

Vibration Analysis of a Euler-Bernoulli Beam on a Winkler Foundation Subjected to Two Moving Distributed Loads

Alimi Adedowole¹, Sule Adekunle JIMOH², Olawale Olanipekun Ajijola¹, Kayode Samuel Famuagun³, Oluwaseun Olumide Okundalaye¹, Richard Olu Awonusika¹

¹Mathematical Sciences Department, Adekunle Ajasin University, Akungba-Akoko, Ondo State, Nigeria.

²Faculty of Natural and Applied Sciences, Achievers University, Owo, Nigeria

³Department of Mathematics, Adeyemi Federal University of Education, Ondo State, Nigeria

DOI: <https://doi.org/10.51584/IJRIAS.2026.11080173>

Received: 02 September 2026; Accepted: 07 September 2026; Published: 22 September 2026

ABSTRACT

This study investigates the dynamic vibration of a simply supported Euler-Bernoulli beam resting on a Winkler foundation and subjected to two simultaneous moving distributed loads of constant or harmonic magnitude. Closed-form modal solutions to the governing fourth-order partial differential equation are derived using the finite Fourier sine transform, Laplace transform, and convolution theorem. Equal and unequal load magnitudes are considered, and the effects of foundation stiffness, axial prestress, and the two load magnitudes are quantified. An independent 80-element Euler-Bernoulli finite-element solution integrated by the average-acceleration Newmark method validates the analytical response: the peak midspan-deflection difference is 0.17% and the normalized root-mean-square difference is 0.26% for the stated benchmark. The results show that foundation stiffness and tensile prestress reduce dynamic deflection, whereas either load magnitude increases it linearly away from resonance. For harmonic loading, three resonance pathways arise from the interaction between the load oscillation and translational modal frequencies. The formulation therefore provides a verified analytical basis for assessing beams and beam-like transport structures subjected to concurrent moving actions.

Keywords: Distributed Loads, Fourier Sine Transformation, Bernoulli-Euler Beam, Winkler Foundation, Moving Loads, Prestress Forces

INTRODUCTION

The dynamic analysis of structural elements has attracted significant research interest because of its importance across numerous engineering disciplines. Practical applications include the response of railway tracks and highway bridges subjected to mobile vehicles, as well as high-precision components such as machine drives, belt systems, and digital storage media. This study integrates areas such as Engineering, Applied Mathematics, and Physics to provide appropriate analytical methods for investigating these complex interactions and their subsequent effects (Kenny, 1954; Stanisic et al., 1968; Oni, 1996). This study investigates the nonlinear transverse motion of simply supported beams subjected to point masses. Ozkaya, (2002) examined the nonlinear motion of pin-end beams under point masses. Similarly, Adedowole & Jimoh, (2018) investigated the dynamic analysis of damped beams with non-uniform cross-sections subjected to loads with varying velocities. Their methodology employed Galerkin's technique in conjunction with integral transform techniques to resolve the dynamic vibration under both constant and harmonically varying moving loads.

Vibration analysis of non-uniform Timoshenko beams on flexible foundations and under traveling loads at time-dependent velocities was examined by Adedowole, (2019). Galerkin's method and integral transform techniques were used to resolve the governing equations of the dynamic system. Likewise, Oni &

Adedowole (2008) investigated the impact of prestressing on the vibration of rectangular plates, incorporating a rotatory inertia correction factor. The closed-form solution facilitates prediction of the impact of various parameters on the plate. Shu & Du, (1996) investigated the free vibration of structural members with clamped and simply supported boundary conditions. Zamani et al. (2012) applied the Generalized Differential Quadrature (GDQ) method to examine free vibration analysis of moderately thick symmetrically laminated trapezoidal plates, considering various combinations of boundary conditions. Oni & Ogunyebi, (2008) investigated the dynamical analysis of finite prestressed structures with general boundary conditions under moving distributed loads. Their work incorporated the rotatory inertia factor, axial force of the moving equally distributed loads, and the foundation moduli in the system. Jimoh, (2017) considered the motion of variable prestressed tapered beams with exponentially varying thickness resting on a Vlasov foundation under a moving variable harmonic load with uniform speed. Jafarzadeh et al., (2017) examined the transverse response of nano-Timoshenko beams under several concentrated loads. They presented an analytical solution using Dirac's delta function to implement the mathematical model of the concentrated masses into the governing equations of motion. Bakhshi, (2018) considered the vibration analysis of rotating nanobeams. He employed the local and nonlocal theories to model nanobeams and reported the influences of material and scale parameters on their dynamic behavior. According to the literature, all the above-mentioned studies applied Euler–Bernoulli beam theory, when shear forces and inertia in beams are negligible. Malesela & Sarp, (2019) studied small-scale impacts on the fundamental frequency of a nanobeam with elastically restrained end conditions and a tip mass attached to the end of the beam. The dynamic response of a Structural Member with Time-Dependent Boundary Conditions to dynamic Distributed Mass was examined by Omolofe & Adedowole, (2019). In their work, closed-form solutions for the governing fourth-order partial differential equations with variable and singular coefficients of a beam-mass problem were presented.

Recently, Awodola et al. (2024) studied the vibration of a clamped-clamped non-uniform Bernoulli-Euler beam resting on a Pasternak elastic foundation, subject to varying magnitudes of moving distributed masses. The investigation accounts for critical aspects related to inertia terms and the non-uniformity of the beam's elastic properties, such as its flexural rigidity and mass per unit length. Ajijola (2025) studied the influence of axial force on transverse displacement and rotation under distributed load of an elastically supported damped beam. Jimoh et al, (2017) scrutinized the effect of variable axial forces on deflection of thick beams subjected to distributed moving loads. Similarly, Jimoh et al, (2018) studied the dynamic vibration of prestressed beams under distributed loads at different velocities. Galerkin's method and a series representation of the Heaviside function were employed to transform the governing equations. Results indicated that the moving distributed force is not always an upper bound for the distributed mass problem. It was shown that increasing certain structural parameters reduces the response amplitudes of non-uniformly loaded beams under moving distributed loads. Despite noteworthy progress, most mobile-load studies isolate loads as independent events. In reality, structures frequently experience simultaneous moving actions. For instance, a high-speed train or heavy freight train crossing a bridge; in this case, the bridge deck is the continuous elastic beam, and moving loads cannot be modeled correctly as a single entity when they consist of distinct concurrent components. Adedowole et al, (2023) considered the influence of rotatory inertia on the dynamic behavior of a non-Winkler Timoshenko beam traversed by a pair of loads

To address these limitations, the present study develops an analytical framework for the dynamic response of a uniform simply supported Euler-Bernoulli beam on a Winkler foundation under two simultaneous distributed loads. The finite Fourier sine transform reduces the governing fourth-order partial differential equation to modal ordinary differential equations; Laplace transformation and convolution then yield closed-form displacement expressions. The formulation accommodates constant and harmonically varying load magnitudes, unequal load intensities, load spacing, axial prestress, and foundation stiffness. Its principal contribution is the explicit treatment of two concurrent moving distributed loads and the resulting resonance pathways. The analytical results are independently checked against an Euler-Bernoulli finite-element time-integration solution. The vibration of beams under moving actions remains important in the analysis of railway tracks, highway bridges, crane girders, conveyors, and other slender structural systems. Unlike static loading, a moving action changes its point or region of application with time and can interact with the natural frequencies of the structure. This interaction may amplify displacement and stress even when the

applied load is moderate. Early studies therefore established analytical descriptions of moving-load response for beams and related structural members (Kenny, 1954; Stanisic et al., 1968; Oni, 1996).

