Derivation of Some Properties of New Topp Leone Exponential – Weibul Distribution and Its Application on Cancer Data Sets

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ABSTRACT

The obtainable breast and bladder cancer data sets are asymmetric and skewed in nature with heavy tail and variable hazard function; therefore, a new Topp Leone Exponential – Weibul (TLE-W) distribution with four distinct parameters was developed to fit these asymmetric data sets. Thus, this new distribution was developed by incorporating existing Weibul distribution into Topp Leone Exponential G family of distributions. More so, the respective density and distribution functions of this new Topp Leone Exponential – Weibul distribution were derived alongside with some respective mathematical properties such as moments, quantile function, renyi entropy and order statistics. Also, maximum likelihood estimation and maximum product of spacing methods were used to estimate the distinct four parameters of TLE – W. However, in the simulation study conducted; the results show that the four estimated parameters of Topp Leone Exponential – Weibul distribution are consistent as the bias and root mean square error approach zero. Conclusively, the breast and bladder cancer survival time data sets were used to validate the results obtained from MLE methods on this new distribution. The results show that Topp Leone Exponential – Weibul distribution best fit the two cancer data sets compare to the competitive distributions used in this study. Perhaps, this new distribution would be useful to model positive real life time data sets that are asymmetrical in nature with heavy tails and as well handle data's volatility during instability.

Keywords: Topp Leone Exponential – Weibul distribution; Mathematical properties; Maximum Likelihood Estimation; Simulation study; Application to breast and bladder cancer data sets.

INTRODUCTION

The statistical inference and modeling real-life data sets with probability distributions are of great important applications in many fields such as medicine, engineering, biological science, management, and agriculture among others. Most often data sets from the medical sciences are modeled to understand the time taken in response of treatment to drugs reaction with respect to a certain cancer. Though, cancer refers to a group of diseases characterized by the uncontrolled growth and spread of abnormal cells. So, the common types of cancer includes: carcinoma (skin or tissue), sarcoma (bones or muscles), leukemia (blood forming tissue) and lymphoma (immune system). Then, a distribution was proposed to model the remission times (in months) of a random sample of 128 bladder cancer patients and the survival times of 121 patient with breast cancer obtained from a large hospital in a period from 1929 – 1938. However, the popularly known probability distribution that are often use to model such data sets in place of exponential distribution is the weibull distribution; it had been used to model a life time data sets in various areas of application. Weibul distribution consists of the shape and scale parameters. Its complementary cumulative distribution function is a stretched exponential function. The weibull distribution is related to a number of other probability distributions; in particular, it interpolates between the exponential distribution and the Rayleigh distribution (Rayleigh distribution). If the quantity X is a "time-to-failure", the Weibull distribution gives a distribution for which the failure rate is proportional to a power of time.

The shape parameter is that power plus one, and so this parameter can be interpreted directly according to Jiang, R. and Murthy, D.N.P. (2011). However, the cumulative distribution function (cdf) and probability density function (pdf) of the Weibul distribution are defined as follows:

$$H(z) = 1 - \exp - \left\{ \frac{z}{\vartheta} \right\}^{\delta} \tag{1}$$

and

$$h(z) = \frac{\delta}{\vartheta} \left(\frac{z}{\vartheta} \right)^{\delta - 1} \exp - \left\{ \frac{z}{\vartheta} \right\}^{\delta} \tag{2}$$

Where z is the random variable, δ and ϑ are the parameters

Though, the weibull distribution has been extended with various families of distributions as aimed to have new distributions with improved flexibility in modeling life time data sets. These distributions functions are more flexible to model real life time data sets, some notable ones are; Exponentiated weibull distribution by Pal, M. *et. al.,* (2006), Exponentiated kumaraswamy exponentiated weibull distribution by Noor A. I and Mundher A. K. (2018), The exponential–weibull lifetime distribution by Gauss M. *et.al.,* (2013), Poisson burr X weibull distribution by Abouelmagd T. H. M. et. al., (2019), A New generalized weighted weibull distribution by Salman, A. *et. al.* (2019), Mashall olkin extended weibull distribution by Ghitany et al. (2005), Beta weibull distribution by Lee et al. (2007), the flexible weibull distribution by Bebbington et al. (2007), Kumaraswamy weibull distribution by Cordeiro et al. (2010), Truncated weibull distribution by Zhang and Xie (2011), The Topp-Leone generated weibull distribution by Aryal et al. (2016), Generalized weibull distributions by Lai, (2014), The Kumaraswamy-transmuted exponentiated modified weibull distribution by AlBabtain et al. (2017), Generalized flexible weibull distribution by Ahmad and Iqbal (2017), A reduced new modified weibull distribution by Almalki (2018), and the transmuted exponentiated additive weibull distribution by Nofal et al. (2018). Therefore, in this paper, a new distribution called Topp Leone Exponential Weibull (TLE – W) distribution was developed to model the asymmetric remission time of breast and survival time of bladder cancer data sets with skewness and heavy tails.

METHODOLOGY

This section discusses the fundamental methods used to derive the cdf and pdf of the new distribution, alongside with some of its respective mathematical properties.

A new Topp Leone Exponential – Weibul (TLE – W) distribution:

As weibull distribution is not a suitable model to explain the non-monotone hazard rate function (hrf), such as unimodal, U-shaped or bathtub form according to Salman, A. et. al.. (2019); although, various generalized classes of distributions have been developed and used to explain a wide range of data sets phenomena. However, series of T-X family of distributions were studied to have family of distribution that can be extended by certain distribution to have new compound distributions by Alzaatreh et, al. (2013). Thus, in this study; two families of distributions have been examined that is; Topp Leone and Exponential families, then mixed together with distinct X distribution that is, weibul distribution. Therefore, to do that, the concept of Bourguignon *et. al.,* (2014) on the cdf of Exponential – G family of distributions as defined in equation (3) which can be obtained using the exponential distribution as stated by Oguntunde *et. al.,* (2017) is considered; so, the cdf of exponential family of distributions is defined as:

$$F(z;\tau,\Pi) = 1 - \exp\left\{ -\tau \left(\frac{H(z;\Pi)}{\overline{H}(z;\Pi)} \right) \right\} \tag{3}$$

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Here, we consider the possibility of generating the compound family of distributions called Topp Leone Exponential – X distributions by considering T to be a hybridized Topp Leone and Exponential random variable; then, Weibul distribution is considered as the baseline distribution. As a result, the new distribution is now Topp Leone Exponential Weibul (TLE – W) distribution. So, the cdf together with pdf for T – X class of distributions using Topp Leone Exponential (generator) with two extra shape and scale parameters is derived. To derive this hybridized family of distributions, thus the pdf of Topp Leone distribution with one parameter is considered, and then integrated over the Exponential – G family of distributions as shown in the proof below.

Proof

Consider the pdf of Topp Leone distribution with one parameter defined as $f(z) = 2\varpi z^{\sigma-1}(1-z)(2-z)^{\varpi-1}$

$$\text{thus,} \quad G(z;\varpi,\tau,\Pi) = 2\sigma\int_0^{Tx} z^{\omega-1}(1-z)(2-z)^{\varpi-1}\, dz$$

$$= 2\varpi\int_0^{Tx}(1-z)\left(2z-z^2\right)^{\varpi-1}\, dz$$

$$\text{Where:} \quad T_z = F(z;\tau,\Pi) = 1-\exp\left\{-\tau\left(\frac{H(z;\Pi)}{1-H(z;\Pi)}\right)\right\}$$

let $k = 2z - z^2$, $\frac{dk}{dz} = 2(1-z)$, then, $dk = 2(1-z)dz$ Therefore, $dz = \frac{dk}{2(1-z)}$

if $z=0$, then $k=(2\times 0)-0^2=0$. Also, if $z=T_z$, then $k = 2T_z - T_z^{\,2}$

$$\text{Though,} \quad \int_0^{2T_z-T_z^{\,2}} \sigma[k]^{\varpi-1}\, dk = \left[\frac{\varpi[k]^{\varpi-1+1}}{\varpi-1+1}\right]_0^{2T_z-T_z^{\,2}} = \left[k^{\varpi}\right]_0^{2T_z-T_z^{\,2}}$$

