

# Mean Gravitational Potential Energy of Neutrons in Isothermal Neutron-Star Atmospheres: A Statistical-Mechanical Approach

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## ABSTRACT

We develop a statistical-mechanical derivation of the mean gravitational potential energy per particle in an infinitely extended isothermal atmosphere and establish the precise domain in which the result  $\langle U \rangle = kBT$  can be used as a benchmark for compact-star surface layers. The classical model assumes a dilute, non-degenerate, non-interacting gas in plane-parallel geometry, constant temperature, and a locally uniform gravitational acceleration. Direct integration of the Boltzmann distribution and the canonical partition function both yield  $\langle U \rangle = kBT$  and a scale height  $H = kBT/(mg)$ . For a canonical  $1.4 M_{\odot}$ , 12 km neutron star at  $10^6$  K, the Newtonian and local general-relativistic surface gravities are  $1.290e+12$  and  $1.594e+12 \text{ ms}^{-2}$ , respectively, giving free-neutron benchmark scale heights of 6.388 and 5.172 mm. Because  $H/R \approx 4.31e-07$ , the local constant-gravity and plane-parallel approximations are geometrically well motivated, although the absolute surface gravity requires relativistic correction. The Maxwell–Boltzmann validity domain is quantified using the thermal de Broglie wavelength and neutron degeneracy parameter  $n\lambda T^3$ ; at  $10^6$  K the conservative  $n\lambda T^3 = 0.1$  boundary occurs near  $3.2 \times 10^4 \text{ g cm}^{-3}$  for an ideal free-neutron gas. We then separate this dilute benchmark from the strongly degenerate crust and core, introduce relativistic Fermi energies, and formulate the Tolman–Oppenheimer–Volkoff equations required for global stellar structure. Comparison with established neutron-star atmosphere calculations shows that the analytical result is best interpreted as a limiting hydrostatic/statistical-mechanical benchmark, not as a complete photospheric model. The formulation therefore provides a transparent bridge from elementary statistical mechanics to relativistic compact-star physics.

**Keywords:** isothermal atmosphere; gravitational potential energy; neutron star; Boltzmann statistics; Fermi–Dirac statistics; Tolman–Oppenheimer–Volkoff equations; general relativity

## INTRODUCTION

Atmospheric stratification in a gravitational field is one of the simplest settings in which microscopic statistical mechanics produces a macroscopic spatial structure. In a classical isothermal atmosphere, the competition between thermal agitation and gravitational confinement generates an exponential density profile [1]. The same calculation also gives a finite mean gravitational potential energy even when the formal atmosphere extends to infinite height. This result is pedagogically important because it links the Boltzmann factor, the canonical ensemble, hydrostatic equilibrium, and the virial/equipartition viewpoint within a single exactly solvable model.

Applying this model to neutron stars requires care. A neutron star possesses a compactness large enough that general-relativistic corrections to the surface gravity are significant, while its dense crust and core are governed by quantum degeneracy and nuclear interactions rather than Maxwell–Boltzmann statistics [2]. In addition, realistic neutron-star photospheres are generally modeled as ion–electron plasmas whose spectra require frequency-dependent radiative transfer, realistic opacities, hydrostatic equilibrium, and often magnetic-field effects [3]. Consequently, a free-neutron isothermal atmosphere should be interpreted as a statistical-mechanical benchmark rather than as a literal photospheric composition.

The purpose of this work is therefore fourfold: first, to derive the classical result  $\langle U \rangle = kBT$  using both direct Boltzmann averaging and the canonical partition function; second, to state explicitly the assumptions under

which the result is valid; third, to quantify the transition from classical to degenerate matter using  $n\lambda T^3$  and  $EF/kBT$ ; and fourth, to embed the local surface calculation consistently within the relativistic neutron-star framework by distinguishing local surface gravity from the global Tolman–Oppenheimer–Volkoff (TOV) problem [4]. This separation avoids comparing thermal atmospheric energies with Fermi energies drawn from physically different density regimes.

Neutron stars are among the most compact astrophysical objects in the Universe, representing the final evolutionary stage of massive stars following core-collapse supernovae [5]. Possessing masses typically between 1.1 and 2.3 solar masses compressed into radii of approximately 10–14 km, neutron stars exhibit average densities exceeding  $10^{17} \text{ kg m}^{-3}$ , comparable to the density of atomic nuclei [6]. Under these extreme conditions, gravity dominates all other long-range interactions and profoundly influences the thermodynamic and statistical behavior of matter. Consequently, neutron stars provide an exceptional laboratory for studying the interplay between gravitation, quantum mechanics, nuclear physics, and statistical mechanics.