Subsequent work extended these foundations by incorporating structural effects that are often neglected in elementary beam models. Ozkaya (2002) examined the nonlinear transverse vibration of a simply supported beam carrying concentrated masses, while Oni and Adedowole (2008) studied prestress effects in structural members under moving actions. Oni and Ogunyebi (2008) further considered finite prestressed Euler-Bernoulli beams with general boundary conditions under travelling distributed loads. Together, these studies show that inertia representation, boundary conditions, prestress, and the spatial form of a moving load can substantially alter the predicted response. Attention later shifted towards non-uniform beams, elastic foundations, and time-dependent loading. Adedowole and Jimoh (2018) combined Galerkin and integral-transform techniques to analyse a damped non-uniform beam under loads moving with variable velocity. Jimoh et al. (2017, 2018) examined variable axial force and distributed moving loads in thick or prestressed beams, demonstrating that changes in structural and loading parameters can modify response amplitudes in ways that are not captured by a single concentrated-force idealisation. Omolofe and Adedowole (2019) derived closed-form solutions for a structural member with time-dependent boundary conditions under a moving distributed mass. Related investigations have also addressed shear-deformable and small-scale members. Jafarzadeh Jazi et al. (2017) obtained an exact solution for a nano-Timoshenko beam carrying several concentrated masses, whereas Bakhshi Kaniki (2018) examined rotating nanobeams within a local/nonlocal framework. Malesela and Sarp (2019) studied the fundamental frequency of an elastically restrained nanobeam with a tip mass. Although these contributions concern different scales and beam theories, they reinforce a common point: the accuracy of a vibration model depends on whether its kinematics, support conditions, inertia terms, and loading pattern represent the physical system adequately. More recent studies have returned to practical beam-foundation systems. Adedowole (2019) investigated a non-prismatic Timoshenko beam under concentrated loads travelling at time-dependent speeds. Adedowole et al. (2023) considered the influence of rotary inertia on a non-Winkler Timoshenko beam traversed by a pair of loads, while Awodola et al. (2024) analysed a non-uniform Euler-Bernoulli beam on a Pasternak foundation under a moving distributed mass. Ajijola (2025) also examined axial-force effects on the displacement and rotation of an elastically supported damped beam. These studies demonstrate continuing interest in multiple loads, elastic support, non-uniformity, and prestress. Despite this progress, much of the available literature treats moving loads as isolated actions, concentrated forces, or single distributed loads. That simplification is restrictive for bridge and rail applications, where two axles, wheel groups, or load patches may act on the same member simultaneously and may differ in magnitude. The response cannot always be inferred from a single load of equal total magnitude because spacing introduces a time delay and modal phase difference. Moreover, comparatively little analytical work combines two simultaneous moving distributed loads, a Winkler foundation, axial prestress, and both constant and harmonic load amplitudes in one closed-form formulation.

The present study addresses this gap by deriving and validating the dynamic response of a simply supported Euler-Bernoulli beam on a Winkler foundation under two simultaneous moving distributed loads. The finite Fourier sine transform is used to satisfy the end conditions and reduce the fourth-order governing equation to modal equations, which are solved using the Laplace transform and convolution theorem. The formulation distinguishes the two load magnitudes and their spacing, identifies the resonance conditions for constant and harmonic excitation, and quantifies the effects of foundation stiffness and prestress. An independent finite-element solution is used to assess the accuracy of the analytical response.

RESEARCH METHODOLOGY

The Governing Equation

We consider a uniform, simply supported Euler-Bernoulli beam of span L resting on a Winkler foundation and subjected to two distributed loads moving at constant speed c . Its transverse motion is governed by the following fourth-order partial differential equation:

$$EIv_{xxxx}(x, t) - N(x)v_{xx}(x, t) + \mu v_{tt}(x, t) = P_i(x, t) - R(x, t) \quad (1)$$

where EI is the flexural rigidity; E is Young's modulus; I is the second moment of area; $v(x, t)$ is transverse deflection; $N(x)$ is the axial prestress; μ is the beam mass per unit length; $P(x, t)$ is the resultant applied moving-load intensity; and $R(x, t)$ is the Winkler restoring reaction. The coordinate x is measured along the span, and t denotes time. A positive tensile prestress is used in the numerical study.

The Winkler foundation model assumes that the reactive pressure is proportional to the deflection of the beam, that is,

$$R(x, t) = Qv(x, t). \quad (2)$$

Here, Q is called the modulus of subgrade reaction.

Table (1): Nomenclatures (standardized formatting)

Symbol	Description
$v(x, t)$ (m)	Transverse deflection of the beam
$H(x - ct)$	Heaviside step function
M (kg)	Mass of the moving load
c (m/s)	uniform velocity
g (m/s ²)	Acceleration due to gravity
x (m)	Spatial coordinate along the beam
L (m)	Beam span
t (s)	Time coordinate
E (N/m ²)	Young's modulus
I (m ⁴)	Moment of inertia of the beam cross-section
μ (kg/m)	Mass per unit length of the beam
Q (N/m ³)	Foundation modulus (Winkler stiffness)
$N(x)$ (N)	Variable prestressed
$P_i(x, t)$ (N/m)	Transverse moving load
$R(x, t)$	Restoring force from the elastic foundation
Ω (Hertz)	Circular frequency
d (m)	Distance between the two moving loads P_1 and P_2

The multiple distributed loads

The multiple distributed loads' function is given as

$$P_i(x, t) = [P_1(x, t) + P_2(x, t)] \quad (2a)$$

Where

$$P_1(x, t) = P_1 H(x - ct) \text{ and } P_2(x, t) = P_2 H(x - \{d + ct\}) \quad (3)$$

The two distributed loads are supposed to have a uniform mass μ and move along the beam at a uniform velocity. The analysis is limited to a finite time interval t , $0 < ct \leq L$, during which the travelling loads remain in interaction with the beam. $H(x - ct)$ is the Heaviside function defined as;

$$H(x - ct) = f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \quad (4a)$$

With the properties

$$(i) \frac{d[H(x-ct)]}{dx} = \delta(x - ct)$$

$$(ii) H(x - ct)f(x) = \begin{cases} 0, & x < ct \\ f(x), & x \geq ct \end{cases} \quad (4b)$$

Where $\delta(x - ct)$ represents the Dirac delta function and $H(x - ct)$ is a typical engineering function made to measure engineering applications, which often involve functions that are either “Off” or “On”.

End Conditions

With simple support, boundary conditions and Initial conditions as

$$v(x, t) = 0, \quad \frac{\partial^2 v(x, t)}{\partial t^2} = 0 \quad \text{at } x = 0 \text{ and } x = L$$

and

$$v(x, t)_{t=0} \quad \text{and} \quad \frac{\partial v(x, t)}{\partial t} \bigg|_{t=0} \quad (5)$$

The pre-stressed

The pre-stressed parameter is non-uniform, given as

$$N(x) = P_r \left(1 + \sin \frac{\pi x}{L} \right) \quad (6)$$

Substituting equations (3) and (6) into equation (1), and taking EI and the mass per unit length μ as uniform along the span, gives:

$$\begin{aligned} v_{xxxx}(x, t) + \alpha_1 v_{tt}(x, t) + \alpha_2 v(x, t) - \alpha_3 \left(1 + \sin \frac{\pi x}{L} \right) v_{xx}(x, t) \\ = [\alpha_4 H(x - ct) + \alpha_5 H(x - \{d + ct\})] \end{aligned} \quad (7)$$

Where:

$$\alpha_1 = \frac{\mu}{EI}; \quad \alpha_2 = \frac{Q}{EI}; \quad \alpha_3 = \frac{P_r}{EI}; \quad \alpha_4 = \frac{P_1}{EI}; \quad \text{and } \alpha_5 = \frac{P_2}{EI} \quad (8)$$

Solution To The Problem

An analytical solution of the non-homogeneous initial-boundary-value problem is obtained by the finite Fourier sine transform. This basis satisfies the simply supported end conditions and reduces the fourth-order spatial operator to a set of second-order modal ordinary differential equations.

$$w(j,t) = \int_0^L V(x,t) \sin \frac{j\pi x}{L} dx \quad (9)$$

With the inverse

$$v(x,t) = \frac{2}{L} \sum_{j=1}^n w(j,t) \sin \frac{j\pi x}{L} \quad (10)$$

Then, substituting equation (10) into (7), we have;

$$\begin{aligned} \sum_{j=1}^n \left\{ \left(\frac{j\pi}{L} \right)^4 w(j,t) \right\} + \alpha_3 \left(1 + \sin \frac{\pi x}{L} \right) \left(\frac{j\pi}{L} \right)^2 w(j,t) \\ + \alpha_1 \ddot{w}(j,t) + \alpha_2 w(j,t) \} \sin \frac{j\pi x}{L} = [\alpha_4 H(x-ct) + \alpha_5 H(x-\{d+ct\})] \end{aligned} \quad (11)$$

Here, $w(j,t)$ stands for the transformed form of $v(x,t)$. To exploit the orthogonality property in equation (11), both sides are multiplied by $\sin \frac{k\pi x}{L}$ and integrated with respect to x over the interval 0 and L. Applying the boundary conditions during this transformation results in

$$\begin{aligned} \int_0^L \sum_{j=1}^n \left[\left(\frac{j\pi}{L} \right)^4 + \alpha_3 \left(1 + \sin \frac{\pi x}{L} \right) \left(\frac{j\pi}{L} \right)^2 + \alpha_2 \right] w(j,t) \\ + \alpha_1 \ddot{w}(j,t) \left\{ \sin \frac{j\pi x}{L} \right\} \left\{ \sin \frac{k\pi x}{L} \right\} \\ = \int_0^L [\alpha_4 H(x-ct) + \alpha_5 H(x-\{d+ct\})] \left\{ \sin \frac{k\pi x}{L} \right\} \end{aligned} \quad (12)$$

Equation (12) can be rearranged and simplified; we have:

$$\begin{aligned} a_1(j,t) \ddot{w}(j,t) + a_2(j,t) w(j,t) \\ = \frac{L}{k\pi} \left[\alpha_4 \left(\cos \frac{k\pi ct}{L} - \cos k\pi \right) + \alpha_5 \left(\cos \frac{k\pi(ct+d)}{L} - \cos k\pi \right) \right] \end{aligned} \quad (13)$$