Therefore, $G(z;\varpi,\tau,\Pi) =$

$$= \left[2\left(1-\exp\left\{-\tau\left(\frac{H(z;\Pi)}{1-H(z;\Pi)}\right)\right\}\right)-\left(1-\exp\left\{-\tau\left(\frac{H(z;\Pi)}{1-H(z;\Pi)}\right)\right\}\right)^2\right]^{\varpi}$$

$$= \left[2\left(1-\exp\{-\tau p\}\right)-\left(1-\exp\{-\tau p\}\right)^2\right]^{\varpi}, \qquad \text{let } p = \left(\frac{H((z;\Pi))}{1-H((z;\Pi))}\right)$$

$$= \left[2\left(1-\exp\{-\tau p\}\right)-\left[\left(1-\exp\{-\tau p\}\right)\left(1-\exp\{-\tau p\}\right)\right]\right]^{\varpi}$$

$$= \left[2\left(1-\exp\{-\tau p\}\right)-\left[1-\exp\{-\tau p\}-\exp\{-\tau p\}+\left(\exp\{-\tau p\}\right)^2\right]\right]^{\varpi}$$

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$$= \left[2\left(1-\exp\{-\tau p\}\right)-\left[1-2\exp\{-\tau p\}+\left(\exp\{-\tau p\}\right)^2\right]\right]^{\varpi}$$

$$= \left[2\left(1-\exp\{-\tau p\}\right)\left[-1+2\exp\{-\tau p\}-\left(\exp\{-\tau p\}\right)^2\right]\right]^{\varpi}$$

$$= \left[2-1-2\exp\{-\tau p\}+2\exp\{-\tau p\}-\left(\exp\{-\tau p\}\right)^2\right]^{\varpi}$$

$$= \left[1-\left(\exp\{-\tau p\}\right)^2\right]^{\varpi}$$

$$\text{hence, } G(z;\varpi,\tau,\Pi)=\left[1-\exp\left\{-2\tau\left(\frac{H(z;\Pi)}{1-H(z;\Pi)}\right)\right\}\right]^{\varpi} \quad (4)$$

for $0<z<\infty, 0<\varpi,\tau<\infty$

The differentiation of equation (4) gives rise to the pdf of the family of distributions thus,

$$g(z;\varpi,\tau,\Pi)=\frac{2\varpi\tau\left[h(z;\Pi)\right]}{\left[\overline{H}(z;\Pi)\right]^2}\exp\left\{-2\tau\left(\frac{H(z;\Pi)}{\overline{H}(z;\Pi)}\right)\right\}\left[1-\exp\left\{-2\tau\left(\frac{H(z;\Pi)}{\overline{H}(z;\Pi)}\right)^2\right\}\right]^{\varpi-1} \quad (5)$$

Thus, the support domain for TLE-G family of distributions is $(0,\infty)$

when $z\to 0$, $G(z;\varpi,\tau,\Pi)=\left[1-exp\{-2\tau(0)\}\right]^{\varpi}=\left[1-exp\{0\}\right]^{\varpi}=1-1=0$

when $z\to\infty$, $G(z;\varpi,\tau,\Pi)=\left[1-exp\{-2\tau(\infty)\}\right]^{\varpi}=\left[1-exp\{\infty\}\right]^{\varpi}=\left[1-0\right]^{\varpi}=1$

Where T_x is the Exponential T – X distributions defined in equation (3), z is the random variable, ϖ and τ are the parameters, while Π is the parameter of the base line distribution. Therefore, the derivation of the cdf and pdf of the new Topp Leone Exponential – Weibul (TLE-W) distribution is done as equation (1) and (2) were substituted into (4) and (5), and then equation (6) is considered as used in equation (7) and (8).

$$\frac{1-\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}{\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}=\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1 \quad (6)$$

Hence, the cdf and pdf of the new distribution called TLE-W are respectively defined as

$$G(z;\varpi,\tau,\delta,\vartheta)=\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi} \quad (7)$$

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$$g(z;\varpi,\tau,\delta,\vartheta)=\frac{2\varpi\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta-1}\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}}{\vartheta z^{2}\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi-1} \tag{8}$$

Therefore, if a random variable Z follows TLE-W(z,ζ) with density function given in equation (8) where $\zeta=(\varpi,\tau,\delta,\vartheta)$ is a vector of parameter's, then the survival function $Sv(z,\zeta)$, hazard function $hz(x,\zeta)$ and cumulative hazard function $Ch(x,\zeta)$ for TLE-W distribution are defined as follows:

$$Sv(z,\zeta)=1-\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi} \tag{9}$$

$$hz(z,\zeta)=\frac{2\varpi\tau\delta\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}}{\vartheta z^{2}\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}\left[1-\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi}\right]} \times\left(\frac{z}{\vartheta}\right)^{\delta-1}\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi-1} \tag{10}$$

$$Ch(z,\zeta)=-\ln\left[1-\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi}\right] \tag{11}$$

Important Expansion for TLE – W distribution

This section introduced a useful expansion for the pdf of TLE-W distribution, by using the generalized binomial and taylor series expansions in equation (12), thus if $|z|\prec 1$ and $k\succ 0$ is a real non integer, the power series holds:

$$(1-z)^{k-1}=\sum_{m=0}^{\infty}\frac{(-1)^{m}\Gamma(k)}{m!\Gamma(k-m)}z^{m} \tag{12}$$

Thus, the idea of equation (12) is applied to the last term in (8), so equation (8) we have;

$$g(z;\varpi,\tau,\delta,\vartheta)=\frac{2\varpi\tau\delta}{\vartheta z^{2}\exp-\left\{\frac{z}{\vartheta}\right\}^{\phi}}\left(\frac{z}{\vartheta}\right)^{\delta-1}\sum_{m=0}^{\infty}(-1)^{m}\frac{\Gamma(\varpi)}{m!\Gamma(\varpi-m)}\left[\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{m+1},$$

Furthermore, the idea of exponential expansion is applied to the last term of the above expression, we have:

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$$=\frac{2\varpi\tau\delta}{\vartheta z^{2}\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left(\frac{z}{\gamma}\right)^{\delta-1}\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}}{m!n!\Gamma(\varpi-m)}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)^{n},$$

$$=\frac{2\varpi\tau\delta}{\vartheta z^{2}\exp-\left\{\frac{z}{\vartheta}\right\}^{\phi}}\left(\frac{z}{\vartheta}\right)^{\delta-1}\sum_{m,n,p=0}^{\infty}(-1)^{m+n+p}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}n!}{m!n!p!\Gamma(\varpi-m)(n-p)!}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}\right)^{(n-p)},$$

$$=\frac{2\varpi\tau\delta}{\vartheta z^{2}\exp-\left\{\frac{z}{\vartheta}\right\}^{\phi}}\left(\frac{z}{\vartheta}\right)^{\delta-1}\sum_{m,n,p,q=0}^{\infty}(-1)^{m+n+p}\frac{\Gamma(\varpi)(2\varpi)^{n}(m+1)^{n}n!(n-p)^{q}}{m!n!p!q!\Gamma(\varpi-m)}\left(\left(\frac{z}{\vartheta}\right)^{\delta}\right)^{q},$$

$$=\frac{2\varpi\tau\delta}{\vartheta z^{2}}\left(\frac{z}{\vartheta}\right)^{\delta-1}\sum_{m,n,p,q=0}^{\infty}(-1)^{m+n+p}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}n!(n-p)^{q}}{m!n!p!q!\Gamma(\varpi-m)}\left(\left(\frac{z}{\vartheta}\right)^{\delta}\right)^{p}\left(\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}\right)^{-1},$$

Therefore,

$$f(z;\varpi,\tau,\delta,\vartheta)=\sum_{m,n,p,q,r=0}^{\infty}\Phi_{m,n,p,q,r}\left(\frac{z}{\vartheta}\right)^{\delta(q+1)-1}\exp\left\{\frac{z}{\vartheta}\right\}^{\delta} \tag{13}$$

where $\Phi_{m,n,p,q,r}=(-1)^{m+p+q}\dfrac{2\varpi\tau\delta}{\vartheta z^{2}}\dfrac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}n!(n-p)^{q}}{m!n!p!q!r!\Gamma(\varpi-m)}$

Mathematical Properties

In this subsection, some mathematical properties of the TLE-W distribution are provided such as moments, quantile function, rényi entropy and order statistics.

