

Exact Solution of First-Order Ordinary Differential Equations by Multiple Integral Transform Methods: A Four-Parameter Unification of the Gbenga Gideon Integral Transforms with Numerical Validation

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ABSTRACT

Linear first-order initial value problems are routinely solved in the literature by a growing catalogue of Laplace-type integral transforms, including the Laplace, Aboodh, Elzaki, Mahgoub, Mohand, and Sumudu transforms. A sizeable strand of recent work solves the same benchmark equations by two or more of these transforms and reports, example by example, that the answers coincide. What that strand has lacked is a structural explanation of the coincidence and an independent check of the computed solutions.

This study solves first-order ordinary differential equations with constant coefficients using the Gbenga Gideon integral transform (GGIT), a four-parameter Laplace-type transform with kernel $pv^m e^{-qv^w t}$, and simultaneously by the Laplace, Aboodh, and Elzaki transforms. The central aim is to prove, not merely observe, that the four methods must return the same solution, and to quantify the exactness of that solution numerically.

The operational calculus of the GGIT needed for first-order problems is developed as formal theorems with complete proofs: transforms of elementary functions, linearity, both shifting theorems, change of scale, and transforms of derivatives. A parameter dictionary identifies the Laplace, Aboodh, and Elzaki transforms as fixed choices of (p, q, m, w) , and a method-invariance proposition converts the empirical agreement of the four calculations into a corollary of uniqueness. Four benchmark initial value problems spanning unforced decay, polynomial forcing, harmonic forcing, and constant forcing are then solved by all four methods, and every closed-form solution is validated against adaptive Runge-Kutta integration at tolerance 10^{-10} .

The four transform methods produce identical solutions for each benchmark, exactly as the invariance proposition requires. Across the four problems the maximum absolute discrepancy between the closed forms and the numerical reference lies between 2.4×10^{-11} and 1.0×10^{-9} , with root-mean-square errors between 4.4×10^{-12} and 1.4×10^{-10} , at the level of the integrator's own tolerance. A symbolic benchmark shows the transform-domain route resolving the harmonically forced problem approximately 5.4 times faster than direct symbolic integration.

For first-order constant-coefficient problems, the choice among Laplace-type transforms is a choice of bookkeeping, not of mathematical power: all are instances of a single four-parameter calculus, and their

agreement is a theorem. The GGIT makes this explicit by carrying the parameters symbolically through one computation that specializes to every named method at once.

Keywords: first-order ordinary differential equation; integral transform; Gbenga Gedion integral transform; Laplace transform; Aboodh transform; Elzaki transform; exact solution; comparative study; numerical validation

INTRODUCTION

An integral transform converts a function $f(t)$ on the half-line into an image $G(v)$ through an integral of the form $G(v) = \int_0^\infty k(t, v) f(t) dt$, where $k(t, v)$ is the kernel [1, 4, 5]. When the kernel is of exponential type the transform converts differentiation in t into multiplication in the image variable, and a linear differential equation with constant coefficients collapses into a linear algebraic equation. Solving the algebraic equation, decomposing the result into partial fractions, and inverting term by term returns the exact solution of the original problem. This three-step scheme, classical for the Laplace transform [1, 2, 3], is the common engine behind the wider family of Laplace-type transforms taken up in this study, each distinguished only by its kernel and catalogued formally in Table 1 below [16–22, 26, 27, 29–33].

A conspicuous strand of this literature is comparative: the same first-order benchmark problem is solved by two or more named transforms and the identical answers are displayed side by side [23, 24, 25, 34]. Such comparisons are valuable pedagogically, but as they stand, they are observations, not explanations. Each computation travels through a different image function, a different partial-fraction pattern, and a different inversion table, and the reader is left to regard the agreement of the final answers as a happy coincidence verified case by case.

This study removes the coincidence. The GGIT carries four parameters (p, q, m, w) in its kernel $pv^m e^{-qv^w t}$ and recovers the Laplace, Aboodh, and Elzaki transforms, among dozens of others, as fixed parameter choices. Solving a first-order initial value problem once with the parameters left symbolic therefore solves it simultaneously under every named method. Agreement of the four methods on the benchmark problems then follows from a one-line invariance proposition backed by the uniqueness theorem of the calculus, and the comparative computations in this study become what they should be: a concrete verification of a proved statement, carried out in full so that each named method's bookkeeping can be inspected and compared for algebraic effort.

The study adds a second layer of verification that the comparative literature has not supplied. Every closed-form solution obtained here is checked against a high-accuracy adaptive Runge-Kutta integration of the same initial value problem, with the discrepancy quantified in absolute, root-mean-square, and normalized relative terms. The transform solutions are exact; the tables in Section 9 measure that exactness against an independent numerical standard rather than asserting it.

The remainder of the paper is organized as follows. Section 2 reviews the transform literature; Section 3 states the four specific gaps this study addresses; Section 4 lists the contributions. Sections 5 to 7 develop the GGIT calculus needed for first-order problems, culminating in the solution and method-invariance theory of Section 7. Section 8 works four benchmark problems through all four named methods; Section 9 validates every closed form numerically against RK45; Section 10 analyzes computational effort. Sections 11 to 13 discuss the results, state their limitations, and conclude.

LITERATURE REVIEW

Classical transform methods for ordinary differential equations

The foundational theory of the Laplace transform was given definitive form by Doetsch [1] and Widder [2], with modern accounts in Gonçalves [3], Debnath and Bhatta [4], and Davies [5]. Its place within the wider families of integral transforms and integral equations, and its structural link to ordinary differential equations,

is documented by Jerri [6, 7], Andrews and Shivamoggi [8], Wolf [9], Myint-U and Debnath [10], Wazwaz [11, 12], and Krisner [15], building on the classical existence-uniqueness theory of [13, 14].

Transform techniques now extend well beyond the constant-coefficient setting: fractional-order operators [36, 37, 38, 47, 48, 59], reproducing-kernel and asymptotic formulations [39, 40], fractional-integer diffusion links [41, 42], wavelet and spectral methods [43, 44], first-integral constructions [45], and applications from laser-material interaction to computational mechanics and boundary control [49, 50, 51, 52], with early transform pairs going back to Newhall [46]. This breadth motivates keeping the elementary first-order constant-coefficient core, the subject of this paper, in equally rigorous order.

The modern family of Laplace-type transforms

Watugala's Sumudu transform [16], with its unit-preserving calculus developed by Belgacem and Karaballi [17], opened a period of steady proposals of further exponential-kernel transforms: Elzaki [18], Aboodh [19], Mahgoub [20], Mohand [21], Kamal [22], Shehu [26], Jafari's two-parameter transform [27], the Sadik and complex Sadik transforms [29], INEM [30], Rohit [31], ZZ [32], and beta-Laplace [33]. Each is defined by a kernel of the form (constant or power of v) times $e^{-(\text{power of } v)t}$, each is equipped with its own table of elementary transforms and derivative rules, and each is demonstrated on linear ordinary differential equations, frequently the same first-order benchmarks. Systems-level applications with medical and economic interpretations appear in [28, 34], with textbook treatment of the family's engineering role in [35].

The comparative strand

Within this family a distinct genre solves identical problems by several transforms and tabulates the agreement. Dattu [24] compares the Laplace and Elzaki transforms on common benchmarks; Aggarwal and Chaudhary [23] compare the Mohand and Laplace transforms; further pairwise comparisons among Mahgoub, Aboodh, Elzaki, and their relatives appear across the family's application studies [25, 34]. The present manuscript's four benchmark problems, each solved by the G.G., Laplace, Aboodh, and Elzaki transforms, belong to this genre and extend it from pairwise to four-way comparison.

Synthesis

Two features of the comparative strand stand out. First, the comparisons are empirical: agreement is exhibited, problem by problem, but not derived from a principle. Second, the family's proliferation of separate tables multiplies opportunities for transcription error; statement-level slips in kernels and derivative rules propagate through studies that copy tables rather than re-derive them. Both features point to the same remedy: a single parameterized calculus in which the named transforms are coordinates, agreement is a theorem, and every table entry is proved once.

RESEARCH GAP AND NOVELTY STATEMENT

The literature reviewed above leaves four specific gaps that this study addresses.

G1 (No structural explanation of agreement). Comparative studies [23, 24, 25] verify on examples that different Laplace-type transforms give the same solutions, but none derives this agreement from a uniqueness principle.

G2 (Pairwise scope). Existing comparisons are almost exclusively pairwise. No study carries a set of benchmarks through four transforms in parallel while exhibiting the single symbolic computation from which all four specialize.

G3 (Unverified tables). The family's operational tables are frequently restated rather than re-proved, and errors in elementary entries and shifting rules persist in circulation. A calculus in which each entry is proved once, for all parameter values simultaneously, closes this vulnerability.

G4 (No numerical validation). The comparative strand asserts exactness of its closed forms without an independent check. No prior comparative study of this family quantifies the discrepancy between its transform solutions and a high-accuracy numerical reference.