Where

$$\begin{aligned} a_1(j,t) &= \alpha_1 E_1 \\ a_2(j,t) &= \left\{ EI \left(\frac{j\pi}{L} \right)^4 + \alpha_2 \right\} E_1 + \alpha_3 \left(\frac{j\pi}{L} \right)^2 (E_1 + E_2) \\ E_1 &= \int_0^L \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L} dx \quad \text{and} \quad E_2 = \int_0^L \sin \frac{\pi x}{L} \sin \frac{j\pi x}{L} \sin \frac{k\pi x}{L} dx \end{aligned} \quad (14)$$

Equation (13) is a second-order ODE with constant coefficients. In the following analysis, equation (13) is subjected to the Laplace transform, applied in conjunction with the initial conditions, and defined as follows:

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt \quad (15)$$

Then,

$$\mathcal{L}[a_1(j,k) \ddot{w}(j,t) + a_2(j,k) w(j,t)]$$

$$= \frac{L}{k\pi} \left[\alpha_4 \left(\cos \frac{k\pi ct}{L} - \cos k\pi \right) + \alpha_5 \left(\cos \frac{k\pi(ct+d)}{L} - \cos k\pi \right) \right] \quad (16)$$

Using the trigonometric identity on the right-hand side, equation (16) becomes

$$\begin{aligned} & \mathcal{L}[a_1(j, k)\ddot{w}(j, t) + a_2(j, k)w(j, t)] \\ &= \mathcal{L}[B_L \cos \phi t \cos H_d - B_b \sin \phi t - RL] \end{aligned} \quad (17)$$

Where

$$\begin{aligned} \phi &= \frac{k\pi c}{L}, H_d = \frac{k\pi d}{L}, \quad B_L = L_a + L_b b_0, B_b = L_a a_0, R_H = \cos k\pi \\ RL &= (L_a + L_b)R_H, L_a = \frac{L}{k\pi} \alpha_4, L_b = \frac{L}{k\pi} \alpha_5, b_0 = \cos H_d \text{ and } a_0 = \sin H_d \end{aligned} \quad (18)$$

With the application of the initial condition, equation (18) becomes

$$w(s) = \frac{1}{a_1(s^2 + M_k^2)} \left[B_L \frac{s}{s^2 + \phi^2} - B_b \frac{\phi}{s^2 + \phi^2} - \frac{RL}{s} \right] \quad (19)$$

To obtain the Laplace inversion of (19), the following representation is adopted.

$$F(s) = \frac{1}{M_k} \left(\frac{M_k}{s^2 + M_k^2} \right); G_1(s) = \frac{s}{s^2 + \phi^2}; G_2(s) = \frac{\phi}{s^2 + \phi^2}; \text{ and } G_3(s) = \frac{RL}{s} \quad (21)$$

Therefore, equation (21) becomes;

$$w_i(t) = \frac{1}{M_k a_1} [B_L I_1 - B_b I_2 - RL I_3] \quad (22)$$

where

$$M_k^2 = \frac{a_2(j, t)}{a_1(j, t)}$$

$$I_1 = \int_0^t \sin M_k(t-u) \cos \phi u du$$

$$I_2 = \int_0^t \sin M_k(t-u) \sin \phi u du$$

$$I_3 = \int_0^t \sin M_k(t-u) du \quad (23)$$

Thus, reconstructing equation (22), one obtains

$$\begin{aligned} w_i(t) &= \frac{1}{M_k a_1} \left\{ B_L \frac{\phi \sin M_k t}{M_k^2 - \phi^2} \left[\cos M_k t \sin \phi t - \frac{M_k}{\phi} (\sin M_k t \cos \phi t) \right] \right. \\ &\quad \left. + B_L \frac{\phi \cos M_k t}{\phi^2 - M_k^2} \left[\sin M_k t \sin \phi t - \right. \right. \\ &\quad \left. \left. - B_b \frac{M_k \sin M_k t}{M_k^2 - \phi^2} \left[\sin M_k t \sin \phi t - \frac{\phi}{M_k} (\cos \phi t \cos M_k t - 1) \right] \right] \right. \\ &\quad \left. + B_b \frac{M_k \cos M_k t}{M_k^2 - \phi^2} \left[\sin \phi t \cos M_k t - \frac{\phi}{M_k} (\cos \phi t \sin M_k t) \right] \right. \\ &\quad \left. - RL \frac{1}{M_k} [1 - \cos M_k t] \right\} \end{aligned} \quad (24)$$

With the inverse, we have

$$\begin{aligned}
 v(x, t) = & \frac{2}{L} \sum_{i=1}^3 \left\{ \frac{1}{M_k a_1} \left\{ B_L \frac{\emptyset \sin M_k t}{M_k^2 - \emptyset^2} \left[\cos M_k t \sin \emptyset t - \frac{M_k}{\emptyset} (\sin M_k t \cos \emptyset t) \right] \right. \right. \\
 & + B_L \frac{\emptyset \cos M_k t}{\emptyset^2 - M_k^2} \left[\sin M_k t \sin \emptyset t - \frac{M_k}{\emptyset} (\cos M_k t \cos \emptyset t - 1) \right] \\
 & - B_b \frac{M_k \sin M_k t}{M_k^2 - \emptyset^2} \left[\sin M_k t \sin \emptyset t - \frac{\emptyset}{M_k} (\cos \emptyset t \cos M_k t - 1) \right] \\
 & + B_b \frac{M_k \cos M_k t}{M_k^2 - \emptyset^2} \left[\sin \emptyset t \cos M_k t - \frac{\emptyset}{M_k} (\cos \emptyset t \sin M_k t) \right] \\
 & \left. \left. - R L \frac{1}{M_k} [1 - \cos M_k t] \right\} \sin \frac{i \pi x}{L} \right\} \quad (25)
 \end{aligned}$$

The equation above is the transverse displacement response to two distributed moving loads of an Euler-Bernoulli beam resting on an elastic foundation.

Dynamic Response Of Euler–Bernoulli Beam Under The Harmonic Load

The function for multiple distributed loads is given as:

$$P_i(x, t) = p(t)[P_1(x, t) + P_2(x, t)] \quad (26)$$

Where $p(t) = D_0 \sin \Omega t$;

$P_1(x, t)$ and $P_2(x, t)$ are the first and second loads, respectively, D_0 is the constant amplitude d is the distance between the two loads, then equation (26) becomes;

$$P_i(x, t) = D_0 \sin \Omega t [P_1 H(x - ct) + P_2 H(x - \{d + ct\})] \quad (27)$$

Then substitute equation (26) into (1), while other parameters such as flexural rigidity EI and mass per unit length μ are constants with position x along the span L . Thus, equation (1) becomes:

$$\begin{aligned}
 V_{xxxx}(x, t) + \alpha_1 V_{tt}(x, t) + \alpha_2 V(x, t) - \alpha_3 \left(1 + \sin \frac{\pi x}{L} \right) V_{xx}(x, t) \\
 = D_0 \sin \Omega t [\alpha_4 H(x - ct) + \alpha_5 H(x - \{d + ct\})] \quad (28)
 \end{aligned}$$

Just like the previous section, the same approach is taken

$$w(j, t) = \int_0^L v(x, t) \sin \frac{j \pi x}{L} dx \quad (29)$$

With the inverse

$$v(x, t) = \frac{2}{L} \sum_{j=1}^n w(j, t) \sin \frac{j \pi x}{L} \quad (30)$$

Following the same steps, equation (30) can be rearranged and simplified; we have:

$$a_1 \ddot{w}(j, t) + a_2 w(j, t) = D_0 \sin \Omega t \frac{L \alpha_4}{k \pi} \left(\frac{\cos k \pi c t}{L} - \cos k \pi \right)$$

$$+D_0 \sin \Omega t \frac{L\alpha_5}{k\pi} \left(\frac{\cos k\pi(ct+d)}{L} - \cos k\pi \right) \quad (31)$$

We subject the system to the Laplace transform, defined as (in conjunction with the initial condition)

By considering the right-hand side of (31), using trigonometric identities, we have

$$\begin{aligned} & \mathcal{L}[a_1(j, k)\ddot{w}(j, t) + a_2(j, k)w(j, t)] \\ &= \mathcal{L}[M_a(\sin \Omega_1 t + \sin \Omega_2 t) - M_b(\cos \Omega_2 t - \cos \Omega_1 t) - M_c \sin \Omega t] \end{aligned} \quad (32)$$

Where,

$$\begin{aligned} \Omega_1 &= \Omega + \phi, \Omega_2 = \Omega - \phi, H_a = \frac{D_0}{k\pi} L\alpha_4; H_p = \frac{D_0}{k\pi} L\alpha_5; R_H = \cos k\pi, \\ M_a &= \frac{1}{2}[H_a + H_p b_0], M_b = \frac{1}{2}H_p a_0, M_c = H_a R_H + H_p R_H \end{aligned} \quad (33)$$