Moments

Suppose Z is a random variable with TLE-W distribution, then the raw moment, say $\mu_n^{'}$, is given by

$$\mu_k^{'}=E\left(z^{k}\right)=\int_{-\infty}^{\infty}z^{k}g(z;\varpi,\tau,\delta,\vartheta)\,dz \tag{14}$$

$$=\frac{2\varpi\tau\delta}{\vartheta\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left(\frac{z}{\vartheta}\right)^{\phi-1}\sum_{a=0}^{\infty}(-1)^{m}\frac{\Gamma(\varpi)}{m!\Gamma(\varpi-m)}\left[\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{m+1}\int_{-\infty}^{\infty}\frac{z^{k}}{z^{2}}dz$$

$$=\frac{2\varpi\tau\delta}{\vartheta\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left(\frac{z}{\vartheta}\right)^{\delta-1}\sum_{m,n=0}^{\infty}(-1)^{m+n}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}}{m!n!\Gamma(\varpi-m)}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)^{n}\int_{-\infty}^{\infty}z^{k-2}dz$$

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$$=\frac{2\varpi\tau\delta}{\vartheta\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left(\frac{z}{\vartheta}\right)^{\vartheta-1}\sum_{m,n,p=0}^{\infty}(-1)^{m+n+p}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}n!}{m!n!p!\Gamma(\varpi-a)(n-p)!}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}\right)^{(n-p)}\int_{-\infty}^{\infty}z^{k-2}dz$$

$$=\frac{2\varpi\tau\delta}{\vartheta\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left(\frac{z}{\vartheta}\right)^{\delta-1}\sum_{m,n,p,q=0}^{\infty}(-1)^{m+n+p}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}n!(n-p)^{q}}{m!n!p!q!\Gamma(\varpi-m)}\left(\left(\frac{z}{\vartheta}\right)^{\delta}\right)^{q}\int_{-\infty}^{\infty}z^{k-2}dz$$

$$=\frac{2\varpi\tau\delta}{\vartheta}\left(\frac{z}{\vartheta}\right)^{\delta-1}\sum_{m,n,p,q=0}^{\infty}(-1)^{m+n+p}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{n}n!(n-p)^{q}}{m!n!p!q!\Gamma(\varpi-a)}\left(\left(\frac{z}{\vartheta}\right)^{\delta}\right)^{p}\left(\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}\right)^{-1}\int_{-\infty}^{\infty}z^{k-2}dz$$

$$=\sum_{m,n,p,q,r=0}^{\infty}\Omega_{m,n,p,q,r}\left(\frac{z}{\vartheta}\right)^{\delta(p+1)-1}\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}\int_{-\infty}^{\infty}z^{k-2}dz \quad (15)$$

where $\Omega_{m,n,p,q,r}=(-1)^{m+n+p}\frac{2\varpi\tau\delta}{\vartheta}\frac{\Gamma(\varpi)(2\tau)^{n}(m+1)^{b}n!(n-p)^{q}}{m!n!p!q!r!\Gamma(\varpi-m)}$

Quantile function

Consider the cdf of TLE – G family of distributions as defined above in equation (4) above, its quantile function is derived as follows:

$$Q(u)=G(z;\varpi,\tau,\gamma)=\left[1-\exp\left\{-2\tau\left(\frac{H(z;\gamma)}{1-H(z;\gamma)}\right)\right\}\right]^{\varpi} = u$$

$$\exp\left\{-2\tau\left(\frac{H(z;\gamma)}{1-H(z;\gamma)}\right)\right\} = 1-u^{\frac{1}{\varpi}} \quad\Rightarrow\quad -2\tau\left(\frac{H(z;\gamma)}{1-H(z;\gamma)}\right) = \log\left(1-u^{\frac{1}{\varpi}}\right)$$

$$\frac{H(z;\gamma)}{1-H(z;\gamma)} = \left(-\frac{1}{2\tau}\right)\log\left(1-u^{\frac{1}{\varpi}}\right) \quad\Rightarrow\quad k. \quad \text{Thus,} \quad H(z;\gamma)= k\left(1-H(z;\gamma)\right)$$

$$\frac{H(z;\gamma)}{1-H(z;\gamma)} = k \quad\Rightarrow\quad H(z;\gamma)=k\left(1-H(z;\gamma)\right)$$

$$H(z;\gamma)=k-kH(z;\gamma) \quad\Rightarrow\quad H(z;\gamma)+kH(z;\gamma)=k \quad\Rightarrow\quad H(z;\gamma)[1+k]=k$$

Hence, $H(z;\gamma)= \frac{k}{(1+k)}$ (16)

Where $H(z;\gamma)$ is the cdf of the baseline distribution; that is the weibul distribution.

Therefore, the quantile function of TLE-W distribution is derived as thus: consider equation (1) as defined above and substitute in equation (16), we have:

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$$1-\exp-\left(\left\{\frac{z}{\vartheta}\right\}\right)^{\delta}=\frac{k}{1+k} \quad \Rightarrow \quad \exp-\left(\left\{\frac{z}{\vartheta}\right\}\right)^{\delta}= 1-\frac{k}{1+k}$$

$$\exp-\left\{\frac{z}{\vartheta}\right\}=\left[1-\frac{k}{1+k}\right]^{\frac{1}{\delta}} \quad \Rightarrow \quad -\left\{\frac{z}{\vartheta}\right\}=\log\left(\left[1-\frac{k}{1+k}\right]^{\frac{1}{\delta}}\right)$$

Therefore, $z=\dfrac{-\vartheta}{\log\left[\left(1-\left(\frac{k}{1+k}\right)\right)^{\frac{1}{\delta}}\right]}$ (17)

Renyi entropy

It is used as indices of diversity and quantifies the uncertainty or randomness of a system. The Renyi Entropy for TLE-W distribution is defined as

$$I_R(v)=(1-v)^{-1}\log\int_{-\infty}^{\infty} g^{v}(z;\varpi,\tau,\delta,\vartheta)\,dz \qquad for\ v\succ 0\ \ and\ \ v\neq 1$$

Consider equation (8) defined above for the TLE-W distribution.

$$g^{v}(z)=\frac{(2\varpi\tau\delta)^{v}\left(\frac{z}{\vartheta}\right)^{v(\delta-1)}\exp\left\{-2\tau v\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}}{(\vartheta z^{2})^{v}\exp-v\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{v(\varpi-1)}$$

$$=\frac{(2\varpi\tau\delta)^{v}}{(\vartheta z^{2})^{v}\exp-v\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left(\frac{z}{\vartheta}\right)^{v(\vartheta-1)}\sum_{m=0}^{\infty}(-1)^{m}\frac{\left[v(\varpi-1)\right]!}{m!\left[v(\varpi-1)-m\right]}\left[\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{v(\varpi-1)+1}$$

$$=\frac{(2\varpi\tau\delta)^{v}}{(\vartheta z^{2})^{v}\exp-v\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left(\frac{z}{\vartheta}\right)^{v(\vartheta-1)}\sum_{m,n=1}^{\infty}(-1)^{m+n}\frac{\left[v(\varpi-1)\right]!(2\tau)^{n}\left(v(\varpi-1)+1\right)^{n}}{m!n!\left[v(\varpi-1)-m\right]}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)^{n}$$

$$=\frac{(2\varpi\tau\delta)^{v}}{(\vartheta z^{2})^{v}\exp-v\left\{\frac{z}{\vartheta}\right\}^{\phi}}\left(\frac{z}{\vartheta}\right)^{v(\vartheta-1)}\sum_{m,n,p=1}^{\infty}(-1)^{m+n+p}\frac{\left[v(\varpi-1)\right]!(2\tau)^{n}\left(v(\varpi-1)+1\right)^{n}b!}{m!n!p!\left[v(\varpi-1)-m\right](n-p)!}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}\right)^{n-p}$$

$$=\frac{(2\varpi\tau\delta)^{v}}{(\vartheta z^{2})^{v}}\left(\frac{z}{\vartheta}\right)^{q\delta+v(\vartheta-1)}\sum_{m,n,p,q=0}^{\infty}(-1)^{m+n+p}\frac{\left[v(\varpi-1)\right]!(2\tau)^{n}\left(v(\varpi-1)+1\right)^{n}n!(n-p)^{q}}{m!n!p!q!\left[v(\varpi-1)-m\right](n-p)!}\exp v\left\{\frac{z}{\vartheta}\right\}^{\delta}$$

$$=\frac{(2\varpi\tau\delta)^{v}}{(\vartheta z^{2})^{v}}\left(\frac{z}{\vartheta}\right)^{q\delta+v(\vartheta-1)}\sum_{m,n,p,q=1}^{\infty}(-1)^{m+n+p}\frac{\left[v(\varpi-1)\right]!(2\tau)^{n}\left(v(\varpi-1)+1\right)^{n}n!(n-p)^{q}}{m!n!p!q!\left[v(\varpi-1)-m\right](n-p)!}\exp v\left\{\frac{z}{\vartheta}\right\}^{\delta}$$