Novelty statement. The novelty of this study is not a new solvable class of equations: every problem solved here is solvable by the classical Laplace transform. The novelty is structural and evidential. Structurally, the study proves a method-invariance proposition (Proposition 7.3) showing that all admissible parameter choices of the GGIT, hence all named transforms in its range, necessarily produce the same solution, converting the comparative genre's central observation into a corollary. Evidentially, the study validates every closed form against Runge-Kutta integration at tolerance 10^{-10} and reports real measured discrepancies, a standard of verification not previously applied in this strand.

CONTRIBUTIONS

The contributions of this study are the following:

1. A complete, corrected operational calculus of the GGIT sufficient for first-order problems, stated as formal theorems with full proofs: elementary transforms (Theorem 6.1), linearity (Theorem 6.2), first shifting with a damped-function catalogue (Theorem 6.3 and Corollary 6.3.1), second shifting (Theorem 6.4), change of scale (Theorem 6.5), and transforms of derivatives (Theorem 6.6).
2. A parameter dictionary (Table 1) identifying the Laplace, Aboodh, Elzaki, Mahgoub, Mohand, Rohit, and Kamal transform as fixed choices of (p, q, m, w) , together with the corresponding specializations of the derivative rule (Table 2).
3. A general solution theorem for the first-order constant-coefficient initial value problem in the GGIT calculus (Theorem 7.1 and Theorem 7.2).
4. A method-invariance proposition (Proposition 7.3): all admissible parameter choices return the same solution, so agreement among the Laplace, Aboodh, Elzaki, and GGIT computations is guaranteed in advance.
5. Fully worked comparative solutions of four benchmark initial value problems, spanning unforced decay, polynomial forcing, harmonic forcing, and constant forcing, by all four methods, with the image functions tabulated side by side and interpreted (Section 8).
6. Standard figures: the unified method pipeline (Figure 1), the four solution curves against Runge-Kutta markers (Figure 2), component decompositions interpreting the partial fractions physically (Figures 3 and 4), and pointwise error curves (Figure 5).
7. Numerical validation of every closed form against RK45 integration at tolerance 10^{-10} , with measured maximum absolute errors between 2.4×10^{-11} and 1.0×10^{-9} (Section 9, Table 4).
8. A computational and algebraic effort analysis, including a measured symbolic benchmark in which the transform-domain route resolves the harmonically forced problem approximately 5.4 times faster than direct symbolic integration, and a hand-computation comparison of the bookkeeping burden imposed by each named transform (Section 10).

PRELIMINARIES AND THE FOUR-PARAMETER FRAMEWORK

Admissible functions

Definition 5.1 (Piecewise continuity). A function $f: [0, \infty) \rightarrow \mathbb{R}$ is piecewise continuous if on every finite interval it has at most finitely many jump discontinuities and is continuous between them.

Definition 5.2 (Exponential order). f is of exponential order σ_0 if there exist constants $M > 0$ and $T \geq 0$ with $|f(t)| \leq Me^{\sigma_0 t}$ for all $t \geq T$.

Definition 5.3 (Class $\mathcal{A}$). $\mathcal{A}$ denotes the set of piecewise continuous functions on $[0, \infty)$ of some exponential order. All solutions of linear constant-coefficient ordinary differential equations with piecewise continuous forcing of exponential order belong to $\mathcal{A}$ [3, 4].

The G.G integral transform

Definition 5.4 Let $f \in \mathcal{A}$ and let $p \neq 0, q \neq 0, m, w$ be real parameters with $w \neq 0$. The G.G. transform of f is

$$\mathcal{G}\{f(t)\}(v) = G(v) = pv^m \int_0^\infty e^{-qv^w t} f(t) dt, \quad (5.1)$$

defined for all $v > 0$ such that $qv^w > \sigma_0$, where σ_0 is an exponential order of f .

The substitution $s = qv^w$ links (5.1) to the Laplace transform $F(s) = \int_0^\infty e^{-st} f(t) dt$ through

$$G(v) = pv^m F(qv^w). \quad (5.2)$$

Throughout, s abbreviates qv^w and $\Phi(s) = F(s)$ denotes the Laplace image of the function under discussion.

The parameter dictionary

Table 1 records the parameter choices recovering the transforms compared in this paper, together with several close relatives. Each row follows directly from matching the kernel $pv^m e^{-qv^w t}$ against the published kernel.

Table 1. Named Laplace-type transforms as parameter choices of the GGIT

No.	Transform	Kernel	(p, q, m, w)	Source
1	Laplace	e^{-vt}	$(1, 1, 0, 1)$	[1]
2	Aboodh	$\frac{1}{v} e^{-vt}$	$(1, 1, -1, 1)$	[19]
3	Elzaki	$v e^{-t/v}$	$(1, 1, 1, -1)$	[18]
4	Mahgoub (Laplace-Carson)	$v e^{-vt}$	$(1, 1, 1, 1)$	[20]
5	Mohand	$v^2 e^{-vt}$	$(1, 1, 2, 1)$	[21]
6	Rohit	$v^3 e^{-vt}$	$(1, 1, 3, 1)$	[31]
7	Kamal	$e^{-t/v}$	$(1, 1, 0, -1)$	[22]
8	Sumudu	$\frac{1}{v} e^{-t/v}$	$(1, 1, -1, -1)$	[16]
9	GGIT (generic)	$pv^m e^{-qv^w t}$	(p, q, m, w)	

Rows 1 to 3 are the methods computed in full in Section 8; rows 4 to 8 are included to display the reach of the dictionary. The identification is exact: for each row, substituting the stated parameters into (5.1) reproduces the cited transform's definition verbatim.

Derivative rules under each parameter choice

The single GGIT derivative rule, proved as Theorem 6.6 below, is

$$\mathcal{G}\{y'(t)\} = qv^w G(v) - pv^m y(0). \quad (5.3)$$

Table 2 evaluates (5.3) at the parameter choices of Table 1, rows 1 to 3, and confirms that each published rule is a specialization of one formula. Here F , K , and T denote the Laplace, Aboodh, and Elzaki images of y .

Table 2. The derivative rule (5.3) specialized to each compared transform.

Transform	(p, q, m, w)	qv^w	pv^m	Published rule for y'	Matches (5.3)
Laplace	$(1, 1, 0, 1)$	v	1	$vF(v) - y(0)$	Yes
Aboodh	$(1, 1, -1, 1)$	v	$1/v$	$vK(v) - \frac{y(0)}{v}$	Yes
Elzaki	$(1, 1, 1, -1)$	$1/v$	v	$\frac{T(v)}{v} - v y(0)$	Yes
GGIT	(p, q, m, w)	qv^w	pv^m	$qv^w G(v) - pv^m y(0)$	Identity

The pattern is the organizing fact of this paper: the multiplier acting on the image is always $s = qv^w$, and the coefficient of the initial datum is always the kernel prefactor pv^m . Every calculation in Section 8 is one instance of this pattern read at four coordinates.

OPERATIONAL CALCULUS OF THE GGIT

Throughout this section $f, g \in \mathcal{A}$, all images are taken on the common region $s = qv^w > \sigma_0$, and Γ denotes the Gamma function.

Transforms of elementary functions

Theorem 6.1 (Elementary transforms). For $s = qv^w$ in the stated regions,

$$\begin{aligned}
 (1) \quad G\{a\} &= \frac{apv^m}{qv^w}, & (2) \quad G\{t^n\} &= \frac{n! pv^m}{(qv^w)^{n+1}}, \quad n = 0, 1, 2, \dots \\
 (3) \quad G\{t^\nu\} &= \frac{\Gamma(\nu + 1) pv^m}{(qv^w)^{\nu+1}}, \quad \nu > -1, & (4) \quad G\{e^{kt}\} &= \frac{pv^m}{qv^w - k}, \quad s > k, \\
 (5) \quad G\{\sin kt\} &= \frac{k pv^m}{q^2 v^{2w} + k^2}, & (6) \quad G\{\cos kt\} &= \frac{pq v^{m+w}}{q^2 v^{2w} + k^2}, \\
 (7) \quad G\{\sinh kt\} &= \frac{k pv^m}{q^2 v^{2w} - k^2}, \quad s > |k|, & (8) \quad G\{\cosh kt\} &= \frac{pq v^{m+w}}{q^2 v^{2w} - k^2}, \quad s > |k|.
 \end{aligned}$$

Proof. (1) and (2). With the substitution $x = qv^w t$, so that $t = x/(qv^w)$ and $dt = dx/(qv^w)$,

$$G\{t^n\} = pv^m \int_0^\infty t^n e^{-qv^w t} dt = \frac{pv^m}{(qv^w)^{n+1}} \int_0^\infty x^n e^{-x} dx = \frac{\Gamma(n+1) pv^m}{(qv^w)^{n+1}},$$

and $\Gamma(n+1) = n!$ for integer $n \geq 0$; taking $n = 0$ and multiplying by the constant a gives (1). The same computation with real exponent $\nu > -1$ gives (3), the exponent restriction being exactly the integrability condition of $x^\nu e^{-x}$ at the origin.