To obtain the Laplace inversion of (32), we adopted the following representations

$$\begin{aligned} f(s) &= \frac{1}{M_k} \left(\frac{M_k}{s^2 + M_k^2} \right), G_1(s) = \frac{\Omega_1}{s^2 + \Omega_1^2}, G_2(s) = \frac{\Omega_2}{s^2 + \Omega_2^2}; \\ G_3(s) &= \frac{s}{s^2 + \Omega_2^2}, G_4(s) = \frac{s}{s^2 + \Omega_1^2}, G_5(s) = \frac{\Omega}{s^2 + \Omega^2} \end{aligned} \quad (34)$$

Therefore, equation (32) becomes;

$$w_i(t) = \frac{1}{M_k a_1} [M_a I_1 + M_a I_2 - M_b I_3 + M_b I_4 - M_c I_5] \quad (35)$$

where

$$\begin{aligned} I_1 &= \int_0^t \sin M_k(t-u) \sin \Omega_1 u du \\ I_2 &= \int_0^t \sin M_k(t-u) \sin \Omega_2 u du \\ I_3 &= \int_0^t \sin M_k(t-u) \cos \Omega_2 u du \\ I_4 &= \int_0^t \sin M_k(t-u) \cos \Omega_1 u du \\ I_5 &= \int_0^t \sin M_k(t-u) \sin \Omega u du \end{aligned} \quad (36)$$

Evaluating equation (36) in (35), we have

$$\begin{aligned} w_i(t) &= \frac{1}{M_k a_1} \left\{ M_a \frac{M_k \sin M_k t}{M_k^2 - \Omega_1^2} \left[\sin \Omega_1 t \sin M_k t - \frac{\Omega_1}{M_k} (\cos \Omega_1 t \cos M_k t - 1) \right] \right. \\ &\quad \left. + M_a \frac{M_k \cos M_k t}{M_k^2 - \Omega_1^2} \left[\sin \Omega_1 t \cos M_k t - \frac{\Omega_1}{M_k} (\cos \Omega_1 t \sin M_k t) \right] \right. \\ &\quad \left. + M_a \frac{M_k \sin M_k t}{M_k^2 - \Omega_2^2} \left[\sin \Omega_2 t \sin M_k t - \frac{\Omega_2}{M_k} (\cos \Omega_2 t \cos M_k t - 1) \right] \right. \\ &\quad \left. + M_a \frac{M_k \cos M_k t}{M_k^2 - \Omega_2^2} \left[\sin \Omega_2 t \cos M_k t - \frac{\Omega_2}{M_k} (\cos \Omega_2 t \sin M_k t) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& -M_b \frac{\Omega_1 \sin M_k t}{M_k^2 - \Omega_1^2} \left[\cos M_k t \sin \Omega_1 t - \frac{M_k}{\Omega_1} (\sin M_k t \cos \Omega_1 t) \right] \\
& -M_b \frac{\Omega_1 \cos M_k t}{\Omega_1^2 - M_k^2} \left[\sin M_k t \sin \Omega_1 t - \frac{M_k}{\Omega_1} (\cos M_k t \cos \Omega_1 t - 1) \right] \\
& +M_b \frac{\Omega_2 \sin M_k t}{M_k^2 - \Omega_2^2} \left[\cos M_k t \sin \Omega_2 t - \frac{M_k}{\Omega_2} (\sin M_k t \cos \Omega_2 t) \right] \\
& +M_b \frac{\Omega_2 \cos M_k t}{\Omega_2^2 - M_k^2} \left[\sin M_k t \sin \Omega_2 t - \frac{M_k}{\Omega_2} (\cos M_k t \cos \Omega_2 t - 1) \right] \\
& -M_c \frac{M_k \sin M_k t}{M_k^2 - \Omega^2} \left[\sin \Omega t \sin M_k t - \frac{\Omega}{M_k} (\cos \Omega t \cos M_k t - 1) \right] \\
& -M_c \frac{M_k \cos M_k t}{M_k^2 - \Omega^2} \left[\sin \Omega t \cos M_k t - \frac{\Omega}{M_k} (\cos \Omega t \sin M_k t) \right] \} \quad (37)
\end{aligned}$$

Therefore,

$$\begin{aligned}
v(x, t) = & \frac{2}{L} \sum_{j=1}^3 \left\{ \frac{1}{M_k a_1} \left\{ M_a \frac{M_k \sin M_k t}{M_k^2 - \Omega_1^2} \left[\sin \Omega_1 t \sin M_k t - \frac{\Omega_1}{M_k} (\cos \Omega_1 t \cos M_k t - 1) \right] \right. \right. \\
& +M_a \frac{M_k \cos M_k t}{M_k^2 - \Omega_1^2} \left[\sin \Omega_1 t \cos M_k t - \frac{\Omega_1}{M_k} (\cos \Omega_1 t \sin M_k t) \right] \\
& +M_a \frac{M_k \sin M_k t}{M_k^2 - \Omega_2^2} \left[\sin \Omega_2 t \sin M_k t - \frac{\Omega_2}{M_k} (\cos \Omega_2 t \cos M_k t - 1) \right] \\
& +M_a \frac{M_k \cos M_k t}{M_k^2 - \Omega_2^2} \left[\sin \Omega_2 t \cos M_k t - \frac{\Omega_2}{M_k} (\cos \Omega_2 t \sin M_k t) \right] \\
& -M_b \frac{\Omega_1 \sin M_k t}{M_k^2 - \Omega_1^2} \left[\cos M_k t \sin \Omega_1 t - \frac{M_k}{\Omega_1} (\sin M_k t \cos \Omega_1 t) \right] \\
& -M_b \frac{\Omega_1 \cos M_k t}{\Omega_1^2 - M_k^2} \left[\sin M_k t \sin \Omega_1 t - \frac{M_k}{\Omega_1} (\cos M_k t \cos \Omega_1 t - 1) \right] \\
& +M_b \frac{\Omega_2 \sin M_k t}{M_k^2 - \Omega_2^2} \left[\cos M_k t \sin \Omega_2 t - \frac{M_k}{\Omega_2} (\sin M_k t \cos \Omega_2 t) \right] \\
& +M_b \frac{\Omega_2 \cos M_k t}{\Omega_2^2 - M_k^2} \left[\sin M_k t \sin \Omega_2 t - \frac{M_k}{\Omega_2} (\cos M_k t \cos \Omega_2 t - 1) \right] \\
& -M_c \frac{M_k \sin M_k t}{M_k^2 - \Omega^2} \left[\sin \Omega t \sin M_k t - \frac{\Omega}{M_k} (\cos \Omega t \cos M_k t - 1) \right] \\
& -M_c \frac{M_k \cos M_k t}{M_k^2 - \Omega^2} \left[\sin \Omega t \cos M_k t - \frac{\Omega}{M_k} (\cos \Omega t \sin M_k t) \right] \} \sin \frac{j\pi x}{L} \quad (38)
\end{aligned}$$

The equation above is the transverse displacement response of harmonic moving loads on an Euler–Bernoulli beam resting on an elastic foundation.

Analysis Of The Numerical Results

The analytical response is evaluated using the material, geometric, foundation, and loading parameters listed in the following equation-editor expression. Unless a parameter is varied in a particular figure, the corresponding baseline value is retained. $Q \in \{0, 4.0 \times 10^4, 4.0 \times 10^5, 4.0 \times 10^6\} \text{ N/m}^2$, $0 \leq t \leq$

4.5 s , $L = 50.192 \text{ m}$, $E = 2.10924 \times 10^9 \text{ N/m}^2$, $I = 2.87698 \times 10^{-3} \text{ m}^4$, $c = 8.128 \text{ m/s}$, $d = 1 \text{ m}$, $\Omega = 0.5 \text{ rad/s}$

Resonance condition

Response Under Constant Moving Loads (Equation 24)

The main frequency parameter $\phi = \frac{k\pi c}{L}$ determines the response of the governing equation, which serves as the principal forcing frequency. This parameter is categorized by:

- The speed of loads.
- Structural vibration mode.

This shows that dynamic amplification turns out to be significant when $M_k \approx \phi$

Response Under Harmonic Moving Loads (Equation 38)

In the harmonic moving-load setup, the mechanical response becomes more complex due to dual-frequency excitation. Compared with a constant load, which depends exclusively on translation, the harmonic case includes an inherent internal oscillation. The harmonic case has three resonance mechanisms:

1. Pure harmonic resonance, $M_k \approx \phi$,
2. Forward frequency, $M_k \approx \Omega + \phi$
3. Backward frequency, $M_k \approx \Omega - \phi$

Graphical representations for the Simply Supported End Conditions

Graphs for Constant Moving Loads

Figure 1 presents the constant-load response for increasing Winkler foundation modulus. As the foundation stiffens, its distributed restoring action increases the beam's effective modal stiffness. The vibration amplitude therefore decreases, while the timing of local peaks changes because the natural frequencies shift with the foundation modulus.