The outermost layers of neutron stars consist of a thin atmosphere whose thickness is typically only a few centimeters to meters because of the enormous surface gravitational acceleration ( $g \sim 10^{12} \text{ m s}^{-2}$ ) [7]. Despite its small thickness, the atmosphere plays a crucial role in determining the thermal spectrum observed by X-ray telescopes. The atmospheric structure governs photon transport, surface cooling, opacity, and the interpretation of observational data used to estimate neutron-star masses and radii [8]. Accurate thermodynamic models of neutron-star atmospheres are therefore essential for constraining the equation of state of ultra-dense matter.

Statistical mechanics provides the theoretical framework for describing systems containing a large number of particles in thermal equilibrium [9]. Within an isothermal atmosphere, the distribution of particles follows the Boltzmann distribution when degeneracy effects are negligible [10]. The probability of finding a particle at a given height depends exponentially on its gravitational potential energy relative to the available thermal energy [11]. This relationship forms the basis for calculating macroscopic quantities such as density profiles, pressure distributions, internal energy, entropy, and gravitational potential energy.

In the present work, we derive the mean gravitational potential energy per neutron in an infinitely high isothermal atmosphere. Beginning with the canonical partition function, we evaluate the configurational integral and demonstrate analytically that the average gravitational potential energy equals one thermal energy unit,  $\langle U \rangle = k_B T$ .

Although this result is well known for classical gases, its extension to neutron-star atmospheres provides valuable physical insight into the competition between gravity and thermal motion under extreme astrophysical conditions.

The classical derivation is subsequently discussed in the context of neutron stars, where several additional physical processes become important. These include relativistic gravity described by the Tolman–Oppenheimer–Volkoff (TOV) equation, neutron degeneracy pressure arising from Fermi–Dirac statistics, strong nuclear interactions, and intense magnetic fields that may exceed  $10^{11} \text{ T}$  in magnetars [12], [13], [14]. Together, these effects modify the atmospheric structure and determine the global equilibrium of neutron stars.

The objectives of this paper are therefore to: derive the mean gravitational potential energy of neutrons in an isothermal atmosphere using statistical mechanics; formulate the partition function for particles in a gravitational field; relate the statistical-mechanical result to neutron-star atmospheres; discuss relativistic corrections applicable to compact stars; and establish a theoretical foundation for future calculations of neutron-star equations of state and atmospheric thermodynamics.

The results presented herein provide a bridge between classical statistical mechanics and modern compact-object astrophysics. They also serve as the basis for more sophisticated models incorporating general relativity, finite-temperature nuclear matter, neutrino transport, and many-body interactions that govern the interiors and atmospheres of neutron stars.

## THEORETICAL FRAMEWORK

### Idealized classical isothermal atmosphere

Consider particles of mass  $m$  above a reference plane  $z = 0$  in a locally uniform gravitational field  $g$  [15]. The model assumes constant  $T$ , negligible interactions, Maxwell–Boltzmann statistics, plane-parallel geometry, and a potential that is linear over the thermally populated layer [16].

$$U(z) = mgz \quad (1)$$

$$n(z) = n_0 \exp(-mgz/kBT) \quad (2)$$

The normalized mean potential energy is obtained from the first moment of the Boltzmann distribution:

$$\langle U \rangle = \frac{\int_0^\infty mgz \exp(-mgz/kBT) dz}{\int_0^\infty \exp(-mgz/kBT) dz} \quad (3)$$

Writing  $a = mg/(kBT)$ , the elementary integrals  $\int_0^\infty e^{-az} dz = 1/a$  and  $\int_0^\infty z e^{-az} dz = 1/a^2$  immediately give

$$\langle U \rangle = kBT \quad (4)$$

The corresponding mean height and scale height are identical:

$$\langle z \rangle = H = \frac{kBT}{mg} \quad (5)$$

Equation (4) is exact only within the stated classical model. It is the mean potential-energy increment associated with vertical stratification relative to  $z = 0$ ; it is not the total gravitational binding energy of a particle to a neutron star [17].