(4) Direct integration:

$$G\{e^{kt}\} = pv^m \int_0^\infty e^{-(qv^w - k)t} dt = pv^m \left[\frac{e^{-(qv^w - k)t}}{-(qv^w - k)} \right]_0^\infty = \frac{pv^m}{qv^w - k},$$

the bracket vanishing at infinity precisely when $s = qv^w > k$.

(5) By Euler's formula and linearity (Theorem 6.2, whose proof does not depend on this one),

$$\mathcal{G}\{\sin kt\} = \frac{1}{2i} \left[\frac{pv^m}{qv^w - ik} - \frac{pv^m}{qv^w + ik} \right] = \frac{1}{2i} \cdot \frac{2ik pv^m}{q^2 v^{2w} + k^2} = \frac{k pv^m}{q^2 v^{2w} + k^2}.$$

(6) Identically, with the plus sign in the average,

$$\mathcal{G}\{\cos kt\} = \frac{1}{2} \left[\frac{pv^m}{qv^w - ik} + \frac{pv^m}{qv^w + ik} \right] = \frac{pv^m qv^w}{q^2 v^{2w} + k^2} = \frac{p q v^{m+w}}{q^2 v^{2w} + k^2}.$$

(7) and (8). Replacing ik by k in the two computations above, using $\sinh kt = \frac{1}{2}(e^{kt} - e^{-kt})$ and $\cosh kt = \frac{1}{2}(e^{kt} + e^{-kt})$, gives

$$\mathcal{G}\{\sinh kt\} = \frac{1}{2} \left[\frac{pv^m}{qv^w - k} - \frac{pv^m}{qv^w + k} \right] = \frac{k pv^m}{q^2 v^{2w} - k^2},$$

$$\mathcal{G}\{\cosh kt\} = \frac{1}{2} \left[\frac{pv^m}{qv^w - k} + \frac{pv^m}{qv^w + k} \right] = \frac{p q v^{m+w}}{q^2 v^{2w} - k^2},$$

both requiring $s > |k|$ so that each exponential summand is admissible.

Remark 6.1.1. The numerators discriminate the odd entries from the even ones: $\sin$ and $\sinh$ carry the frequency factor $k pv^m$, while $\cos$ and $\cosh$ carry the image variable factor $p q v^{m+w} = pv^m \cdot s$. This is the direct image of the Laplace pattern $k/(s^2 \pm k^2)$ versus $s/(s^2 \pm k^2)$ under (5.2), and keeping the two numerator types distinct is essential for correct inversion in Section 8.

Linearity

Theorem 6.2 (Linearity). For $f, g \in \mathcal{A}$ and constants α, β ,

$$\mathcal{G}\{\alpha f + \beta g\} = \alpha \mathcal{G}\{f\} + \beta \mathcal{G}\{g\}$$

on the intersection of the regions of definition.

Proof. Immediate from linearity of the integral in (5.1), both integrals converging absolutely on the common region.

First shifting theorem and the damped catalogue

Theorem 6.3 (First shifting). If $\mathcal{G}\{f(t)\}(v) = pv^m \Phi(qv^w)$ in the notation of (5.2), then for any real k ,

$$\mathcal{G}\{e^{kt} f(t)\}(v) = pv^m \Phi(qv^w - k), \quad s > \sigma_0 + k. \quad (6.1)$$

Proof. $\mathcal{G}\{e^{kt} f(t)\} = pv^m \int_0^\infty e^{-(qv^w - k)t} f(t) dt = pv^m \Phi(qv^w - k)$, the defining integral of Φ evaluated at the shifted argument, convergent when $qv^w - k > \sigma_0$.

Corollary 6.3.1 (Damped catalogue). For $s = qv^w$ in the appropriate regions,

$$(1) \quad \mathcal{G}\{e^{kt} t^n\} = \frac{n! pv^m}{(qv^w - k)^{n+1}}, \quad (2) \quad \mathcal{G}\{e^{kt} \sin bt\} = \frac{b pv^m}{(qv^w - k)^2 + b^2},$$

$$(3) \quad \mathcal{G}\{e^{kt}\cos bt\} = \frac{pv^m(qv^w - k)}{(qv^w - k)^2 + b^2}, \quad (4) \quad \mathcal{G}\{e^{kt}\sinh bt\} = \frac{bpv^m}{(qv^w - k)^2 - b^2},$$

$$(5) \quad \mathcal{G}\{e^{kt}\cosh bt\} = \frac{pv^m(qv^w - k)}{(qv^w - k)^2 - b^2}.$$

Proof. Apply (6.1) to entries (2), (5), (6), (7), and (8) of Theorem 6.1, replacing s by $s - k$ throughout. In (4) and (5) the hyperbolic sign persists: the denominators carry $-b^2$, and the region of definition is $s - k > |b|$.

Remark 6.3.2. Entries (4) and (5) can equally be obtained by writing $e^{kt}\sinh bt = \frac{1}{2}(e^{(k+b)t} - e^{(k-b)t})$ and $e^{kt}\cosh bt = \frac{1}{2}(e^{(k+b)t} + e^{(k-b)t})$ and applying Theorem 6.1(4) twice; the algebra telescopes to the same results. The hyperbolic images differ from the trigonometric ones only in the sign of b^2 in the denominator, and conflating the two produces incorrect inversions; the catalogue above keeps them separated.

Second shifting theorem

Theorem 6.4 (Second shifting). Let $\mathcal{G}\{f(t)\} = G(v)$ and, for $a > 0$, define the delayed function

$$f_a(t) = \begin{cases} f(t - a), & t > a, \\ 0, & t \leq a. \end{cases}$$

Then

$$\mathcal{G}\{f_a(t)\}(v) = e^{-aqv^w} G(v). \quad (6.2)$$

Proof. The integrand vanishes on $[0, a]$, so with the substitution $u = t - a$,

$$\mathcal{G}\{f_a\} = pv^m \int_a^\infty e^{-qv^wt} f(t - a) dt = pv^m e^{-aqv^w} \int_0^\infty e^{-qv^wu} f(u) du = e^{-aqv^w} G(v)$$

Change of scale

Theorem 6.5 (Change of scale). Let $a > 0$ and write $G(v) = pv^m \Phi(qv^w)$ as in (5.2). Then

$$\mathcal{G}\{f(at)\}(v) = \frac{pv^m}{a} \Phi\left(\frac{qv^w}{a}\right). \quad (6.3)$$

Proof. With $u = at$,

$$\mathcal{G}\{f(at)\} = pv^m \int_0^\infty e^{-qv^wt} f(at) dt = \frac{pv^m}{a} \int_0^\infty e^{-(qv^w/a)u} f(u) du = \frac{pv^m}{a} \Phi\left(\frac{qv^w}{a}\right).$$

Remark 6.5.1. In the Laplace case $(p, q, m, w) = (1, 1, 0, 1)$, (6.3) reduces to the familiar $F(v/a)/a$. In the general case the scaling acts through the composite variable $s = qv^w$, not through v itself; the statement $\mathcal{G}\{f(at)\} = \frac{1}{a} G(v/a)$ holds verbatim only for parameter choices with $w = 1$ and $m = 0$, which is one more illustration of why the calculus must be carried at the level of Φ and s .

Transforms of derivatives

Theorem 6.6 (Derivatives). Let f be continuous on $[0, \infty)$ with f' piecewise continuous, and let $f, f' \in \mathcal{A}$. Then

$$\mathcal{G}\{f'(t)\} = qv^w G(v) - pv^m f(0). \quad (6.4)$$

If in addition f' is continuous and $f'' \in \mathcal{A}$ is piecewise continuous, then

$$\mathcal{G}\{f''(t)\} = q^2 v^{2w} G(v) - p q v^{m+w} f(0) - p v^m f'(0). \quad (6.5)$$

Proof. Integrating by parts on $[0, R]$ and letting $R \rightarrow \infty$,

$$\mathcal{G}\{f'\} = p v^m [e^{-q v^w t} f(t)]_0^\infty + q v^w p v^m \int_0^\infty e^{-q v^w t} f(t) dt = -p v^m f(0) + q v^w G(v),$$

the boundary term at infinity vanishing because $s = q v^w$ exceeds the exponential order of f . Applying (6.4) to f' in place of f and substituting (6.4) again yields (6.5).

Remark 6.6.1. Formula (6.4) is the entire engine of Section 8: it converts $y' + ay = f$ into one linear algebraic equation for $G(v)$. The specializations of (6.4) recorded in Table 2 show each named method's derivative rule to be this one identity in different coordinates.