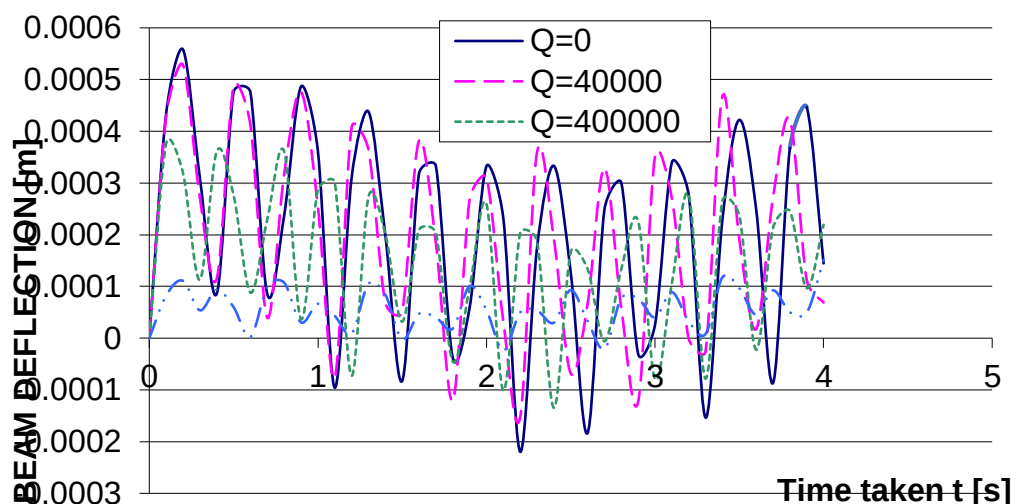


Fig 1. Transverse vibrations of an Euler–Bernoulli beam under the action of two traveling distributed loads

for various values of foundation modulus $Q(\text{N/m}^3)$ when the beam length $L = 50$ and for fixed values of $Pr = 2000\text{N}$, $P_1 = 6000\text{N}$, $P_2 = 7000\text{N}$, $d = 1\text{m}$ and $c = 8.142\text{ m/s}$

Figure 2 shows the constant-load response for different tensile prestress values. Increasing tensile prestress raises the effective geometric stiffness in the adopted sign convention and reduces the maximum transverse displacement. This trend indicates improved resistance to dynamic bending, provided that the member and its connections can sustain the imposed prestress.

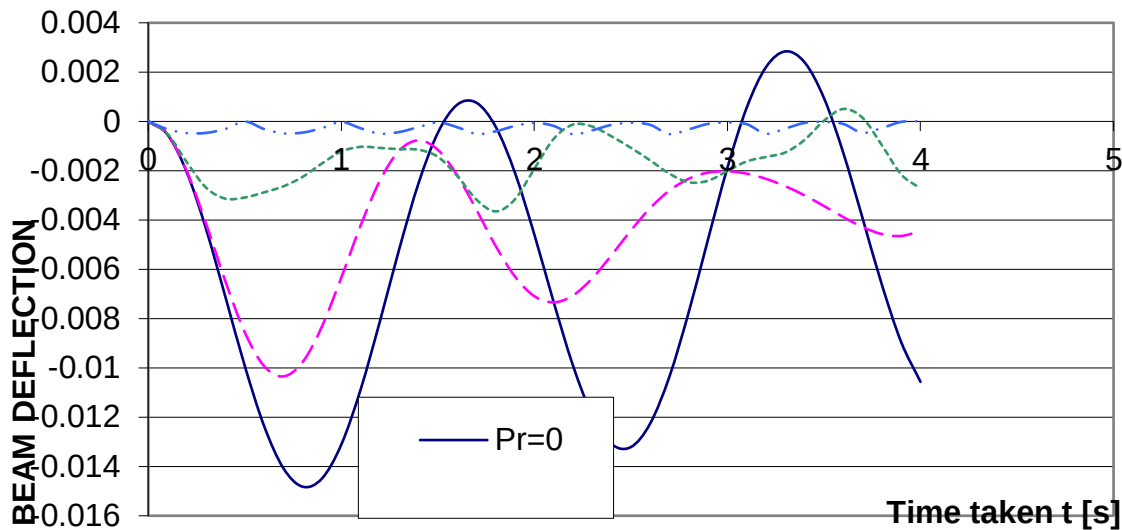


Fig 2. Transverse vibrations of an Euler–Bernoulli beam under the action of two traveling distributed loads for various values of pre-stressed $Pr(\text{N})$ when the beam length $L = 50$ and for fixed values of $Q = 4 \times 10^6\text{N/m}^3$, $P_1 = 6000\text{N}$, $P_2 = 7000\text{N}$, $d = 1\text{m}$ and $c = 8.142\text{ m/s}$

Figure 3 isolates the effect of the first moving distributed load. With all other parameters fixed, increasing the first load magnitude increases the modal forcing and produces a corresponding increase in beam deflection. The approximately proportional change is consistent with the linear governing equation away from resonance.

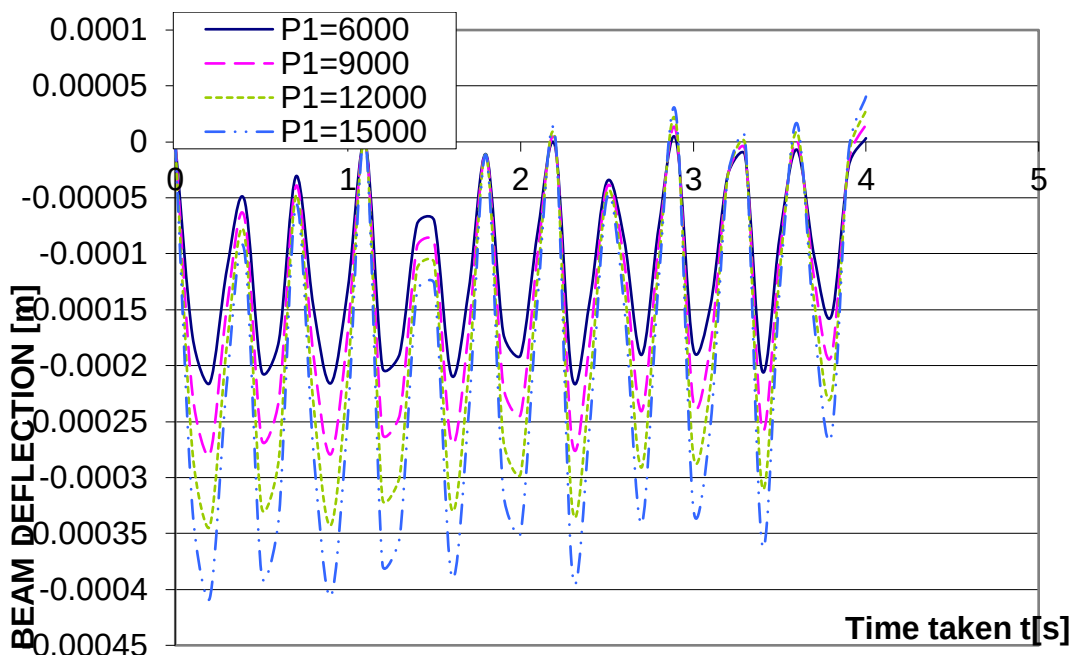


Fig 3. Transverse vibrations of an Euler–Bernoulli beam under the action of two traveling distributed loads for various values of Load $P_1(\text{N})$ when the beam length $L = 50$ and for fixed values of $Q = 4 \times 10^6\text{N/m}^3$,

$P_2 = 7000\text{N}$, $P_r = 2000\text{N}$, $d = 2\text{m}$ and $c = 8.142\text{ m/s}$

Figure 4 isolates the effect of the second moving distributed load. Its contribution also increases with magnitude, but the response does not exactly duplicate that of the first load because the fixed spacing introduces a time delay and changes the phase at which the second load excites each mode.