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$$I_R(v)=(1-v)^{-1}\log\sum_{m,n,p,q=1}^{\infty}\Omega_{m,n,p,q}\int_{-\infty}^{\infty}\exp v\left\{\frac{z}{\vartheta}\right\}^{\delta}dz$$

where $$\Omega_{m,n,p,q}=\frac{(-1)^{m+n+p}(2\varpi\tau\delta)^{v}}{(\vartheta z^2)^{v}}\frac{\left[v(\varpi-1)\right]!(2\tau)^{n}\left(v(\varpi-1)+1\right)^{n}n!(n-p)^{q}}{m!n!p!q!\left[v(\varpi-1)-m\right](n-p)!}\left(\frac{z}{\vartheta}\right)^{q\delta+v(\vartheta-1)} \quad (18)$$

Order Statistics

$$f_{i,k}=\frac{g(z)}{\beta(i,k-i+1)}\sum_{j=0}^{k-i}(-1)^{j}\binom{k-1}{j}G^{j+i-1}(z)$$

$$g(z)G^{j+i-1}(z)=\frac{2\varpi\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta-1}\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}}{\vartheta z^2\exp-\left\{\frac{z}{\vartheta}\right\}^{\phi}}\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi-1}$$

$$\times\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi(j+i-1)}$$

$$=\frac{2\varpi\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta-1}\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}}{\vartheta z^2\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi\left[1+(j+i-1)\right]-1}$$

$$=\frac{2\varpi\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta-1}\left[\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{m+1}}{\vartheta z^2\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\sum_{m=0}^{\infty}(-1)^{m}\binom{\varpi\left[1+(j+i-1)\right]-1}{m}$$

$$=\frac{2\varpi\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta-1}(m+1)^{n}(2\tau)^{n}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)^{n}}{\vartheta z^2\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}n!}\sum_{m,n=0}^{\infty}(-1)^{m+n}\binom{\sigma\left[1+(j+i-1)\right]-1}{m}$$

$$=\frac{2\varpi\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta-1}(n+1)^{n}(2\tau)^{n}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}\right)^{n-p}}{\vartheta z^2\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}n!}\sum_{m,n,p=0}^{\infty}(-1)^{m+n+p}\binom{\varpi\left[1+(j+i-1)\right]-1}{m}\binom{n}{p}$$

$$=\sum_{m,n,p,q=0}^{\infty}(-1)^{m+n+p}\frac{(2\tau)^{n+1}\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta(q+1)-1}(m+1)^{n}}{\vartheta z^2 n!p!\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\binom{\varpi\left[1+(j+i-1)\right]-1}{m}\binom{n}{p}(n-p)^{q}$$

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Therefore, $f_{i,k}=\sum_{j=0}^{k-i}(-1)^{j}\sum_{m,n,p,q=0}^{\infty}(-1)^{m+n+p}\frac{\binom{k-1}{j}}{\beta(i,k-i+1)}\Phi_{m,n,p}$

$$\text{Where, }\ \Phi_{m,n,p}=\frac{(2\tau)^{n+1}\tau\delta\left(\frac{z}{\vartheta}\right)^{\delta(q+1)-1}(m+1)^{n}}{\vartheta z^{2}n!\,p!\exp-\left\{\frac{z}{\vartheta}\right\}^{\delta}}\binom{\varpi\left[1+(j+i-1)\right]-1}{m}\binom{n}{p}(n-p)^{q} \tag{19}$$

Inference

This section discusses two methods of estimations which are Maximum Likelihood Estimate and Maximum Product of Spacing; they are both used to estimate the respective parameters of the new TLE – W distribution. Moreover, these two methods are also used to conduct simulation study as to affirm the consistency of the parameters.

Maximum Likelihood Estimation:

Several approaches are used to estimate a parameter, but the maximum likelihood method is the most commonly used among others. Therefore, the maximum likelihood estimators of the unknown parameters of the TLE-W distribution from complete samples are determined. Let $Z_1,\ldots,Z_n$ be observed values from the TLE-W distribution with a vector of parameters ϕ. The Log-likelihood function can be expressed as

$$\begin{aligned}l(\phi)&=n\log(2)+n\log(\varpi)+n\log(\tau)+n\log(\delta)-n\log(\vartheta)+(\delta-1)\sum_{i=1}^{\infty}\log\left(\left\{\frac{z}{\vartheta}\right\}\right)-2\tau\sum_{i=1}^{\infty}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\\&\quad-2\sum_{i=1}^{\infty}\log(z)+\sum_{i=1}^{\infty}\left(\frac{z}{\vartheta}\right)^{\delta}+(\varpi-1)\sum_{i=1}^{\infty}\log\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]\end{aligned} \tag{20}$$

Differentiating equation (20) above with respect to $\varpi,\tau,\delta,\vartheta$ respectively and equating them to 0, we have:

$$\frac{n}{\varpi}+\sum_{i=1}^{\infty}\log\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]=0$$

$$\frac{n}{\tau}-2\sum_{i=1}^{\infty}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)-2(\varpi-1)\sum_{i=1}^{\infty}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\times\frac{\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}}{\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]}=0$$

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$$\frac{n}{\delta}+\sum_{i=1}^{\infty}\log\left(\left\{\frac{z}{\vartheta}\right\}\right)-2\tau\left(\frac{z}{\vartheta}\right)\exp\left\{\frac{z}{\vartheta}\right\}^{\phi}\sum_{i=1}^{\infty}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)+\left(\frac{z}{\vartheta}\right)\sum_{i=1}^{\infty}\left(\frac{z}{\vartheta}\right)^{\delta}+\left(\varpi-1\right)\left(\frac{z}{\vartheta}\right)\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}$$

$$\times\sum_{i=1}^{\infty}\frac{\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}}{\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]}=0$$

$$-\frac{n}{\vartheta}+\left(\delta-1\right)\sum_{i=1}^{\infty}\log\left(z\right)-2\tau z^{\delta}\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}\sum_{i=1}^{\infty}\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)+\sum_{i=1}^{\infty}z^{\delta}+\left(\varpi-1\right)z^{\delta}\exp\left\{\frac{z}{\vartheta}\right\}^{\phi}\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}$$

$$\times\sum_{i=1}^{\infty}\frac{1}{\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]}=0$$

Maximum Product of Spacing:

If $Z_1 . . . Z_n$ is a random sample from TLE – W distribution having cdf $F\left(z;\varpi,\tau,\delta,\vartheta\right)$. The uniform spacing of a random sample from TLE – W distribution is defined as

$$D_i\left(z;\varpi,\tau,\delta,\vartheta\right) = G\left(z_{(i)};\varpi,\tau,\delta,\vartheta\right)-G\left(z_{(i-1)};\varpi,\tau,\delta,\vartheta\right)$$

$$=\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z_{(i)}}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi}-\left[1-\exp\left\{-2\tau\left(\exp\left\{\frac{z_{(i-1)}}{\vartheta}\right\}^{\delta}-1\right)\right\}\right]^{\varpi} \quad (21)$$

Where $G\left(z_0;\varpi,\tau,\delta,\vartheta\right)=0$ *and* $G\left(z_{n+1};\varpi,\tau,\delta,\vartheta\right)=1$. Clearly $\sum_{i=1}^{n+1}D_i\left(\varpi,\tau,\delta,\vartheta\right)=1$