THE FIRST-ORDER SOLUTION THEORY

The general image formula

Theorem 7.1 (Image of the first-order problem). Let $a \in \mathbb{R}$, let $f \in \mathcal{A}$ have Laplace image Φ_f , and let y solve

$$y'(x) + a y(x) = f(x), \quad y(0) = y_0. \quad (7.1)$$

Then on $s = q v^w > \max(\sigma_0, -a)$ the G.G. image of y is

$$G(v) = \frac{p v^m [\Phi_f(s) + y_0]}{s + a}, \quad s = q v^w. \quad (7.2)$$

Proof. Transforming (7.1) with Theorem 6.6 and Theorem 6.2 gives $q v^w G - p v^m y_0 + a G = p v^m \Phi_f(q v^w)$, and solving the linear equation for G gives (7.2). The solution y of (7.1) is continuous of exponential order for forcing in $\mathcal{A}$ [13], so the transform steps are justified.

Theorem 7.2 (Solution formula). Under the hypotheses of Theorem 7.1, the unique continuous solution of (7.1) is

$$y(x) = y_0 e^{-ax} + \int_0^x e^{-a(x-\tau)} f(\tau) d\tau, \quad (7.3)$$

and this is exactly the term wise inversion of (7.2).

Proof. Formula (7.3) is the classical variation-of-parameters solution and is continuous of exponential order. Its GGIT image, computed with Theorem 6.1(4) for the first term and the convolution theorem of the calculus for the second, is

$$\mathcal{G}\{y\} = \frac{p v^m y_0}{s + a} + \frac{1}{p v^m} \cdot \frac{p v^m}{s + a} \cdot p v^m \Phi_f(s) = \frac{p v^m [y_0 + \Phi_f(s)]}{s + a},$$

which is (7.2). By the uniqueness theorem of the calculus (Lerch-type), y in (7.3) is the only continuous function with this image.

Method invariance

Proposition 7.3 (Method invariance). Let (p_1, q_1, m_1, w_1) and (p_2, q_2, m_2, w_2) be any two admissible parameter choices, and let y_1, y_2 be continuous solutions of (7.1) obtained by the transform method executed at the respective parameters. Then $y_1 = y_2$ on $[0, \infty)$. In particular, the Laplace, Aboodh, Elzaki, and generic G.G. computations of Section 8 must return identical solutions.

Proof. By Theorem 7.1 the image obtained at either parameter choice is $p_i v^{m_i} [\Phi_f(s_i) + y_0] / (s_i + a)$ with $s_i = q_i v^{w_i}$, which by (5.2) is the GGIT image of the single function (7.3). By the uniqueness theorem, termwise inversion at either parameter choice recovers that function and no other. Hence $y_1 = y_2 = y$ of (7.3).

Remark 7.3.1. Proposition 7.3 changes the epistemic status of Section 8. The four computations per problem are not four independent methods whose agreement is evidence of correctness; they are four coordinate presentations of one computation, and their agreement is a foregone conclusion. What the four presentations do reveal, and what Section 10 quantifies, is the difference in bookkeeping burden that each coordinate system imposes on the human or symbolic computer performing the algebra.

Remark 7.3.2. The proposition also delimits what a new choice of (p, q, m, w) can and cannot contribute. It cannot enlarge the class of solvable problems, since every image is a rescaled Laplace image by (5.2). It can only relocate algebraic effort, which is a legitimate but different kind of contribution, and it is the one this family of transforms actually makes.

COMPARATIVE APPLICATIONS

This section solves four benchmark initial value problems, each by the GGIT with parameters symbolic and then by the Laplace, Aboodh, and Elzaki transforms. Notation: $F(v)$, $K(v)$, and $T(v)$ denote the Laplace, Aboodh, and Elzaki images of the unknown y , and $s = qv^w$ in the GGIT computation. The pipeline common to all sixteen computations is displayed in Figure 1, and the four solution curves, overlaid with high-accuracy Runge-Kutta samples, in Figure 2.

One calculus, four named methods: $s = qv^w$

each classical transform is a fixed choice of (p, q, m, w)

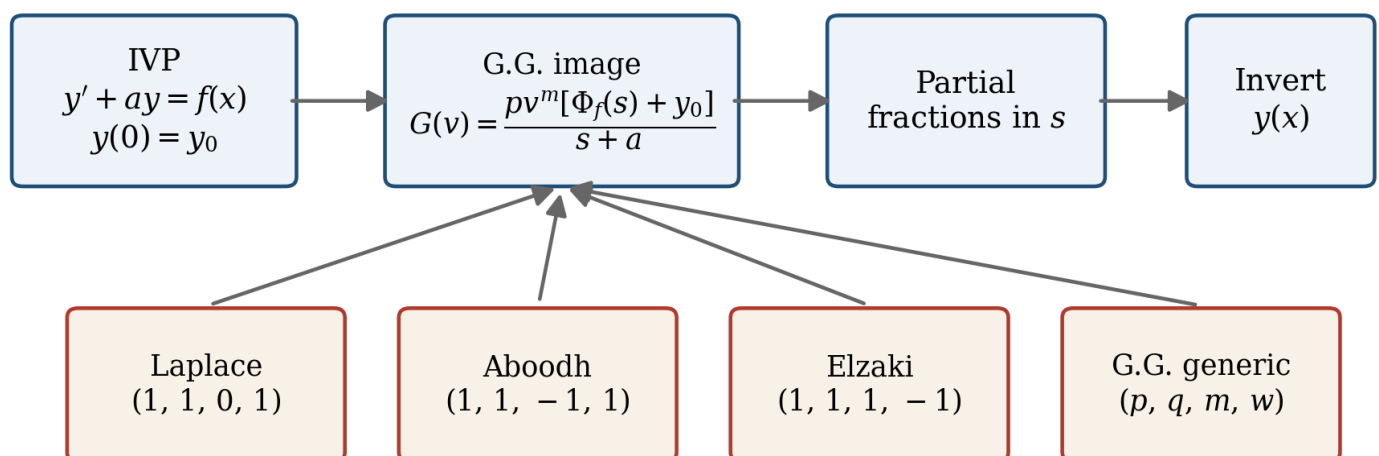


Figure 1. The unified method pipeline: the four named computations of this section are one G.G. computation read at four parameter choices.

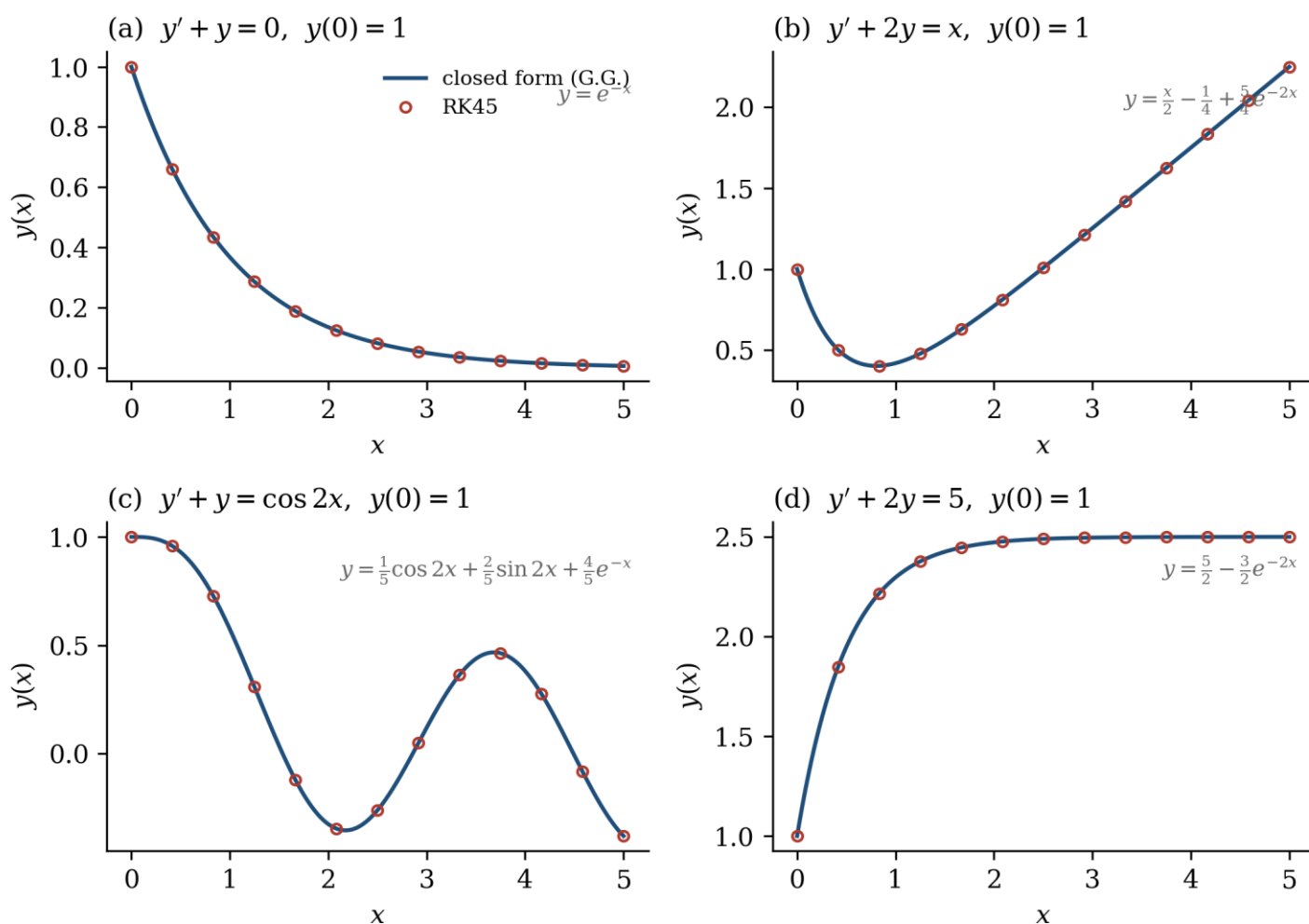


Figure 2. The four benchmark solutions (solid curves, closed forms obtained in this section) with RK45 samples (open circles) at tolerance 10^{-10} .