### Canonical-ensemble derivation

For a single classical particle, the Hamiltonian separates into kinetic and gravitational terms:

$$\mathcal{H} = p^2/(2m) + mgz \quad (6)$$

The configurational part of the one-particle partition function is

$$Z_z = \int_0^\infty \exp(-\beta mgz) dz = \frac{1}{\beta mg} \quad (7)$$

with  $\beta = 1/(kBT)$ . The canonical mean potential energy follows from differentiation of  $\ln Z_z$ :

$$\langle U \rangle = -\partial \ln Z_z / \partial \beta = \frac{1}{\beta} = kBT \quad (8)$$

The independent derivations in Eqs. (3)–(8) establish the internal consistency of the classical result.

### Hydrostatic-equilibrium interpretation

For an ideal gas  $P = nkBT$ , plane-parallel hydrostatic balance gives

$$dP/dz = -\rho g \quad (9)$$

and, for  $\rho = mn$ , integration at constant  $T$  recovers the same exponential profile. Thus, the statistical and continuum descriptions are equivalent under the same assumptions.

## Relativistic Neutron-Star Surface Layer

### Newtonian and local GR surface gravity

For a star of gravitational mass  $M$  and circumferential radius  $R$ , the Newtonian surface gravity is

$$g_N = \frac{GM}{R^2} \quad (10)$$

For a static spherical star described externally by the Schwarzschild metric, the local proper surface acceleration measured by a stationary observer is

$$g_S = \frac{GM}{R^2 \sqrt{1 - 2GM/(Rc^2)}} \quad (11)$$

The associated compactness and gravitational redshift are

$$\mathcal{C} = 2GM/(Rc^2), \quad 1 + zg = (1 - \mathcal{C})^{-1/2} \quad (12)$$

For  $M = 1.4 M_\odot$  and  $R = 12 \text{ km}$ ,  $g_N = 1.290e + 12 \text{ m s}^{-2}$ ,  $\mathcal{C} = 0.3446$ ,  $zg = 0.2352$ , and  $g_S = 1.594e + 12 \text{ m s}^{-2}$ . In the realistic surface-layer section of this paper,  $g_S$  is the adopted gravitational acceleration;  $g_N$  is retained only as a labeled comparison.

### Thin-layer criterion and local linearization

At  $T = 10^6 \text{ K}$  the free-neutron benchmark gives  $HN = 6.388 \text{ mm}$  and  $HGR = 5.172 \text{ mm}$ . The ratio  $HGR/R = 4.310e - 07$  is extremely small. Hence the atmosphere is geometrically thin and the local potential may be linearized over a few scale heights even though the global stellar spacetime is relativistic [18].

$$\Delta U \simeq mgsz \quad (13)$$

Using the local acceleration in the same Boltzmann integral leaves the normalized mean energy unchanged:

$$\langle \Delta U \rangle = kBT \quad (14)$$

At  $10^6 \text{ K}$  this energy is  $kBT = 86.17 \text{ eV}$ . The invariance of Eq. (14) with respect to the numerical value of  $g$  follows because increasing  $g$  reduces the scale height by the same factor by which it increases the potential-energy gradient.

## Quantitative Validity of Maxwell–Boltzmann Statistics

### Thermal wavelength and degeneracy parameter

The classical approximation requires low phase-space occupation. For an ideal neutron gas, define the thermal de Broglie wavelength [19]

$$\lambda_T = h/\sqrt{2\pi mnkBT} \quad (15)$$

and the degeneracy parameter

$$\eta d = n\lambda_T^3 = \frac{\rho\lambda_T^3}{mn} \quad (16)$$

A conservative practical classical criterion is  $\eta d < 0.1$ ;  $\eta d \gtrsim 1$  marks the onset of significant quantum degeneracy. Because  $\lambda_T^3 \propto T^{-3/2}$ , the corresponding density boundary scales as  $T^{3/2}$ .

$$\rho_{MB}(T; \eta d) = \frac{\eta d mn}{\lambda_T^3} \propto T^{3/2} \quad (17)$$

For the ideal free-neutron benchmark,  $\eta d = 0.1$  occurs at approximately  $1.01 \times 10^3$ ,  $3.19 \times 10^4$ , and  $1.01 \times 10^6 \text{ g cm}^{-3}$  at  $10^5$ ,  $10^6$ , and  $10^7 \text{ K}$ , respectively. These values quantify the statistical validity of the ideal neutron gas; they are not a substitute for composition-dependent ionization, Coulomb-coupling, and opacity criteria in a real photosphere.