The maximum product of spacing estimators $\varpi_{MPS},\hat{\tau}_{MPS},\delta_{MPS}\,\&\,\vartheta_{MPS}$ of the parameters $\varpi,\tau,\delta\,\&\,\vartheta$ are obtained by maximizing,

$$H\left(\varpi,\tau,\delta,\vartheta\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i\left(\varpi,\tau,\delta,\vartheta\right) \qquad \text{With respect to } \varpi,\tau,\delta\,\&\,\vartheta,$$

The estimators $\varpi_{MPS},\hat{\tau}_{MPS},\delta_{MPS}\,\&\,\vartheta_{MPS}$ of the parameters $\varpi,\tau,\delta\,\&\,\vartheta$ can be obtained by solving the non-linear equations as follows:

$$\frac{d}{d\left(\varpi,\tau,\delta,\vartheta\right)}H\left(\varpi,\tau,\delta,\vartheta\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i\left(\varpi,\tau,\delta,\vartheta\right)}\left[G\left(z_{(i)};\varpi,\tau,\delta,\vartheta\right)-G\left(z_{(i-1)};\varpi,\tau,\delta,\vartheta\right)\right]=0$$

The individual parameter estimate is obtain as follows

$$\frac{d}{d\left(\varpi\right)}H\left(\varpi,\tau,\delta,\vartheta\right)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i\left(\varpi,\tau,\delta,\vartheta\right)}\left[\Delta_{\varpi}\left(z_{(i)};\varpi,\tau,\delta,\vartheta\right)-\Delta_{\varpi}\left(z_{(i-1)};\varpi,\tau,\delta,\vartheta\right)\right]=0$$

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$$\frac{d}{d(\tau)}H(\tau)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\varpi,\tau,\delta,\vartheta)}\left[\Delta_\tau\left(z_{(i)};\varpi,\tau,\delta,\vartheta\right)-\Delta_\tau\left(z_{(i-1)};\varpi,\tau,\delta,\vartheta\right)\right]=0$$

$$\frac{d}{d(\delta)}H(\delta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\varpi,\tau,\delta,\vartheta)}\left[\Delta_\delta\left(z_{(i)};\varpi,\tau,\delta,\vartheta\right)-\Delta_\delta\left(z_{(i-1)};\varpi,\tau,\delta,\vartheta\right)\right]=0$$

$$\frac{d}{d(\vartheta)}H(\vartheta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\varpi,\tau,\delta,\vartheta)}\left[\Delta_\vartheta\left(z_{(i)};\varpi,\tau,\delta,\vartheta\right)-\Delta_\vartheta\left(z_{(i-1)};\varpi,\tau,\delta,\vartheta\right)\right]=0$$

Simulation Study

Simulation study for different sample size on the efficiency of MLE and MPS method to determine the parameters' consistency of Topp Leone Exponential – Weibull (TLE-W) distribution were carried out. Thus, a random variable Z from Topp Leone Exponential – Weibull (TLE-W) distribution was generated using R Software. Samples of size n = 10; 20; 40; 60; 80 and 100 from TLE-W distribution for some selected combination of parameters were used. This process is repeated N = 1000 time to calculate mean estimate, bias and root mean squared error. The results obtained are given in table 6.1 below. The initial values used for the four parameters of Topp Leone Exponential – Weibull (TLE-W) distribution are $\sigma=1.1,\lambda=1.4,\phi=1.6,\gamma=1.3$. Therefore, from table 3.1; it is observed that when sample size increases the corresponding means values for the four parameters approach the initial values, while the bias and root mean squared error decrease towards zero. Therefore, the parameters of Topp Leone Exponential – Weibull (TLE-W) distribution are consistent as the maximum likelihood method and maximum product of space work very well to estimate the parameters.

Table 3.1: Means, Bias and RMSEs for the TLE-W parameters estimates when $\varpi=1.1,\tau=1.4,\delta=1.6,\vartheta=1.3$

N		MLE ϖ	MLE τ	MLE δ	MLE ϑ	MPS ϖ	MPS τ	MPS δ	MPS ϑ
10	**Mean**	1.2417	0.9786	2.0033	1.3799	1.1550	1.0136	1.7156	1.6096
	Bias	0.1417	-0.4214	0.4033	0.0799	0.0550	-0.3864	0.1156	0.3096
	RMSE	0.6513	0.6158	0.7071	0.2934	0.5919	0.5158	0.5795	0.4485
20	**Mean**	1.1750	1.0227	1.9021	1.4410	1.1492	1.0376	1.6935	1.5749
	Bias	0.0750	-0.3773	0.3021	0.1410	0.0492	-0.3624	0.0935	0.2749
	RMSE	0.5418	0.5524	0.5709	0.2666	0.5106	0.4758	0.4828	0.3635
40	**Mean**	1.1235	1.0528	1.8203	1.4871	1.1342	1.0576	1.6818	1.5610
	Bias	0.0235	-0.3472	0.2203	0.1871	0.0342	-0.3424	0.0818	0.2610
	RMSE	0.3992	0.4501	0.4279	0.2527	0.4192	0.4183	0.3834	0.3124
60	**Mean**	1.1059	1.0488	1.8014	1.4944	1.1166	1.0691	1.6881	1.5589
	Bias	0.0059	-0.3512	0.2014	0.1944	0.0166	-0.330	0.0881	0.2589
	RMSE	0.3394	0.4256	0.3838	0.2383	0.3518	0.4013	0.3241	0.3001
80	**Mean**	1.0922	1.0397	1.7938	1.4962	1.1052	1.0606	1.6979	1.5487
	Bias	-0.0078	-0.3603	0.1938	0.1962	0.0052	-0.339	0.0979	0.2487
	RMSE	0.3202	0.4257	0.3574	0.2371	0.3120	0.3959	0.3078	0.2814
100	**Mean**	1.0770	1.0434	1.7995	1.5014	1.8046	0.9231	0.5997	1.7074
	Bias	-0.0230	-0.3566	0.1995	0.2014	0.1046	-0.4769	-0.0003	0.4074
	RMSE	0.2747	0.4040	0.3498	0.2304	0.2898	0.4842	0.0799	0.4234

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Application to Data Sets

Flexibility illustration of Topp Leone Exponential – Weibull (TLE-W) distribution was carried out in application to real life data sets by comparing its performance with some existing distributions with the feature of weibull distribution. Thus, maximum Likelihood estimation is only considered to estimate the parameters. Therefore, the goodness-of-fit statistic and the MLE's for the models' parameters are presented in Tables 4.1 and 4.2. In comparing the fitted models, this paper firstly considered Akaike information criterion (AIC) and Bayesian Information Criterion (BIC) goodness-of-fit measure. So, the fits of this new Topp Leone Exponential – Weibull (TLE-W) distribution with other competitive distributions such as Topp Leone-Weibull (TL-W), Topp Leone-Generalized –Weibull (TL-GW), Transmuted–Weibull (T-W) and Exponentiated–Generalized Weibull (ET-GW) distributions are compared. However, below are their pdf .

1. Topp Leone Weibull (TL-W) distribution. Diamond et. al., (2021):

$$f_{TL-W}(z) = 2\lambda\alpha\theta z^{\alpha-1}\exp\left\{-2\theta z^{\alpha}\right\}\left(1-\exp\left\{-2\theta z^{\alpha}\right\}\right)^{\lambda-1}$$

2. Topp Leone Generalized Weibull.

$$f_{TL-GW}(z) = 2\sigma\lambda\theta\phi^{\theta}z^{(\theta-1)}\exp\left\{-(\phi z)^{\theta}\right\}\left(1-\exp\left\{-(\phi z)^{\theta}\right\}\right)^{(\sigma\lambda-1)}\left(1-\left(1-\exp\left\{-(\phi z)^{\theta}\right\}\right)^{\lambda}\right)\left(2-\left(1-\exp\left\{-(\phi z)^{\theta}\right\}\right)^{\lambda}\right)^{(\sigma-1)}$$ 3.