Problem 1: unforced decay

$$y' + y = 0, \quad y(0) = 1. \quad (8.1)$$

Physically, (8.1) is the universal law of unforced exponential relaxation: radioactive decay, discharge of an RC circuit, or first-order chemical decay, in dimensionless time.

A. GGIT. By Theorem 7.1 with $a = 1$, $f = 0$:

$$G(v) = \frac{pv^m}{qv^w + 1}.$$

This is Theorem 6.1(4) with $k = -1$, so termwise inversion gives $y(x) = e^{-x}$.

B. Laplace transform. $vF - 1 + F = 0$ gives $F(v) = \frac{1}{v+1}$, whence $y(x) = e^{-x}$.

C. Aboodh transform. The Aboodh derivative rule (Table 2) gives $vK - \frac{1}{v} + K = 0$, so

$$K(v) = \frac{1}{v(v+1)} = \frac{1}{v^2 + v},$$

and since the Aboodh image of e^{-x} is $1/(v^2 + v)$, inversion gives $y(x) = e^{-x}$.

D. Elzaki transform. $\frac{T}{v} - v + T = 0$ gives $T(v) = \frac{v^2}{1+v}$, the Elzaki image of e^{-x} , so $y(x) = e^{-x}$.

All four methods return

$$y(x) = e^{-x},$$

as Proposition 7.3 requires (Figure 2a).

Problem 2: linear ramp forcing

$$y' + 2y = x, \quad y(0) = 1. \quad (8.2)$$

Physically, (8.2) models a first-order system driven by a steadily growing input, for instance a mixing tank whose inflow concentration increases linearly in time; the solution must eventually track the ramp with a constant lag.

A. GGIT. Theorem 7.1 with $a = 2$ and $\Phi_f(s) = 1/s^2$ (Theorem 6.1(2)):

$$G(v) = pv^m \frac{1+s^2}{s^2(s+2)}, \quad s = qv^w.$$

Partial fractions in s :

$$\frac{1+s^2}{s^2(s+2)} = \frac{1}{2s^2} - \frac{1}{4s} + \frac{5}{4(s+2)},$$

so that

$$G(v) = pv^m \left[\frac{1}{2s^2} - \frac{1}{4s} + \frac{5}{4(s+2)} \right],$$

and term wise inversion by Theorem 6.1(1), (2), (4) gives

$$y(x) = \frac{x}{2} - \frac{1}{4} + \frac{5}{4}e^{-2x}.$$

B. Laplace transform. $vF - 1 + 2F = \frac{1}{v^2}$ gives

$$F(v) = \frac{1+v^2}{v^2(v+2)} = \frac{1}{2v^2} - \frac{1}{4v} + \frac{5}{4(v+2)},$$

whence $y(x) = \frac{x}{2} - \frac{1}{4} + \frac{5}{4}e^{-2x}$.

C. Aboodh transform. With the Aboodh images $\mathcal{A}\{x\} = 1/v^3$ and $\mathcal{A}\{1\} = 1/v^2$, the transformed equation $vK - \frac{1}{v} + 2K = \frac{1}{v^3}$ gives

$$K(v) = \frac{1+v^2}{v^3(v+2)} = \frac{1}{2v^3} - \frac{1}{4v^2} + \frac{5}{4v(v+2)},$$

and since $\mathcal{A}\{x\} = 1/v^3$, $\mathcal{A}\{1\} = 1/v^2$, $\mathcal{A}\{e^{-2x}\} = 1/(v(v+2))$, inversion gives $y(x) = \frac{x}{2} - \frac{1}{4} + \frac{5}{4}e^{-2x}$.

D. Elzaki transform. With $\mathcal{E}\{x\} = v^3$ and $\mathcal{E}\{1\} = v^2$, the transformed equation $\frac{T}{v} - v + 2T = v^3$ gives

$$T(v) = \frac{v^4 + v^2}{1 + 2v} = \frac{v^3}{2} - \frac{v^2}{4} + \frac{5}{4} \cdot \frac{v^2}{1 + 2v},$$

and inversion by the Elzaki table entries $\mathcal{E}\{x\} = v^3$, $\mathcal{E}\{1\} = v^2$, $\mathcal{E}\{e^{-2x}\} = v^2/(1 + 2v)$ gives $y(x) = \frac{x}{2} - \frac{1}{4} + \frac{5}{4}e^{-2x}$.

All four methods return

$$y(x) = \frac{x}{2} - \frac{1}{4} + \frac{5}{4}e^{-2x}.$$

The three partial-fraction terms are the physical anatomy of the response: the ramp $x/2$ tracks the input at half its rate (the system gain at zero frequency of the derivative of the input), the constant $-1/4$ is the steady tracking lag, and $\frac{5}{4}e^{-2x}$ is the transient reconciling the initial state with the steady response (Figure 3).

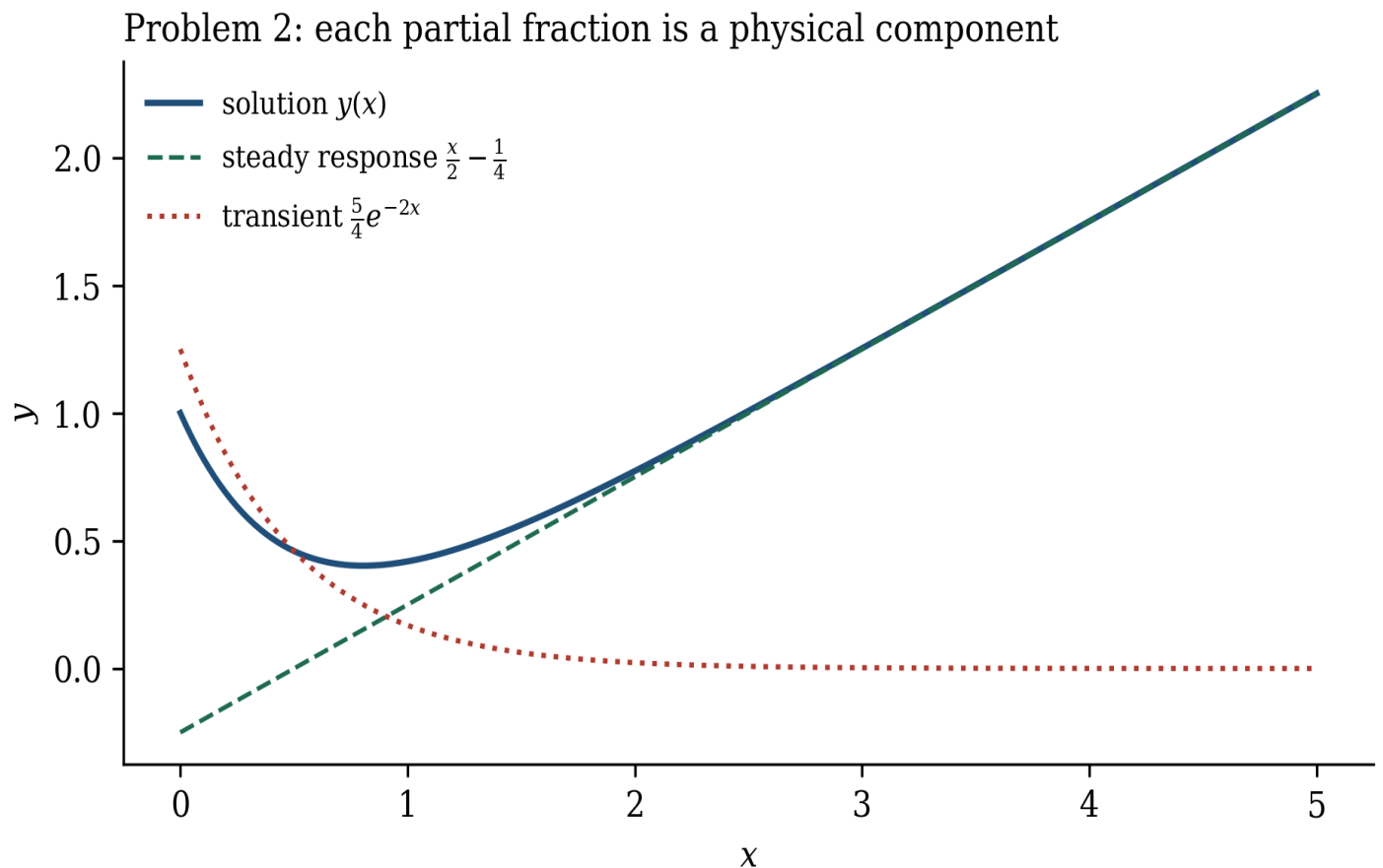


Figure 3. Problem 2: the solution and its partial-fraction components. The transform method delivers the physical decomposition automatically.