### Fermi energy and thermal-to-degenerate transition

For a zero-temperature ideal neutron gas, the Fermi momentum is

$$p_F = \hbar(3\pi^2 n)^{1/3} \quad (18)$$

The relativistic kinetic Fermi energy is

$$E_F = \sqrt{(p_F^2 c^2 + m_n^2 c^4)} - m_n c^2 \quad (19)$$

and the Fermi temperature is

$$T_F = E_F / k_B \quad (20)$$

A useful thermal-degeneracy ratio is  $\theta = k_B T / E_F = T / T_F$ . Classical behavior requires low occupation and  $\theta$  not much smaller than unity, whereas  $\theta \ll 1$  denotes strong degeneracy.

$$\theta = \frac{k_B T}{E_F} = T / T_F \quad (21)$$

This formulation prevents the misleading juxtaposition of an atmospheric value  $k_B T \approx 86 \text{ eV}$  with a representative interior  $E_F \approx 60 \text{ MeV}$ . The latter corresponds to matter at very high density and implies  $T_F \approx 7 \times 10^{11} \text{ K}$ , so at  $10^6 \text{ K}$  such matter is deeply degenerate and belongs to the crust/core regime rather than the dilute atmospheric benchmark.

### General-Relativistic Stellar Interior

#### Tolman–Oppenheimer–Volkoff equations

The plane-parallel atmosphere does not determine the global stellar mass or radius. For a static, spherical, non-rotating perfect-fluid star, the Einstein equations reduce to the TOV system [20]. With  $\epsilon$  denoting total energy density and  $m(r)$  the enclosed gravitational mass,

$$dm/dr = 4\pi r^2 \epsilon / c^2 \quad (22)$$

$$dP/dr = - \frac{G(\epsilon + P)(m + 4\pi r^3 P / c^2)}{c^2 r^2 (1 - 2Gm / (rc^2))} \quad (23)$$

The system is closed only after specifying an equation of state

$$P = P(\epsilon, n_B, T, Y_i, \dots) \quad (24)$$

with boundary conditions  $m(0) = 0$  and  $P(0) = P_c$ . Integration proceeds outward until  $P(R) = 0$ ; the gravitational mass is then  $M = m(R)$ . A sequence of central pressures generates the mass–radius relation.

$$m(0) = 0, \quad P(0) = P_c, \quad P(R) = 0, \quad M = m(R) \quad (25)$$

### Numerical implications and separation of scales

For the adopted M–R pair, compactness and surface redshift are fixed kinematically by Eq. (12), but they do not constitute a TOV solution unless an EOS and central pressure are supplied. This distinction is essential: the local atmosphere calculation uses the already-specified stellar M and R as boundary data, whereas the TOV problem

predicts  $M$  and  $R$  from an interior EOS. Dense-matter calculations must also include nuclear interactions, composition, beta equilibrium, and causality/stability constraints; the ideal Fermi gas is used here only to diagnose degeneracy.

### Relation to Established Neutron-Star Atmosphere Models

Modern neutron-star atmosphere calculations commonly exploit the same geometrical simplification identified here—an atmosphere that is thin compared with the stellar radius—while solving substantially richer physics. Plane-parallel hydrostatic equilibrium is coupled to frequency-dependent radiative transfer, realistic absorption and scattering opacities, ionization balance, and radiative equilibrium. Magnetic atmosphere models additionally require polarization-dependent transport and magnetic modifications of the equation of state and opacity [21].

Accordingly, Eq. (14) is meaningful as a benchmark for the purely gravitational/statistical-mechanical component of stratification. It can test whether a numerical atmosphere code reproduces the expected exponential density law in the dilute, isothermal, ideal-gas, constant-opacity-free limit. It cannot by itself predict emergent X-ray spectra, color corrections, photospheric composition, magnetic polarization, or the detailed thermal structure. This interpretation is consistent with established neutron-star atmosphere literature, where plane-parallel geometry is retained but the radiative-transfer problem supplies the observable spectrum [22].

The benchmark is therefore most useful in code verification and pedagogy: a numerical hydrostatic module initialized in the classical limit should recover  $H = kBT/(mgs)$ , while departures introduced by temperature gradients, composition, radiation pressure, degeneracy, or magnetic fields can then be quantified relative to this exact limiting solution.