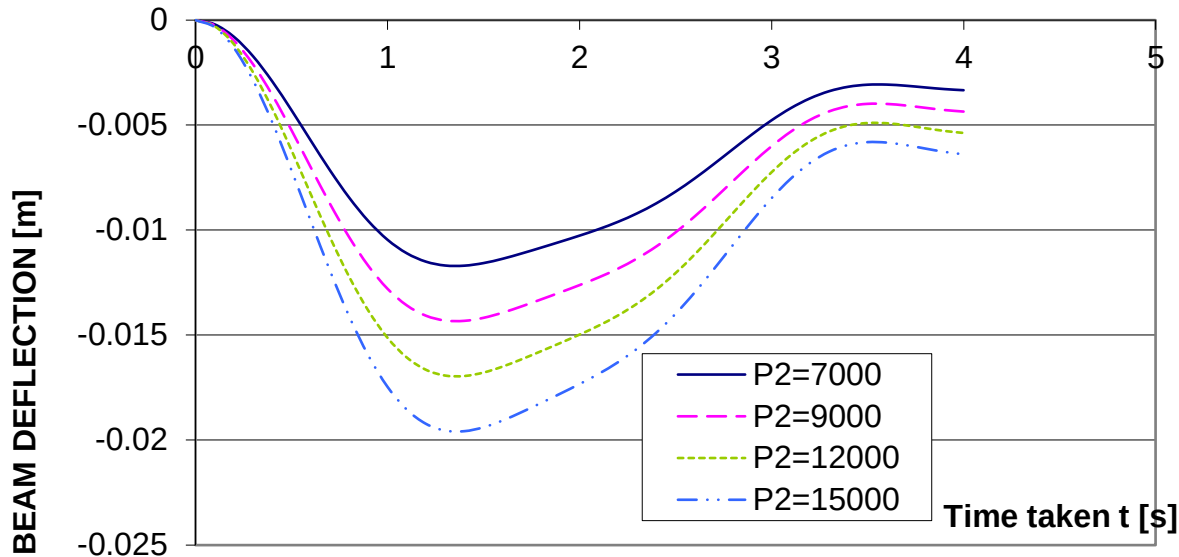


Fig 4. Transverse vibrations of an Euler–Bernoulli beam under the action of two traveling distributed loads for various values of Load $P_2(\text{N})$ when the beam length $L = 50$ and for fixed values of $Q = 4 \times 10^6 \text{N/m}^3$, $P_1 = 6000\text{N}$, $P_r = 2000\text{N}$, $d = 2\text{m}$ and $c = 8.142\text{ m/s}$

Graphs for Harmonic Moving Loads

Figure 5 gives the harmonic-load response for increasing foundation modulus. The stiffer foundation suppresses the response by increasing modal stiffness, although the harmonic modulation produces a smoother low-frequency envelope than in the constant-load case. The result confirms that foundation stiffness remains an effective control parameter under combined translational and harmonic excitation.

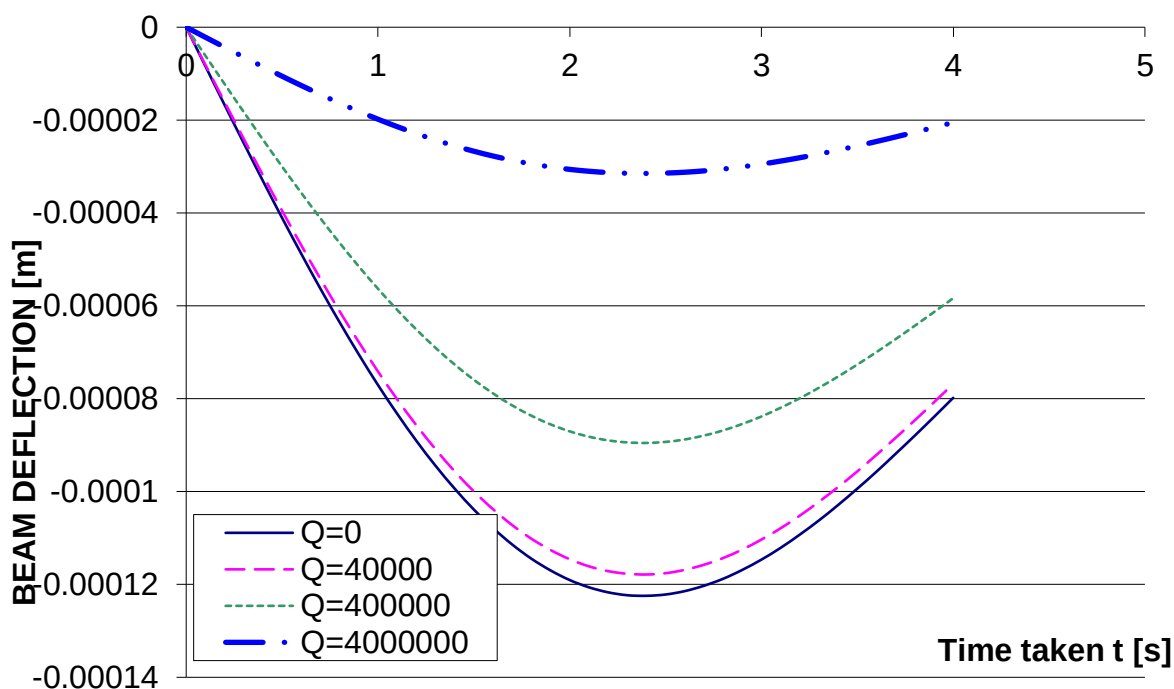


Fig 5. Transverse vibrations of an Euler–Bernoulli beam under the action of two harmonic traveling

distributed loads for various values of foundation modulus $Q(\text{N/m}^3)$ when the beam length $L = 50$ and for fixed values of $Pr = 2000\text{N}$, $P1 = 6000\text{N}$, $P2 = 7000\text{N}$, $d = 1\text{m}$ and $c = 8.142\text{ m/s}$

Figure 6 demonstrates the effect of tensile prestress under harmonic loading. Larger prestress values reduce the maximum deflection and shift the response curve because the structural modal frequencies change while the translational and harmonic excitation frequencies remain fixed.

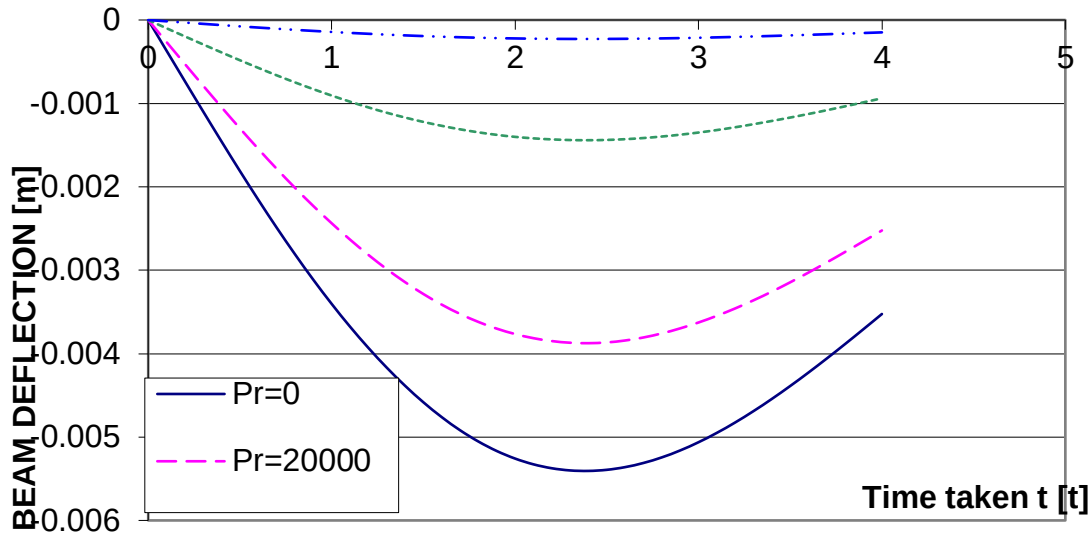


Fig 6. Transverse vibrations of an Euler–Bernoulli beam under the action of two harmonic traveling distributed loads for various values of pre-stressed $Pr(\text{N})$ when the beam length $L = 50$ and for fixed values of $Q = 4 \times 10^6\text{N/m}^3$, $P1 = 6000\text{N}$, $P2 = 7000\text{N}$, $d = 1\text{m}$ and $c = 8.142\text{ m/s}$

Figure 7 shows the harmonic response as the first load magnitude increases. The response amplitude rises because the first load contributes a larger harmonic modal force. The curve shapes remain related because the speed, harmonic frequency, foundation stiffness, spacing, and second load are unchanged.