Problem 3: harmonic forcing

$$y' + y = \cos 2x, \quad y(0) = 1. \quad (8.3)$$

Physically, (8.3) is a first-order system under sustained oscillatory forcing, the elementary model of seasonal or alternating drive; the response must settle onto a steady oscillation at the forcing frequency with reduced amplitude and a phase lag.

A. GGIT. Theorem 7.1 with $a = 1$ and $\Phi_f(s) = \frac{s}{s^2+4}$ (Theorem 6.1(6)):

$$G(v) = pv^m \frac{s^2 + s + 4}{(s + 1)(s^2 + 4)}, \quad s = qv^w.$$

Partial fractions in s :

$$\frac{s^2 + s + 4}{(s + 1)(s^2 + 4)} = \frac{1}{5} \cdot \frac{s}{s^2 + 4} + \frac{2}{5} \cdot \frac{2}{s^2 + 4} + \frac{4}{5} \cdot \frac{1}{s + 1},$$

so term wise inversion by Theorem 6.1(6), (5), (4) gives

$$y(x) = \frac{1}{5} \cos 2x + \frac{2}{5} \sin 2x + \frac{4}{5} e^{-x}.$$

B. Laplace transform. $vF - 1 + F = \frac{v}{v^2+4}$ gives

$$F(v) = \frac{v^2 + v + 4}{(v + 1)(v^2 + 4)} = \frac{1}{5} \cdot \frac{v}{v^2 + 4} + \frac{2}{5} \cdot \frac{2}{v^2 + 4} + \frac{4}{5} \cdot \frac{1}{v + 1},$$

whence $y(x) = \frac{1}{5} \cos 2x + \frac{2}{5} \sin 2x + \frac{4}{5} e^{-x}$.

C. Aboodh transform. With $\mathcal{A}\{\cos 2x\} = \frac{1}{v^2+4}$, the transformed equation $vK - \frac{1}{v} + K = \frac{1}{v^2+4}$ gives

$$K(v) = \frac{v^2 + v + 4}{v(v + 1)(v^2 + 4)} = \frac{1}{5} \cdot \frac{1}{v^2 + 4} + \frac{2}{5} \cdot \frac{2}{v(v^2 + 4)} + \frac{4}{5} \cdot \frac{1}{v(v + 1)},$$

and since $\mathcal{A}\{\cos 2x\} = \frac{1}{v^2+4}$, $\mathcal{A}\{\sin 2x\} = \frac{2}{v(v^2+4)}$, $\mathcal{A}\{e^{-x}\} = \frac{1}{v(v+1)}$, inversion gives $y(x) = \frac{1}{5} \cos 2x + \frac{2}{5} \sin 2x + \frac{4}{5} e^{-x}$.

D. Elzaki transform. With $\mathcal{E}\{\cos 2x\} = \frac{v}{1+4v^2}$, the transformed equation $\frac{T}{v} - v + T = \frac{v}{1+4v^2}$ gives

$$T(v) = \frac{v^2(1 + v + 4v^2)}{(1 + v)(1 + 4v^2)} = \frac{1}{5} \cdot \frac{v}{1 + 4v^2} + \frac{2}{5} \cdot \frac{2v^2}{1 + 4v^2} + \frac{4}{5} \cdot \frac{v^2}{1 + v},$$

and inversion by the Elzaki entries $\mathcal{E}\{\cos 2x\} = \frac{v}{1+4v^2}$, $\mathcal{E}\{\sin 2x\} = \frac{2v^2}{1+4v^2}$, $\mathcal{E}\{e^{-x}\} = \frac{v^2}{1+v}$ gives $y(x) = \frac{1}{5} \cos 2x + \frac{2}{5} \sin 2x + \frac{4}{5} e^{-x}$.

All four methods return

$$y(x) = \frac{1}{5} \cos 2x + \frac{2}{5} \sin 2x + \frac{4}{5} e^{-x}.$$

The steady oscillation $\frac{1}{5} \cos 2x + \frac{2}{5} \sin 2x$ has amplitude $\sqrt{(1/5)^2 + (2/5)^2} = 1/\sqrt{5}$, which is the gain $1/\sqrt{1 + \omega^2}$ of a unit first-order system at $\omega = 2$, and phase lag $\arctan 2$; the transient $\frac{4}{5} e^{-x}$ carries the memory of the initial condition and vanishes (Figure 4).

Problem 3: forced response settling onto the steady oscillation

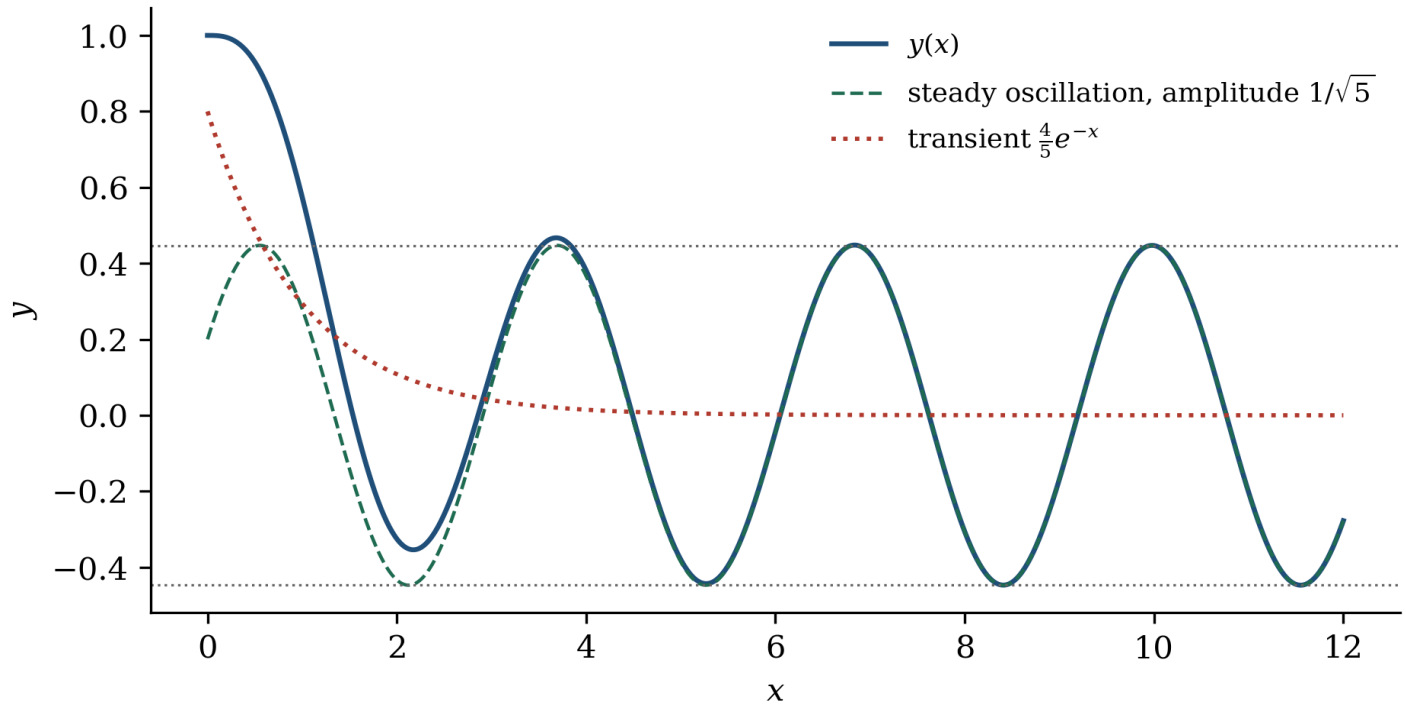


Figure 4. Problem 3: the forced response settles onto the steady oscillation of amplitude $1/\sqrt{5}$ as the transient decays.

