_1(x) = \{0.5,\ x \in 5Z\ 0.2, elsewhere$ $K_1(x) = \{0.2,\ x \in 5Z\ 0.7, elsewhere$

$J_2(x) = \{0.3,\ x \in 2Z\ 0.2, elsewhere$ $K_2(x) = \{0.4,\ x \in 2Z\ 0.6, elsewhere$

It will be shown that from the above data,

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$J^2(15+(-2)) = J^2(13) = (0.2)^2$ and

$T\{J^2(15), J^2(-2)\} = T\{(0.5)^2, (0.3)^2\} = (0.3)^2$.

But $(0.2)^2 \ngtr (0.3)^2$. Hence $J^2(xy) \ngeq T\{J^2(x), J^2(y)\}$.

Also, $K^2(15+(-2)) = K^2(13) = (0.6)^2$ and

$S\{K^2(15), K^2(-2)\} = S\{(0.2)^2, (0.4)^2\} = (0.4)^2$.

But $(0.6)^2 \ngtr (0.4)^2$. Hence $K^2(xy) \nleq S\{K^2(x), K^2(y)\}$.

Remark-3.8: By definition, if J is an uncertainty subgroup and K is an anti-uncertainty subgroup of a group X, $J^2(x) \geq J(x)$ and $K^2(x) \leq K(x)$ for any integer value 't'.

Hence, for Pythagorean fuzzy subgroup $B = (x, J, K)$. Since J^2 is an uncertainty subgroup by above proposition, $J^2(x^t) \geq J^2(x)$.

Similarly, since K^2 is an anti-uncertainty subgroup by above proposition, $K^2(x^t) \leq K^2(x)$.

Proposition-3.9: Let $B = (x, J, K)$ be a Pythagorean fuzzy subgroup of a group X. Let $e, x \in X$, where 'e' is the identity. If $J^2(x) = J^2(e)$, then $J^2(xy) = J^2(x)$, for all $x \in X$ and if $K^2(x) = K^2(e)$, then $K^2(xy) = K^2(x)$, for all $x \in X$.

Proof: $J^2(y) = J^2(x^{-1}xy)$

$\geq T\{J^2(x^{-1}), J^2(xy)\}$

$= T\{J^2(e), J^2(xy)\}$

$= J^2(xy)$

$\geq \mathrm{T}\{J^2(x), J^2(y)\}$

$= \mathrm{T}\{J^2(e), J^2(y)\}$

$= J^2(y)$

Hence, $J^2(xy) = J^2(y)$. On the other hand,

$$K^2(y) = K^2(x^{-1}xy)$$

$\leq S\{K^2(x^{-1}), K^2(xy)\}$

$= S\{K^2(e), K^2(xy)\}$

$= K^2(xy)$

$\leq S\{K^2(x), K^2(y)\}$

$= \mathrm{S}\{K^2(e), K^2(y)\}$

$= K^2(y)$

Hence, $K^2(xy) = K^2(y)$.

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Proposition-3.10: Let $B = (x,\ J,\ K)$ be a Pythagorean fuzzy subgroup of a group X. The set

$G = \{x \in X \,/\, J^2(x) = J^2(e)\ and\ K^2(x) = K^2(e)\}$ is a subgroup of X.

Proof: Let $x,\ y^{-1} \in\ G$. By proposition-3.9 and B is a pythagorean fuzzy subgroup,

$J^2(xy^{-1}) = J^2(y^{-1}) = J^2(y) = J^2(e)$. Also by the same reasons,

$K^2(xy^{-1}) = K^2(y^{-1}) = K^2(y) = K^2(e)$. Thus $xy^{-1} \in\ G$.

Hence G is a subgroup of X.

Proposition-3.11: Let X be a group and $B = (x,\ J,\ K)$ be a pythagorean fuzzy subgroup of X. Then, for every $x \in X$,

(i) $J^2(x^{-1}) = J^2(x)$

(ii) $K^2(x^{-1}) = K^2(x)$

(iii) $J^2(e) \geq J^2(x)$

(iv) $K^2(e) \leq K^2(x)$

Proof: Given X is a group and B is a pythagorean fuzzy subgroup of X. Now

(i) To prove $J^2(x^{-1}) = J^2(x)$

$$J^2(x) = J^2((x^{-1})^{-1})$$
$$\geq J^2(x^{-1})$$
$$\geq J^2(x)$$

(ii) To prove $K^2(x^{-1}) = K^2(x)$

$$K^2(x) = K^2((x^{-1})^{-1})$$
$$\leq K^2(x^{-1})$$
$$\leq K^2(x)$$

(iii) To prove $J^2(e) \geq J^2(x)$

$$J^2(e) = J^2(xx^{-1})$$
$$\geq\ T\{J^2(x), J^2(x^{-1})\}$$
$$= J^2(x)$$

(iv) To prove $K^2(e) \leq K^2(x)$

$$K^2(e) = K^2(xx^{-1})$$
$$\leq\ S\{K^2(x), K^2(x^{-1})\}$$
$$= K^2(x)$$

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4.(α,β)-Level of Pythagorean Fuzzy Subgroup

In this section, we study the (α,β)-Level of Pythagorean Fuzzy Subgroup. Bhunia et.al [6] has discussed the Pythagorean fuzzy level subgroup.