Transmuted Weibul

$$f_{L-W}(z) = \left(\frac{\alpha}{\beta^{\alpha}}\right)z^{(\alpha-1)}\exp\left\{-\left(\frac{z}{b}\right)^{\alpha}\right\}\left(1-\phi+2\phi\exp\left\{-\left(\frac{z}{b}\right)^{\alpha}\right\}\right)$$

4. Exponentiated generalized weibull. Oguntunde et. at., (2015)

$$f_{EG-W}(z) = ab\frac{a}{\beta}\left(\frac{z}{\beta}\right)^{a-1}\exp\left\{-\left(\frac{z}{\beta}\right)^{a}\right\}\left[1-\left(1-\exp\left\{-\left(\frac{z}{\beta}\right)^{a}\right\}\right)\right]^{a-1}\left[1-\left(1-\left(1-\exp\left\{-\left(\frac{z}{\beta}\right)^{a}\right\}\right)\right)^{a}\right]^{b-1}$$

From the results presented in the two tables, the distribution with the lowest AIC and BIC values is Topp Leone Exponential – Weibull (TLE-W) distribution when compared to other competitive distributions. Thus, Topp Leone Exponential – Weibull (TLE-W) distribution best fits the two data sets. The data sets are:

Data Set 1

It is bladder cancer (monthly remission times) of 128 sampled patients initially used by Lee and Wang (2003).

0.08, 5.85, 8.26, 11.98, 19.13, 1.76, 10.34, 14.83, 3.88, 5.32, 7.39, 3.25, 4.50, 2.09, 3.48, 4.87, 0.81, 2.62,11.64, 17.36, 1.40, 3.02, 4.34, 34.26, 0.90 , 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.33, 5.49, 7.66, 11.25, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 17.14, 79.05, 1.35, 2.87, 12.07, 21.73, 2.07, 3.36, 6.93,5.62, 7.87, 3.82, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 14.77, 32.15, 2.64, 5.71, 7.28, 9.74, 14.76, 26.31, 5.32, 7.32, 10.06, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 7.93, 11.79, 18.10, 1.46, 4.40, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 8.65, 12.63, 22.69.

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Data Set 2

It consists survival times of 121 patients with breast cancer gotten from a large hospital in a period from 1929 – 1938 initially used by Lee (1992) and Ramos *et al.,* (2013).

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0

In Tables 4.1 & 4.2, the values of log-likelihood (LL), AIC and BIC are minimum and favorable of TLE-W distribution than other existing distributions, this indicates that the new model (TLE-W) best fit the two cancer data sets. It is depicted from the results that the developed model provides best fit. That is, it is more reliable with these types of data. It is also clear that the TLE-W distribution provides the best fit as compare to TL-W, TLG-W, ETG-W and T-W for the given two cancer data sets. So, the TLE-W model could be chosen as the best model. Also, the reliability of TLE-W distribution as the best fit to the two cancer data sets is displayed on the two graph on figure 2. While, figure 1 depicts the nature of data distributions TLE-W distribution can best handle.

Table 4.1**.** Parameters Estimation for Various Distributions depending on data set 1.

Models	Estimates	$-LL$	AIC	BIC	Rank
TLE-W	$\varpi=7.6289$ $\tau=0.6112$ $\delta=0.2638$ $\vartheta=0.4381$	410.933	829.8660	830.2948	1
TL-W	$\theta=1.2438$ $\lambda=0.9633$ $\varphi=0.0673$	412.7324	831.4648	833.8936	2
T-W	$\alpha=0.6672$ $\phi=3.9606$ $\gamma=-1.1289$	412.0173	830.0346	832.4634	3
TL-GW	$\theta=0.4726$ $\psi=1.2773$ $\phi=1.3735$ $\gamma=0.0451$	420.0049	848.0098	848.4386	4

Table 4.2**.** Parameters Estimation for Various Distributions depending on data set 2.

Models	Estimates	$-LL$	AIC	BIC	Rank
TLE-W	$\varpi=2.5433$ $\tau=0.0501$ $\delta=0.2763$ $\vartheta=1.0189$	578.6631	1165.326	1165.6573	1
TL-W	$\theta=3.0203$ $\lambda=0.6987$ $\varphi=0.0668$	583.3204	1172.641	1174.9719	2
TL-GW	$\theta=9.2632$ $\psi=3.2347$ $\phi=0.2358$ $\gamma=1.2798$	598.4617	1204.923	1205.2545	3
T-W	$\alpha=0.4785$ $\phi=11.0835$ $\gamma=-1.4375$	603.3242	1212.648	1214.9795	4
ET-GW	$\delta=0.8945$ $\pi=1.2991$ $\phi=0.1684$ $\gamma=0.2231$	772.4545	1552.909	1553.2401	5

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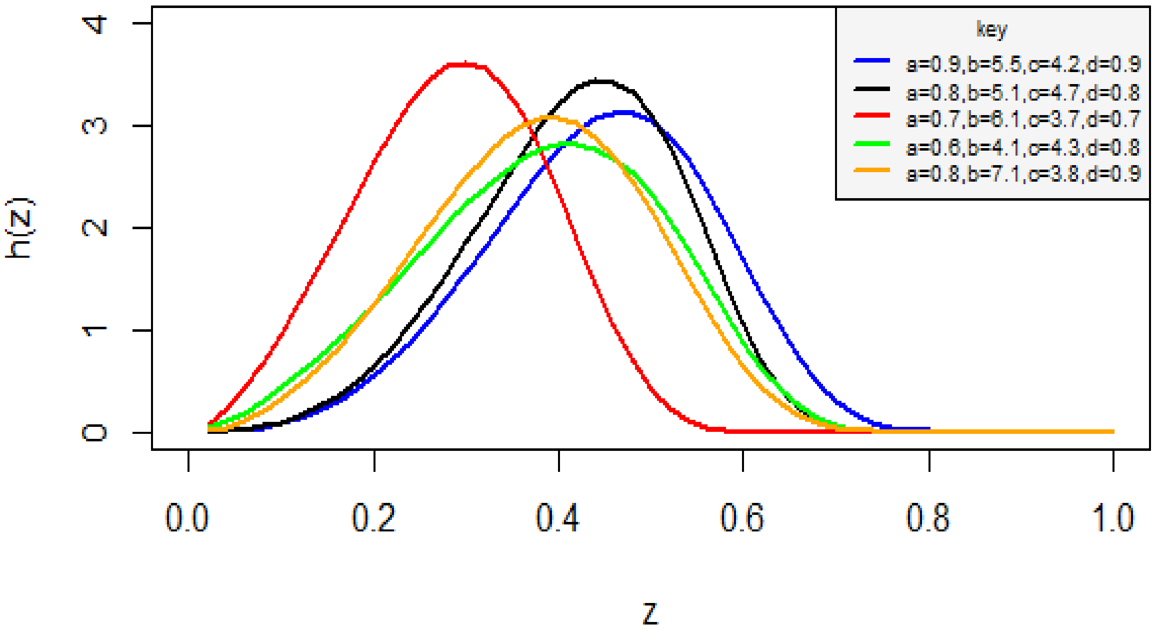
Shape of the new TLE-W distribution