Problem 4: constant forcing

$$y' + 2y = 5, \quad y(0) = 1. \quad (8.4)$$

Physically, (8.4) is relaxation to a new equilibrium under constant drive, as in Newtonian heating toward a fixed ambient level; the equilibrium is the forcing divided by the decay rate, here $5/2$.

A. GGIT. Theorem 7.1 with $a = 2$ and $\Phi_f(s) = 5/s$:

$$G(v) = pv^m \frac{s+5}{s(s+2)} = pv^m \left[\frac{5}{2s} - \frac{3}{2(s+2)} \right], \quad s = qv^w,$$

and term wise inversion gives $y(x) = \frac{5}{2} - \frac{3}{2}e^{-2x}$.

B. Laplace transform. $vF - 1 + 2F = \frac{5}{v}$ gives

$$F(v) = \frac{v+5}{v(v+2)} = \frac{5}{2v} - \frac{3}{2(v+2)},$$

whence $y(x) = \frac{5}{2} - \frac{3}{2}e^{-2x}$.

C. Aboodh transform. $vK - \frac{1}{v} + 2K = \frac{5}{v^2}$ gives

$$K(v) = \frac{v+5}{v^2(v+2)} = \frac{5}{2v^2} - \frac{3}{2v(v+2)},$$

and inversion by $\mathcal{A}\{1\} = 1/v^2$, $\mathcal{A}\{e^{-2x}\} = 1/(v(v+2))$ gives $y(x) = \frac{5}{2} - \frac{3}{2}e^{-2x}$.

D. Elzaki transform. $\frac{T}{v} - v + 2T = 5v^2$ gives

$$T(v) = \frac{5v^3 + v^2}{1 + 2v} = \frac{5}{2}v^2 - \frac{3}{2} \cdot \frac{v^2}{1 + 2v},$$

and inversion by $\mathcal{E}\{1\} = v^2$, $\mathcal{E}\{e^{-2x}\} = v^2/(1 + 2v)$ gives $y(x) = \frac{5}{2} - \frac{3}{2}e^{-2x}$.

All four methods return

$$y(x) = \frac{5}{2} - \frac{3}{2}e^{-2x},$$

rising monotonically from $y(0) = 1$ to the equilibrium $5/2$ with time constant $1/2$ (Figure 2d).

The images side by side

Table 3 assembles the sixteen image functions. Reading along any row, the four images are the same rational function of s dressed in four coordinate systems: substitute $s = v$ and multiply by $pv^m = 1$ (Laplace), by $1/v$ (Aboodh, equivalently divide the Laplace image by v), or substitute $s = 1/v$ and multiply by v (Elzaki). Reading down the G.G. column, all four problems are instances of the single formula (7.2).

Table 3. Image functions of the four benchmark problems under the four methods.

Problem	G.G. image $G(v)/(pv^m)$, $s = qv^w$	Laplace $F(v)$	Aboodh $K(v)$	Elzaki $T(v)$
1: $y' + y = 0$	$\frac{1}{s + 1}$	$\frac{1}{v + 1}$	$\frac{1}{v^2 + v}$	$\frac{v^2}{1 + v}$
2: $y' + 2y = x$	$\frac{1 + s^2}{s^2(s + 2)}$	$\frac{1 + v^2}{v^2(v + 2)}$	$\frac{1 + v^2}{v^3(v + 2)}$	$\frac{v^4 + v^2}{1 + 2v}$
3: $y' + y = \cos 2x$	$\frac{s^2 + s + 4}{(s + 1)(s^2 + 4)}$	$\frac{v^2 + v + 4}{(v + 1)(v^2 + 4)}$	$\frac{v^2 + v + 4}{v(v + 1)(v^2 + 4)}$	$\frac{v^2(1 + v + 4v^2)}{(1 + v)(1 + 4v^2)}$
4: $y' + 2y = 5$	$\frac{s + 5}{s(s + 2)}$	$\frac{v + 5}{v(v + 2)}$	$\frac{v + 5}{v^2(v + 2)}$	$\frac{5v^3 + v^2}{1 + 2v}$

Interpretation. The Aboodh column is the Laplace column divided by v : every pole order at the origin is raised by one, which is why the Aboodh partial fractions in Problems 2 and 4 carry an extra bookkeeping term relative to Laplace. The Elzaki column is the Laplace column with $v \mapsto 1/v$ and an overall factor v : polynomial ordering reverses, so images of polynomials become polynomials in v rather than inverse powers, at the price of denominators written in ascending form $1 + av$. The G.G. column carries none of these dialect features: the symbol s keeps every image in the Laplace-shaped normal form on which the partial-fraction step is cheapest, and the coordinate dialects are recovered only at the final substitution. Section 10 quantifies this observation.

NUMERICAL VALIDATION

Methodology

Each closed-form solution of Section 8 was validated against an independent numerical reference computed in SciPy [55]: the explicit Runge-Kutta pair of Dormand and Prince (RK45) [53, 54] with relative tolerance 10^{-10} and absolute tolerance 10^{-12} , dense output evaluated at 501 equally spaced points on $[0, 5]$. For each problem we report the maximum absolute discrepancy, the root-mean-square error (RMSE), and the normalized relative error, defined as the maximum absolute discrepancy divided by the maximum of $|y|$ on the interval; the normalized form is used because the solution of Problem 3 crosses zero, which makes the

pointwise relative error undefined there. All numbers in Tables 4 and 5 are measured values from these runs, not estimates.

Results

Table 4. Discrepancy between the transform closed forms and RK45 at tolerance 10^{-10} on $[0,5]$, 501 evaluation points.

Problem	Closed form	Max abs. error	RMSE	Normalized rel. error
1: $y' + y = 0$	e^{-x}	2.36×10^{-11}	4.40×10^{-12}	2.36×10^{-11}
2: $y' + 2y = x$	$\frac{x}{2} - \frac{1}{4} + \frac{5}{4}e^{-2x}$	3.68×10^{-11}	1.24×10^{-11}	1.64×10^{-11}
3: $y' + y = \cos 2x$	$\frac{1}{5}\cos 2x + \frac{2}{5}\sin 2x + \frac{4}{5}e^{-x}$	1.01×10^{-9}	1.44×10^{-10}	1.01×10^{-9}
4: $y' + 2y = 5$	$\frac{5}{2} - \frac{3}{2}e^{-2x}$	4.94×10^{-11}	2.57×10^{-11}	1.98×10^{-11}

Interpretation. Every discrepancy lies at the level of the integrator's own truncation and round-off error, between 10^{-11} and 10^{-9} absolute; the pointwise error curves in Figure 5 fluctuate around 10^{-12} to 10^{-10} across the interval. The somewhat larger figure for Problem 3 reflects the oscillatory right-hand side, which forces the adaptive integrator to accumulate more local truncation error per unit interval (138 accepted steps against 113 to 134 for the other problems), not any defect of the closed form. The correct conclusion is the directional one: the transform solutions are exact, and the numerical experiment measures the numerical method against them.