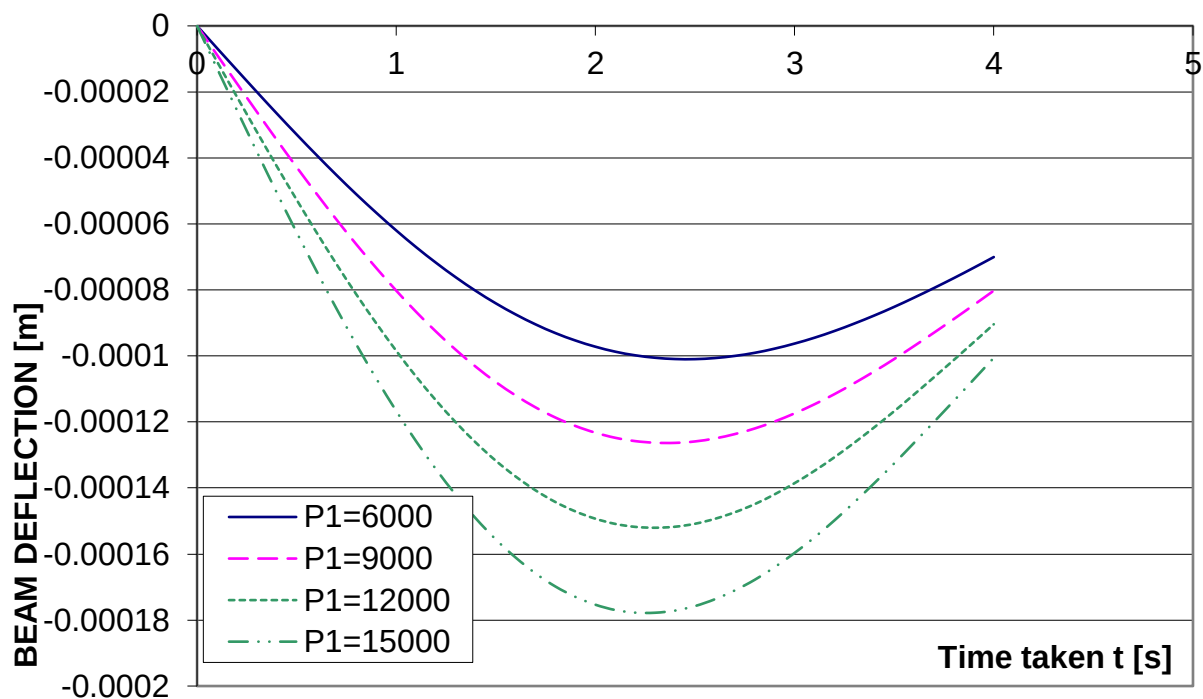


Fig 7. Transverse vibrations of an Euler–Bernoulli beam under the action of two harmonic traveling distributed loads for various values of Load $P1(\text{N})$ when the beam length $L = 50$ and for fixed values of $Q = 4 \times 10^6\text{N/m}^3$, $P2 = 7000\text{N}$, $Pr = 2000\text{N}$, $d = 1\text{m}$ and $c = 8.142\text{ m/s}$

Figure 8 shows the corresponding influence of the second harmonic moving load. Increasing this load raises the response, while its delayed arrival produces a phase-dependent contribution. This confirms that the two moving loads should be retained separately rather than replaced automatically by one load of equal total magnitude.

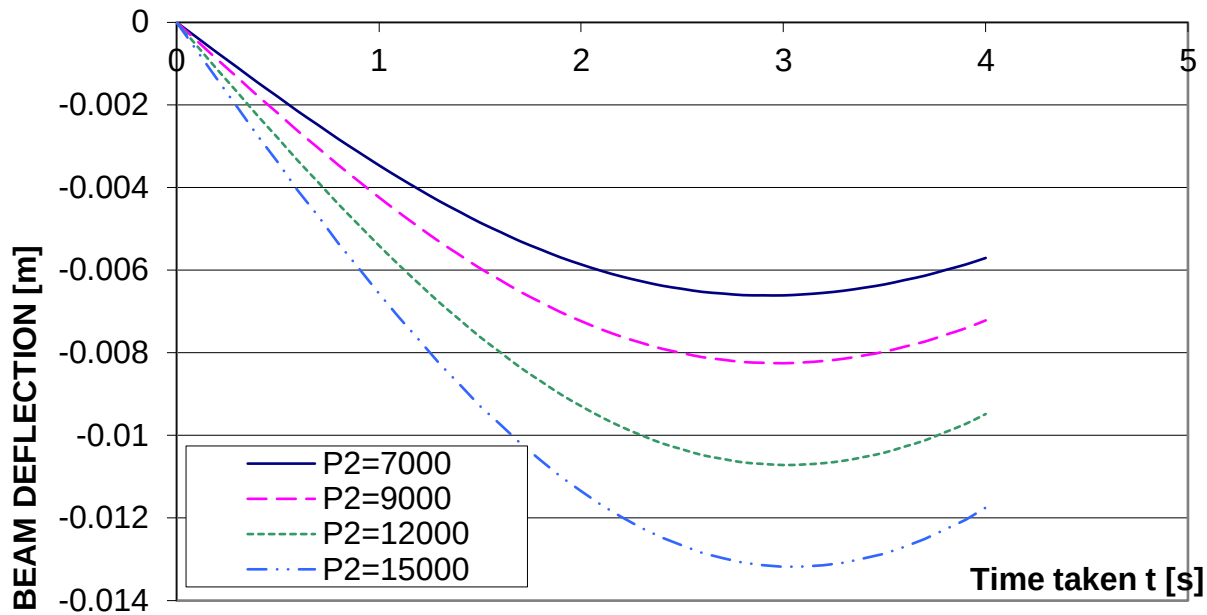


Fig 8. Transverse vibrations of an Euler–Bernoulli beam under the action of two harmonic traveling distributed loads for various values of Load P_2 (N) when the beam length $L = 50$ and for fixed values of $Q = 4 \times 10^6 \text{ N/m}^3$, $P_1 = 6000 \text{ N}$, $P_r = 2000 \text{ N}$, $d = 1 \text{ m}$ and $c = 8.142 \text{ m/s}$

Finite-Element Validation

To verify the derived modal solution independently, the baseline constant-load case was reproduced with a conventional Euler-Bernoulli finite-element model. The beam was divided into 80 cubic Hermite elements. Consistent mass, elastic bending stiffness, geometric stiffness, and Winkler-foundation matrices were assembled, and the simply supported end constraints were imposed. The two moving distributed loads were integrated over their instantaneous one-metre footprints by four-point Gauss quadrature. The semi-discrete system was advanced with the average-acceleration Newmark method using the parameters below.

$$\beta = 1/4, \gamma = 1/2, \Delta t = 0.002 \text{ s}$$

For the corresponding analytical evaluation, 50 sine modes were retained. Both solutions used the same parameter set: $L = 50.192 \text{ m}$, $E = 2.10924 \times 10^9 \text{ N/m}^2$, $I = 2.87698 \times 10^{-3} \text{ m}^4$, $\mu = 2750 \text{ kg/m}$, $Q = 4.0 \times 10^6 \text{ N/m}^2$, $N = 2000 \text{ N}$, $c = 8.128 \text{ m/s}$, $d = 1 \text{ m}$, $P_1 = 6000 \text{ N/m}$, $P_2 = 7000 \text{ N/m}$

Measure	Analytical modal solution	Finite-element solution	Difference
Peak absolute midspan deflection	0.98685 mm	0.98518 mm	0.17%
Root-mean-square time-history difference	-	0.00258 mm	0.26% of FE peak

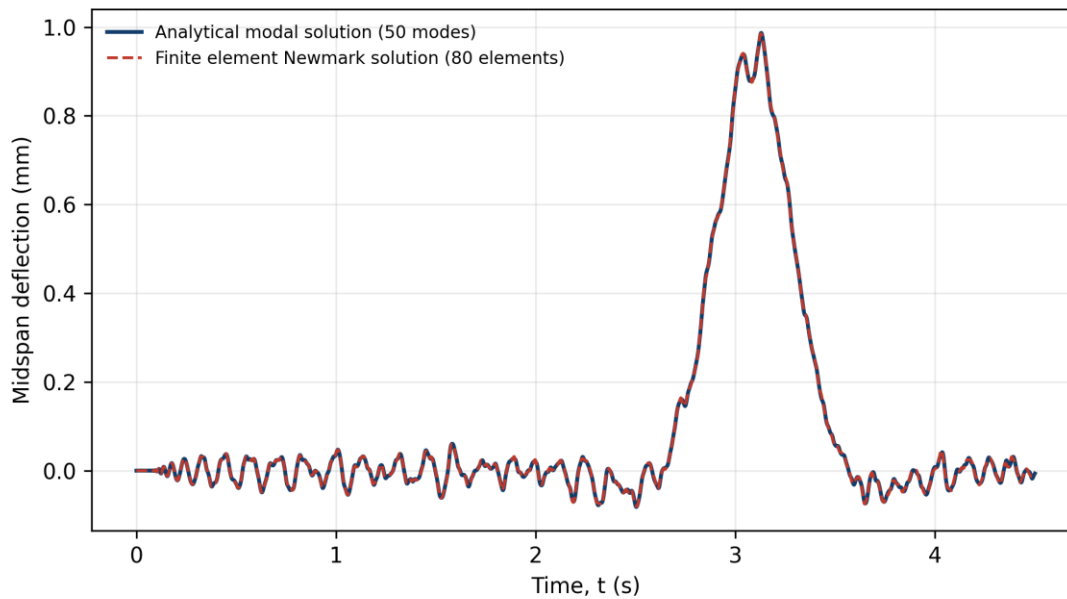


Figure 9. Comparison of the analytical modal and finite-element midspan responses for the validation case.

The two traces are visually coincident over the complete time interval. The peak absolute deflections are 0.98685 mm and 0.98518 mm for the analytical and finite-element solutions, respectively, giving a relative difference of 0.17%. The normalized root-mean-square difference is 0.26%. This close agreement confirms the modal forcing, foundation and prestress terms, boundary-condition implementation, and numerical evaluation of the closed-form solution for the benchmark considered. The small residual difference is attributable to spatial discretization and modal truncation.