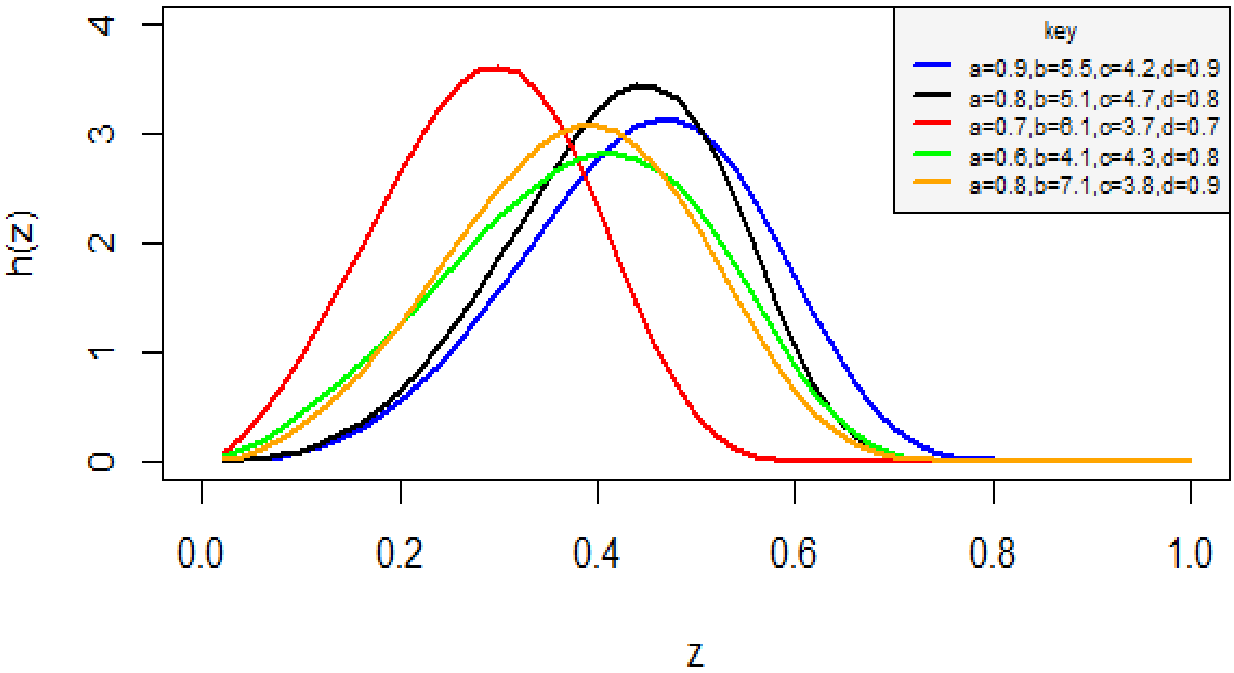
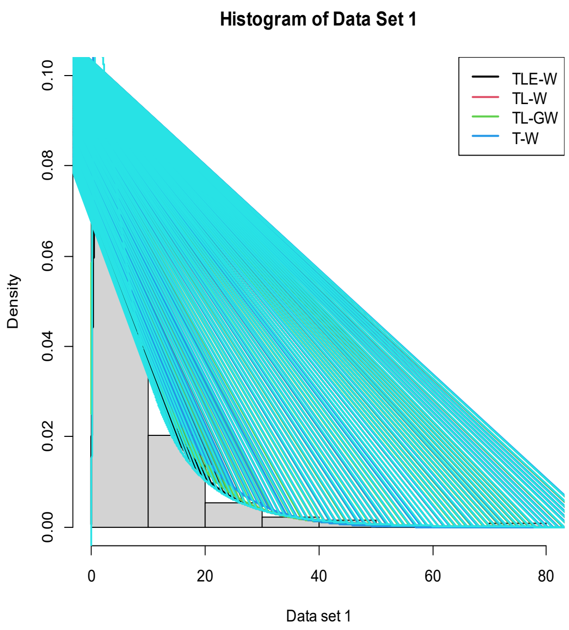


Figure 1 pd function of TLE-W



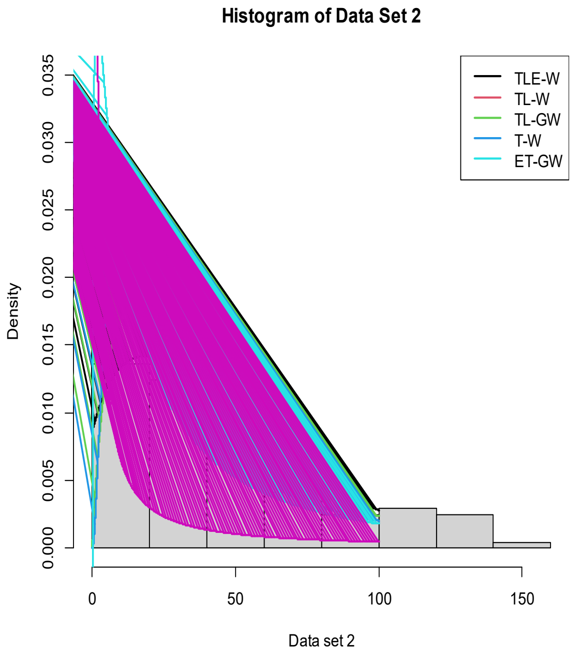


Figure 2: Model fitting graph

Additional Model Diagnostics: Hazard Function, Q-Q/P-P Plots and Further Goodness-of-Fit Statistics

To further strengthen the empirical validation of the Topp Leone Exponential–Weibull (TLE-W) model presented above, this section supplements Tables 4.1 and 4.2 with (i) the fitted hazard function of the TLE-W distribution for each data set, (ii) Quantile-Quantile (Q-Q) and Probability-Probability (P-P) plots comparing the fitted model with the empirical distribution of the data, and (iii) three additional goodness-of-fit statistics -- the Kolmogorov-Smirnov (KS) statistic, the Anderson-Darling statistic (A*) and the Cramér-von Mises statistic (W*) -- together with the associated KS p-value.

So, the additional goodness-of-fit statistics in table 4.3 reports the Kolmogorov-Smirnov (KS) statistic and its p-value, the modified Cramér-von Mises statistic (W*) and the modified Anderson-Darling statistic (A*) for the TLE-W fit to both data sets, computed following the standard modification. For both data sets the KS statistic is small (D = 0.0453 for Data Set 1 and D = 0.0532 for Data Set 2) with correspondingly large p-values (0.945 and 0.865), so the claim that the data arise from the fitted TLE-W distribution is not rejected at the 5% level. Also, the W* and A* statistics are likewise small relative to the conventional critical values used for these tests, reinforcing the conclusion already drawn from the AIC & BIC comparison in Tables 4.1 and 4.2 that the TLE-W distribution gives an adequate and competitive fit to both the bladder and breast cancer survival data sets.

Table 4.3. Kolmogorov-Smirnov (KS), Cramér-von Mises (W*) and Anderson-Darling (A*) goodness-of-fit statistics for the TLE-W fit to the two cancer data sets.

H_0: the data sets follow the TLE-W distribution.

Data set	**KS statistic D (p-value)**	**Cramer-von Mises W***	**Anderson-Darling A***	**Decision (5% level)**
Data Set 1 -- Bladder cancer (n = 128)	0.0453 (p = 0.945)	0.0432	0.2870	Fail to reject H_0: TLE-W fits the data
Data Set 2 -- Breast cancer (n = 121)	0.0532 (p = 0.865)	0.0565	0.4254	Fail to reject H_0: TLE-W fits the data

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Figure 3 displays the fitted hazard function of the TLE-W distribution for the two cancer data sets, evaluated at the parameter estimates in Table 4.1. For Data Set 1 (bladder cancer), the hazard rises from the origin and approaches a slowly increasing, near-constant level over the observed range, consistent with an increasing/near-constant failure-rate pattern typical of remission-time data. For Data Set 2 (breast cancer), the hazard likewise increases with time and flattens at longer survival times. In both cases the hazard shape supports the flexibility claimed for TLE-W in modelling monotone hazard rates, complementing the density shapes shown in Figure 1.

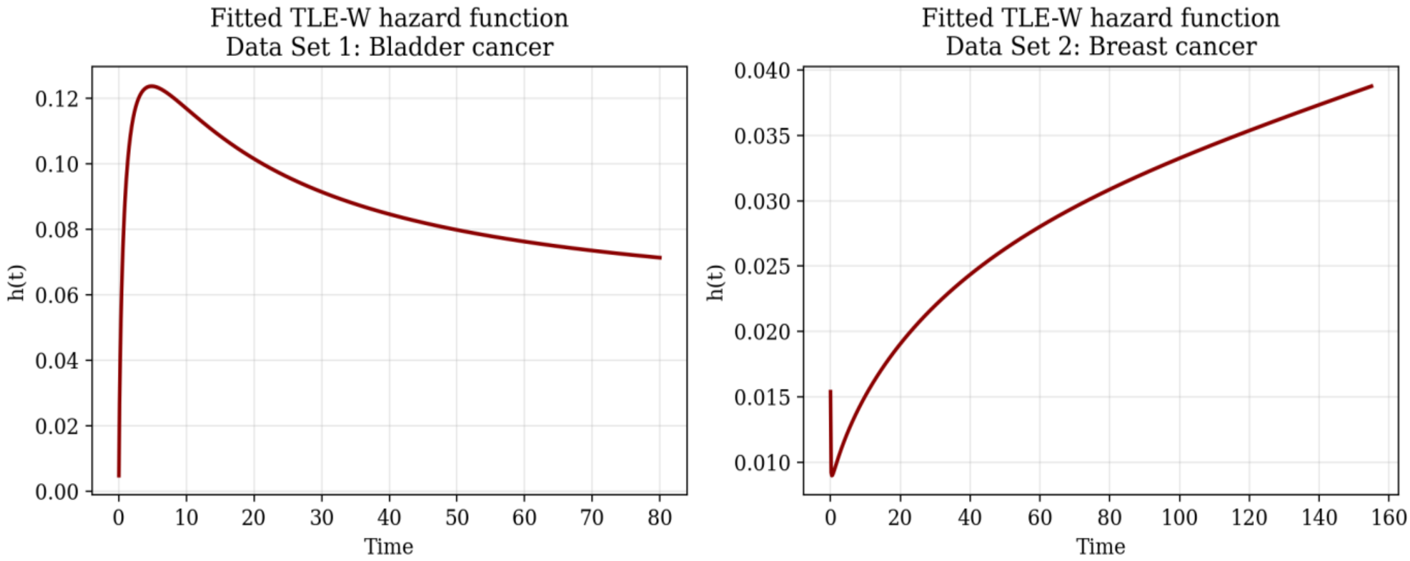

Figure 3: Fitted hazard function h(t) of the TLE-W distribution for the two cancer data sets.