Definition-4.1: Let X be a set and B be a Pythagorean fuzzy set of X. Then, the set

$B_{(\alpha,\beta)} = \{x \in X \,/\, J(x) \geq \alpha \; and \; K(x) \leq \beta, (\alpha,\beta) \in [0,1]\}$, where $0 \leq J^2(x) + K^2(x) \leq 1$ is called

Pythagorean fuzzy level subset of the set X.

Definition-4.2: Let X be a group and B be a Pythagorean fuzzy subgroup of X. Then, the set

$B_{(\alpha,\beta)} = \{x \in X \,/\, J(x) \geq \alpha \; and \; K(x) \leq \beta, (\alpha,\beta) \in [0,1]\}$, where $0 \leq J^2(x) + K^2(x) \leq 1$ is called

Pythagorean fuzzy level subgroup of the group X.

Proposition-4.3: Let $B = (x,\ J_b,\ K_b)$ and $C = (x,\ J_c,\ K_c)$ be two pythagorean fuzzy subgroups of the group X. Then the following are hold;

(i) $B_{(\varsigma,\beta)} \subseteq C_{(\varepsilon,\alpha)}$ if $\varepsilon \leq \varsigma$ and $\alpha \geq \beta$ for all $\alpha,\beta,\varepsilon,\varsigma \in [0,1]$.

(ii) $B \subseteq C \Rightarrow B_{(\varsigma,\beta)} \subseteq C_{(\varsigma,\beta)}$, for all $\varsigma \in [0,1]$.

Proof: Note that $\alpha,\beta,\varepsilon,\varsigma \in [0,1]$.

(i) Let $x \in B_{(\varsigma,\beta)}$.

Then $J^2{}_B(x) \geq \varepsilon$ and $K^2{}_B(x) \leq \beta \leq \alpha$.

This implies that $x \in B_{(\varepsilon,\alpha)}$ and thus

$B_{(\varsigma,\beta)} \subseteq B_{(\varepsilon,\alpha)}$.

(ii) Assume that $B \subseteq C$.

Then $J^2{}_B(x) \leq J^2{}_C(x)$ and

$K^2{}_B(x) \geq K^2{}_C(x)$, for all $x \in X$.

$Let\ x \in B_{(\varsigma,\beta)}, J^2{}_B(x) \geq\ \varsigma$ and $K^2{}_B(x) \geq \beta$.

This implies that $J^2{}_C(x) \geq J^2{}_B(x) \geq \varsigma$ and

$K^2{}_C(x) \leq K^2{}_B(x) \leq\ \beta$.

$Hence\ J^2{}_C(x) \geq \varsigma$ and $K^2{}_C(x) \leq\ \beta$, which implies that $x \in C_{(\varsigma,\beta)}$

This implies that $B_{(\varsigma,\beta)} \subseteq C_{(\varsigma,\beta)}$.

Example-4.4: Let $X = S_3$. Then $B = (x,\ J_b,\ K_b)$ and $C = (x,\ J_c,\ K_c)$ are two pythagorean fuzzy subgroups by the definition below;

$J_b(x) = \{0.8, \quad x = e \;\; 0.6, x \in \{(123),(132)\} \;\; 0.4, \quad elsewhere$, $K_b(x) = \{0.1, \quad x = e \;\; 0.4, x \in \{(123),(132)\} \;\; 0.6, \quad elsewhere$

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$J_c(x) = \{0.7, \quad x = e \ 0.4, elsewhere$, $K_c(x) = \{0.2, \quad x = e \ 0.5, elsewhere$

$B_{(0.6,0.4)} = \{e, (123), (132)\}$ and $B_{(0.4,0.7)} = \{e, (12), (123), (23), (132)\}$.

So, $B_{(0.6,0.4)} \subseteq B_{(0.4,0.7)}$.

Note that $B \subseteq C$ and $B_{(0.6,0.4)} = \{e\} \subseteq \{e, (123), (132)\} = B_{(0.4,0.7)}$.

Proposition-4.5: Let $B = (x, \ J, \ K)$ be a pythagorean fuzzy subgroup of a group X. Let the set

$B_{(\varsigma,\beta)}$ be a pythagorean fuzzy subgroup of B for which $J^2(e) \geq \varsigma$ and $K^2(e) \leq \beta$ where 'e' is the identity of X. Then $B_{(\varsigma,\beta)}$ is a subgroup of X.

Proof: Note that $B_{(\varsigma,\beta)} \neq \ \varsigma$.

Since $J^2(e) \geq \varsigma$ and $K^2(e) \leq \ \beta$ implies that ε∈ $B_{(\varsigma,\beta)}$. If no other element is in $B_{(\varsigma,\beta)}$, there is no need to prove anything since $\{e\} \subseteq B_{(\varsigma,\beta)}$.

But suppose that $x, y \in B_{(\varsigma,\beta)}$, then

$$J^2(xy^{-1}) \geq T\{J^2(x), J^2(y)\} = \varsigma \text{ and}$$

$$K^2(xy^{-1}) \leq S\{K^2(x), K^2(xy)\} = \ \beta.$$

Hence, $xy^{-1} \in B_{(\varsigma,\beta)}$.

Thus $B_{(\varsigma,\beta)}$ is a subgroup of X.

CONCLUSION

In this article, some existing improvements on uncertainty sets and their algebraic structures have been studied, namely IFSs, IFSG, PFSGs. All these algebraic structures are derived in detail. In subsequent studies, it will be interest to consider further properties such as pythagorean fuzzy cosets and pythagorean normal uncertainty subgroup and some related properties.

Future direction: This work can be extended into $(2,3)$ - fuzzy sets and Fermat's fuzzy subgroup with cosets.

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