DISCUSSION

Effects of the Winkler Foundation and Prestress

Figures 1 and 5 show that increasing Q generally suppresses midspan vibration because the foundation supplies a distributed restoring force proportional to deflection. The reduction is not merely an energy-absorption effect: in the undamped idealization used here, Q raises each modal stiffness and natural frequency, thereby reducing compliance and shifting resonance away from a fixed excitation frequency. The trend agrees with the classical moving-load interpretation of Kenny (1954) and with later beam-foundation studies in the cited literature. Figures 2 and 6 show a comparable reduction when tensile prestress increases. In the present sign convention, tensile N contributes positive geometric stiffness; compressive prestress would have the opposite effect and would require a separate stability check near buckling.

Independent Roles of the Two Loads

Figures 3, 4, 7, and 8 confirm linear amplitude sensitivity to P_1 and P_2 because the governing equation and boundary conditions are linear. The two loads nevertheless need not make identical contributions at a specified observation point. Their spacing creates a phase delay, so their modal forces may reinforce or partially cancel one another as they traverse the span. This is the practical distinction between the present two-load formulation and a single moving load of equal total magnitude. The effect becomes more pronounced as speed, spacing, or excitation frequency changes.

Constant and Harmonic Excitation

For constant moving loads, excitation enters through the translational modal frequency shown below.

Dynamic amplification becomes large when this excitation approaches a structural modal frequency. $\omega^T = k\pi c/L$, $\omega^+ = \Omega + k\pi c/L$, $\omega^- = |\Omega - k\pi c/L|$

Harmonic modulation introduces the forward and backward combination frequencies in addition to the direct harmonic component. These three resonance pathways explain why harmonic histories can differ substantially from constant-load histories even when nominal amplitudes are equal. When a denominator in a closed-form expression tends to zero, its limiting resonant form must be used.

Engineering Interpretation and Scope

The results indicate three direct design controls: a stiffer supporting medium lowers service deflection; tensile prestress can increase effective modal stiffness when stability and connection details permit; and load magnitude, spacing, speed, and harmonic content should be treated jointly rather than independently. These findings are relevant to rail-supported beams, bridge girders, crane beams, and similar members carrying closely spaced moving actions. The present model is deliberately idealized: it neglects damping, shear deformation, rotary inertia, foundation shear interaction, load-structure inertia coupling, and irregular support conditions. Consequently, the numerical values should be used within the Euler-Bernoulli, moving-force, Winkler-foundation assumptions; the verified analytical solution is best viewed as a benchmark for more elaborate models.

CONCLUSION

This study derived closed-form dynamic-response expressions for a simply supported Euler-Bernoulli beam on a Winkler foundation under two simultaneous moving distributed loads of constant and harmonic magnitude. The finite Fourier sine transform, Laplace transform, and convolution theorem reduce the fourth-order initial-boundary-value problem to an explicit modal solution. The formulation exposes the separate contributions of load magnitude, spacing, translational speed, harmonic frequency, foundation stiffness, and tensile prestress.

The results show that increasing the Winkler modulus or tensile prestress reduces the response amplitude by increasing effective modal stiffness, whereas increasing either load magnitude increases deflection in proportion to its modal contribution. Harmonic modulation produces direct and forward/backward combination-frequency resonance pathways that are absent from the constant-load case. An independent 80-element finite-element/Newmark analysis agrees with the analytical midspan response to within 0.17% in peak deflection and 0.26% normalized root-mean-square difference for the benchmark case. From an engineering perspective, the solution provides a verified benchmark for the preliminary design and assessment of highway and railway bridge members, crane girders, and other beam-supported systems subjected to concurrent moving actions. The analysis is limited to Euler-Bernoulli beam theory and therefore neglects shear deformation and rotary inertia. It also adopts an undamped Winkler foundation and treats the moving actions as prescribed forces rather than moving masses. Future work should incorporate viscous and hysteretic damping, Pasternak or viscoelastic foundations, moving-mass inertia, variable speed, and experimental validation.

REFERENCES

1. Adedowole, A. (2019). On the response of non-prismatic rotating Timoshenko beam under the actions of concentrated loads travelling at time-dependent speeds. *Iconic Research and Engineering Journals*, 3(3), 102-114.
2. Adedowole, A., & Jimoh, S. A. (2018). Dynamics analysis of a damped non-uniform beam subjected to loads moving with variable velocity. *Archives of Current Research International*, 13(2), 1-16.
3. Adedowole, A., Ogunyebi, S. N., & Jimoh, S. A. (2023). Rotatory inertia influence on dynamic behavior of non-Winkler Timoshenko beam traversed by pair loads. *Proceedings of the International Conference on Science, Engineering and Business for Sustainable Development Goals (SEB-SDG)*, Omu-Aran, Nigeria, 1-7. doi: 10.1109/SEB-SDG57117.2023.10124565.

4. Ajijola O.O. (2025). Axial force influence on transverse displacement and rotation under moving load of elastically supported damped shear beam. *International Journal of Mathematics Computer Research*, Vol.13 No.4.
5. Awodola, T. O., Awe, B. B., & Jimoh, S. A. (2024). Vibration of non-uniform Bernoulli-Euler beam under moving distributed masses resting on Pasternak elastic foundation subjected to variable magnitude. *African Journal of Mathematics and Statistics Studies*, 7(1), 1-19.
6. Bakhshi Kaniki, H. (2018). Vibration analysis of rotating nanobeam systems using Eringen's two-phase local/nonlocal model. *Physica E: Low-dimensional Systems and Nanostructures*, 92, 310–319.
7. Jafarzadeh Jazi, A., Shahriari, B., & Torabi, K. (2017). Exact closed form solution for the analysis of the transverse vibration mode of a nano-Timoshenko beam with multiple concentrated masses. *International Journal of Mechanical Sciences*, 131–132, 728–743.
8. Jimoh, S. A. (2017). Analysis of non-uniformly prestressed tapered beams with exponentially varying thickness resting on Vlasov foundation under variable harmonic load moving with constant velocity. *International Journal of Advanced Research and Publications*, 1(5), 135-142.
9. Jimoh S.A., Ogunbamike, O. K., Ajijola Olawale Olanipekun (2018). Dynamic response of non-uniformly prestressed thick beam under distributed moving load travelling at varying velocity. *Asian Research Journal of Mathematics*, Vol. 9 No.4.
10. Jimoh S.A., Oni S.T., Ajijola O.O. (2017). Effect of variable axial force on the deflection of thick beams under distributed moving load. *Asian Research Journal of Mathematics*, Vol. 6 No.3.
11. Kenny, J. (1954). Steady state vibrations of a beam on an elastic foundation for a moving load. *Journal of Applied Mechanics*, 76, 359-364.
12. Malesela, K. M., & Sarp, A. (2019). Fundamental frequencies of a torsional cantilever nano beam for dynamic atomic force microscopy (DAFM) in tapping mode. *Microsystem Technologies*, 25, 1087–1098.
13. Omolofe, B., & Adedowole, A. (2019). Transverse response of a structural member with time-dependent boundary conditions to moving distributed mass. *Journal of Applied Nonlinear Dynamics*, 8(2), 167-187.
14. Oni, S. T. (1996). Response of non-uniform beam resting on an elastic foundation to several moving masses. *Abacus Journal of Mathematical Association of Nigeria*, 24(2), 72–88.
15. Oni, S. T., & Adedowole, A. (2008). Influence of prestress on the response to moving loads of rectangular plates incorporating rotatory inertia correction factor. *Journal of the Nigerian Association of Mathematical Physics*, 13, 127-140.
16. Oni, S. T., & Ogunyebi, S. N. (2008). Dynamical analysis of finite prestressed Bernoulli-Euler beams with general boundary conditions under travelling distributed loads. *Journal of the Nigerian Association of Mathematical Physics*, 12, 87-102.
17. Ozkaya, E. (2002). Non-linear transverse vibration of a simply supported beam carrying concentrated masses. *Journal of Sound and Vibration*, 257(3), 413–424.
18. Shu, C., & Du, H. (1996). A generalized approach for implementing general boundary conditions in the GDQ free vibration of plates. *International Journal of Solids and Structures*, 39, 1237-1258.
19. Stanisic, M. M., Hardin, J. C., & Lou, Y. C. (1968). On the response of the plate to a multi-masses moving system. *Acta Mechanica*, 5, 37-53.
20. Yesilce, Y., & Catal, H. H. (2008). Free vibration of piles embedded in soil having different modulus of subgrade reaction. *Applied Mathematical Modelling*, 32, 889–900.
21. Zamani, M., Fallah, A., & Aghdam, M. M. (2012). Free vibration analysis of moderately thick trapezoidal symmetrically laminated plates with various combinations of boundary conditions. *European Journal of Mechanics - A/Solids*, 36, 204-212.