More so, Figures 4 and 5 present the Q-Q and P-P plots of the fitted TLE-W distribution for Data Set 1 and Data Set 2 respectively. In each Q-Q plot, the sample order statistics are plotted against the corresponding theoretical quantiles obtained by inverting the fitted TLE-W cdf; in each P-P plot, the empirical cumulative probabilities (i - 0.5)/n are plotted against the fitted cdf evaluated at the ordered observations. For both data sets, the points lie close to the 45-degree reference line across the whole range, with only minor departures in the extreme upper tail (largest observations, e.g. 79.05 months for Data Set 1 and 154 months for Data Set 2), indicating that the TLE-W distribution provides an adequate description of the bulk and tail behaviour of both data sets.

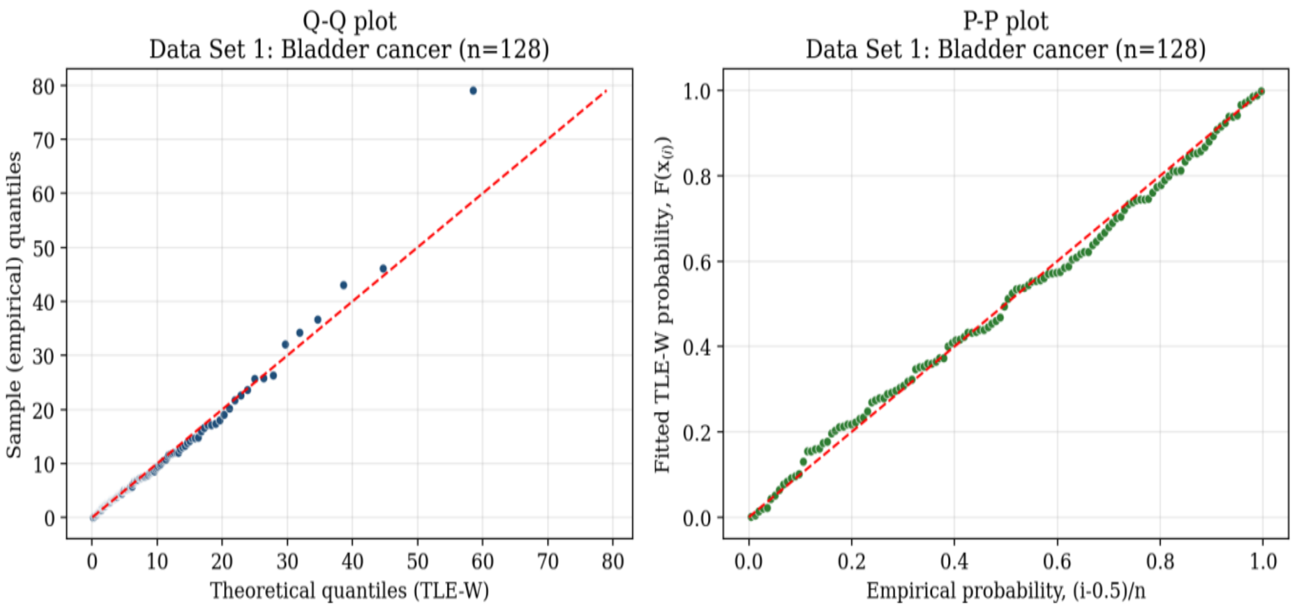

Figure 4: Q-Q and P-P plots of the fitted TLE-W distribution -- Data Set 1 (bladder cancer, n = 128).

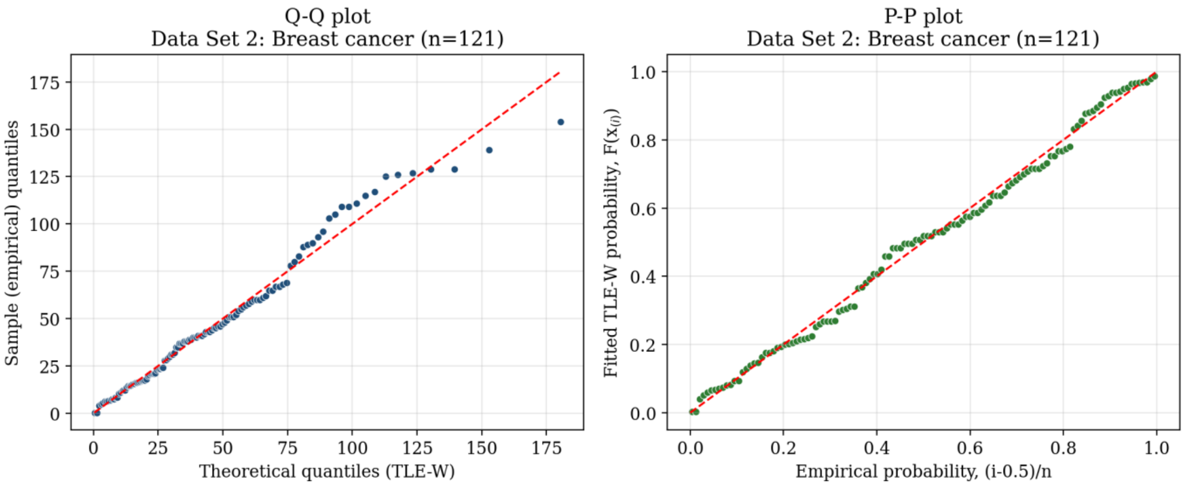

Figure 5: Q-Q and P-P plots of the fitted TLE-W distribution -- Data Set 2 (breast cancer, n = 121).

CONCLUSION

A new distribution called Topp-Leone Exponential – Weibull (TLE–W) distribution with four distinct parameters was developed. That is, a new distribution that possesses the features of topp leone, exponential and weibull distributions with four distinct parameters. This Topp-Leone Exponential–Weibul distribution is an additional significant model to the literature of applied statistics and biostatistics. With the use of Binomial expansion theorem, some mathematical properties of this new distribution were derived. The efficiency performance of the four parameters is consistent, however tested through simulation study with the method of maximum likelihood estimate and maximum product of spacing which showed that the parameters are consistent. The empirical analysis of this new TLE – W distribution presented in the discussion depict that the TLE – W density provides a visually compelling and quantitatively superior fit and flexibility to both the Cancer data sets, An evidence is the close display of the empirical histograms, the minimum values of AIC and BIC togetherness the values of other goodness of fit statistics. Also, the hazard functions exhibit flexible, non-monotonic shapes that reflect realistic bathtub risk patterns that are inaccessible to other simpler at most three-parameter competitors such as Transmuted Weibul; only posses the features of transmuted and weibul distribution, Topp Leone Weibul; only possesses the features of Topp Leone and Weibull distribution, Topp Leone Generalized Weibul only possesses the features of Topp Leone and generalized Weibull distribution and Exponentiated Generalized Weibul only possesses the features of exponentiated and generalized weibul distribution. Though, the additional characteristics of exponential distribution in this new four parameters structure distribution (TLE-W) helps to capture the exponential trend in the two data sets which are statistically identifiable and computationally tractable, with MLEs and MPS that can be obtained through standard numerical optimization routine.

Conflict of Interest

This research work has not been sent to any journal for either review or publication, it is free from any copy right.

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Author contribution

A. A Sanusi and F. N Musa did the introduction write up, derived the methodologies, inference and interpretations. A U Shelleng did all the applications parts of the work.

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