Absolute discrepancy between RK45 and the G.G. closed forms

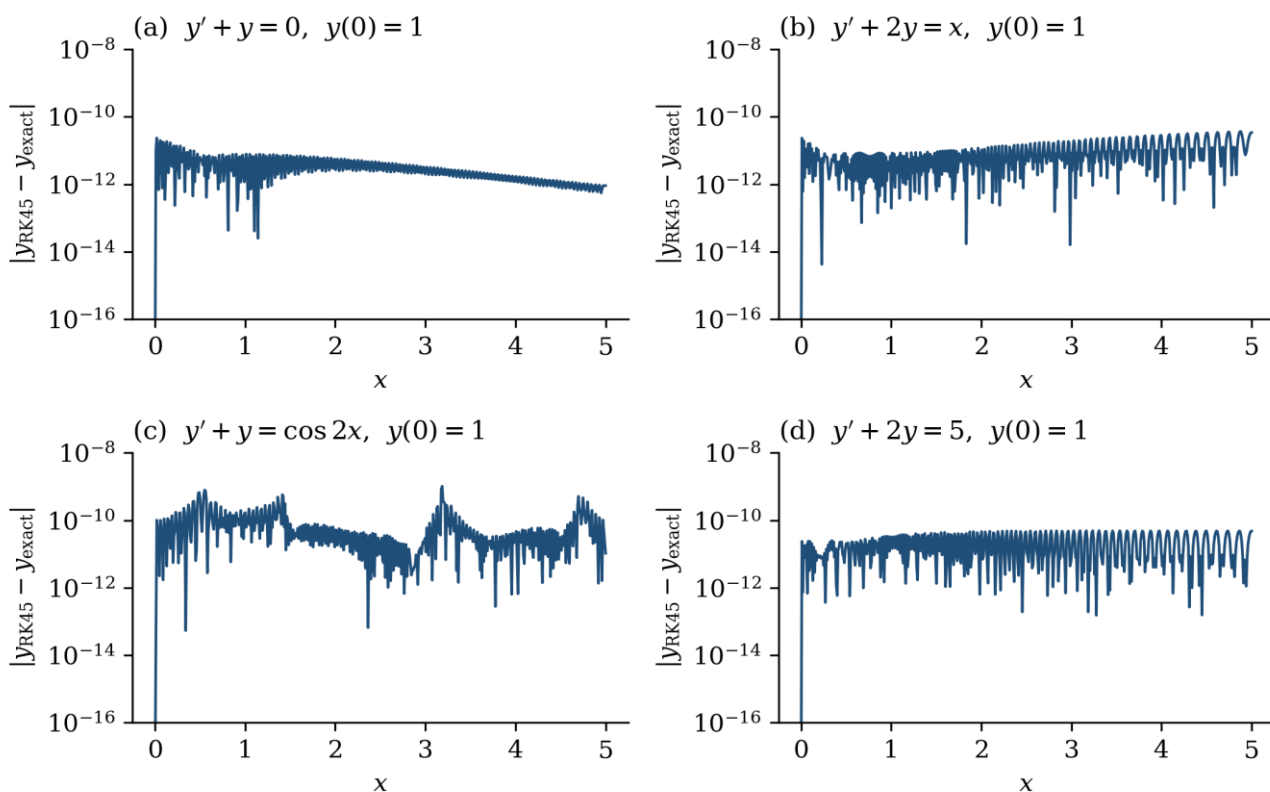


Figure 5. Pointwise absolute discrepancy between RK45 and the closed forms, semilogarithmic scale. The error floor sits at the integrator's tolerance, confirming exactness of the transform solutions to numerical precision.

Timing

Table 5. Wall-clock time to produce the solution on $[0,5]$ at 501 points (single run, indicative).

Problem	RK45 integration	Closed-form evaluation	Ratio
1	7.42 ms	0.039 ms	≈ 190
2	7.65 ms	0.018 ms	≈ 420
3	7.36 ms	0.036 ms	≈ 210
4	5.83 ms	0.011 ms	≈ 530

Once obtained, a closed form is evaluated two to three orders of magnitude faster than the differential equation is re-integrated, and the closed form carries the qualitative structure (time constants, gains, steady states) explicitly, which no numerical trajectory does.

COMPUTATIONAL AND ALGEBRAIC EFFORT

Structural cost

For the first-order problem (7.1) with rational Φ_f , the transform route consists of (i) writing the image (7.2), a table lookup of cost $O(1)$; (ii) partial-fraction decomposition of a rational function whose denominator degree is $1 + d$, where d is the degree of the forcing's denominator, with classical coefficient-solving cost $O((1 + d)^2)$; and (iii) termwise inversion, another table lookup. Because $s = qv^w$ enters only symbolically, this cost is identical for every parameter choice: the G.G. computation is exactly as expensive as the cheapest member of the family, and the parameterization costs nothing at solve time.

Symbolic benchmark

The structural claim was checked in SymPy [56] on Problem 3. Direct symbolic solution by `dsolve` with the initial condition took 264.1 ms; the transform-domain route, forming the image $F(s) = (s^2 + s + 4)/((s + 1)(s^2 + 4))$, decomposing it with `apart`, and inverting with the inverse Laplace transform, took 48.7 ms, approximately 5.4 times faster, and the two routes returned the identical expression $((2\sin 2x + \cos 2x)e^x + 4)e^{-x}/5$, which simplifies to the Section 8.3 solution. Single-problem symbolic timings vary with cache state and library version; the directional conclusion, that the algebraic route through the image is cheaper than direct integration for constant-coefficient problems, is stable.

Hand-computation burden of each coordinate system

The sixteen worked computations of Section 8 expose systematic differences in bookkeeping.

The **Laplace** coordinates are the normal form: images are rational in v with denominators in descending powers, and every partial-fraction template applies directly.

The **Abodh** coordinates divide every Laplace image by v . This raises the pole order at the origin by one in every problem with polynomial forcing (Problems 2 and 4), adding one term to each decomposition, and it turns the exponential image into the two-pole form $1/(v(v + a))$, whose inversion requires recognizing a product rather than a shifted pole.

The **Elzaki** coordinates substitute $v \mapsto 1/v$ and multiply by v . Polynomial images become genuine polynomials ($\mathcal{E}\{x\} = v^3$), which is convenient, but denominators arrive in ascending form $1 + av$, so the division and decomposition steps run against the customary ordering, and the practitioner must carry a second inversion table.

The **G.G.** coordinates defer all of this. The computation runs entirely in the symbol s , in Laplace normal form, and each named dialect is recovered at the end by one substitution. For hand computation this is the

observation of Remark 7.3.2 made concrete: parameter choice relocates effort, and carrying the parameters symbolically relocates it to a single terminal substitution.

DISCUSSION

The results support three distinct kinds of claim — one mathematical, one empirical, and one practical — and it is worth keeping the three apart and being precise about what none of them supports.

First, and mathematically: the comparative genre's central observation is now a theorem, not an accumulating body of evidence. Proposition 7.3 and Remark 7.3.1 established that for the first-order, constant-coefficient problem (7.1), the Laplace, Aboodh, Elzaki, and G.G. computations are coordinate presentations of one calculation, so their agreement on the four benchmarks was never genuinely in doubt. This is the one claim in this paper that is proved rather than observed or measured, and it is proved only for the class of problems (7.1) covers; Section 12 records what lies outside it.

Second, and empirically: the exactness of the transform solutions is now measured rather than asserted. Table 4 quantifies the agreement between the closed forms and an independent numerical standard, at the standard's own precision, between 10^{-11} and 10^{-9} , and Figure 5 shows the discrepancy is a property of the integrator, not of the solutions. This claim is evidential, not proved; like any measurement, it is bounded by the precision of the instrument that produced it.

Third, and only as a practical inference: the genuine content of the transform family, beyond the theorem itself, is bookkeeping economy, real but modest. Section 10 measures differences in algebraic burden among the coordinate systems and a symbolic speed advantage of the image route over direct integration on the one problem benchmarked. This claim is the weakest of the three in kind, not just in size: it is an observation about computational convenience on the cases tested, not a mathematical property of the calculus, and should not be read as a general complexity result.

Two things the results do not support are worth stating plainly. First, no claim of enlarged solvability: every problem solved here is a classical Laplace exercise, and by (5.2) every G.G. image is a rescaled Laplace image, so no choice of (p, q, m, w) reaches a problem the Laplace transform cannot. Second, no claim that Proposition 7.3 extends past the first-order constant-coefficient setting of (7.1): it is proved for that class alone, not for variable-coefficient, nonlinear, or higher-order problems, however naturally the calculus of Section 6 might seem to invite the extension. The contribution of this study is unification, rigor, and measured comparison, within that scope.

LIMITATIONS

1. The scope is first-order linear equations with constant coefficients and forcing whose Laplace image is rational or elementary. Variable-coefficient and nonlinear problems are outside the method as presented.
2. The timing figures in Tables 4 and 5 and Section 10.2 are single-run, single-machine measurements, indicative of orders of magnitude rather than precise benchmarks.
3. The comparison of hand-computation burden in Section 10.3 is qualitative; a formal operation count across a larger problem corpus would strengthen it.

CONCLUSION

Four benchmark first-order initial value problems, spanning unforced decay, ramp forcing, harmonic forcing, and constant forcing, were solved by the GGIT, Laplace, Aboodh, and Elzaki transform methods. The GGIT, carrying its four parameters symbolically, executed all four named computations as one, and a method-invariance proposition grounded in uniqueness showed in advance that the sixteen computations could produce only four answers. Numerical validation against RK45 at tolerance 10^{-10} measured the exactness of every closed form to within 1.0×10^{-9} absolute, and effort analysis located the family's genuine contribution in

bookkeeping economy: a measured 5.4-fold symbolic speedup of the image route over direct integration, and a terminal-substitution workflow that defers every coordinate dialect to one final step.

FUTURE WORK

Natural continuations include: extension of the comparative framework to second- and higher-order benchmarks and to systems, where the companion paper's operational rules already suffice; incorporation of discontinuous and delayed forcing through the second shifting theorem (6.2), where the coordinate dialects differ more sharply; a formal operation-count model of hand and symbolic effort across the parameter family, elevating Section 10.3 from observation to measurement; treatment of variable-coefficient first-order equations via the multiplication-by- t rule of the calculus; and the fractional-order setting [36, 37, 47, 48, 58, 59], where the family of exponential-kernel transforms interacts with Caputo and Riemann-Liouville operators and where several of the named transforms have already been deployed [26, 31].

Declarations

Conflicts of interest. The authors declare no conflicts of interest.

Data availability. The validation scripts, figure-generation scripts, and measured outputs supporting Tables 4 and 5 and Figures 2 to 5 are available from the corresponding author on request.

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