## RESULTS AND DISCUSSION

The central analytical result is independent of the absolute magnitude of the locally constant gravitational acceleration:  $\langle U \rangle = kBT$ . Gravity controls the vertical length scale  $H$ , not the normalized mean gravitational energy. This property follows from the linearity of  $U(z)$  and the exponential Boltzmann weight. The result is therefore robust within its model assumptions but should not be extrapolated beyond them.

For the canonical neutron-star parameters, the GR correction increases the local surface gravity from  $1.290e + 12$  to  $1.594e + 12 \text{ m s}^{-2}$ , a factor of 1.235, and reduces the free-neutron benchmark scale height from 6.388 to 5.172 mm. The very small  $H/R$  ratio quantitatively supports plane-parallel geometry and local constant gravity, while the approximately 24% gravity correction demonstrates why Newtonian and relativistic surface gravities should not be interchanged without declaration.

The degeneracy analysis supplies the missing density–temperature boundary. At fixed temperature  $\eta d$  increases linearly with density, while the boundary density rises as  $T^{3/2}$ . Thus, an atmosphere can be geometrically thin yet statistically classical, whereas deeper material can become degenerate without any failure of the thin-layer approximation. Geometrical thinness and quantum degeneracy are therefore independent validity tests and must be assessed separately.

The Fermi-energy analysis also resolves the apparent 86 eV versus 60 MeV inconsistency. These energies characterize different physical regions:  $kBT$  is a thermal/local-stratification scale at  $10^6 \text{ K}$ , whereas a tens-of-MeV Fermi energy is a dense-matter scale. Once density is treated as an explicit variable, the transition between these regimes is continuous and quantitatively defined rather than represented by unrelated sample numbers.

### Graphical results

Figures 1–6 are presented together in the Results and Discussion section. They quantify the classical barometric profile, the Newtonian-to-general-relativistic correction to the surface scale height, the compactness dependence of the relativistic gravity factor, the density–temperature boundary of Maxwell–Boltzmann statistics, the growth of the neutron degeneracy parameter with density, and the corresponding Fermi-energy scale.

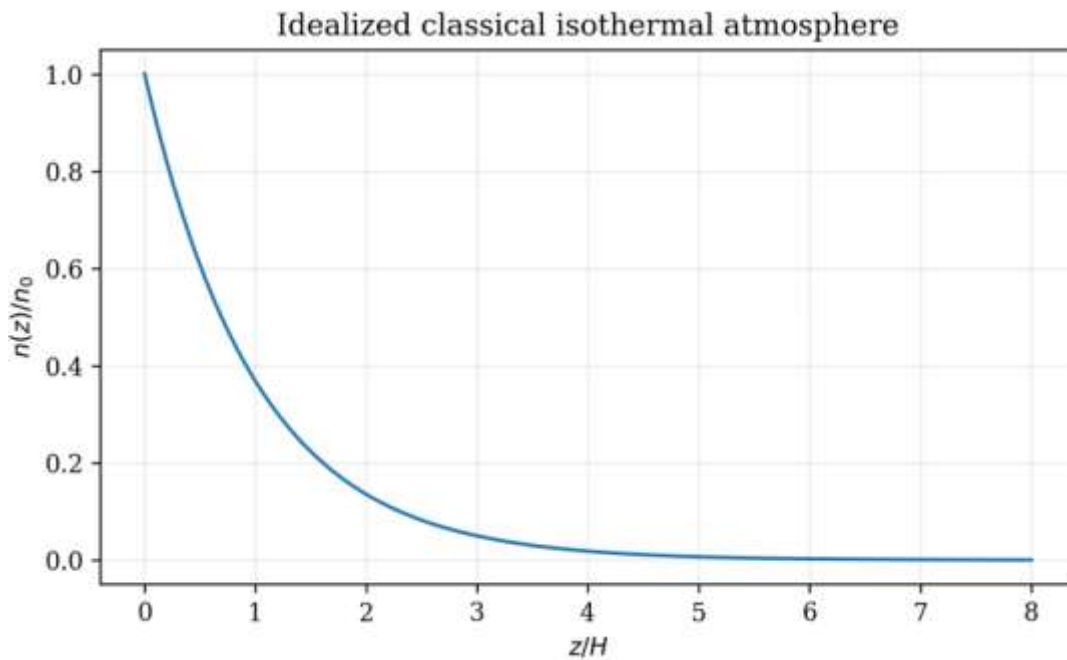


Fig. 1 Normalized classical Boltzmann density profile. The exponential tail gives a finite first moment even for a formally infinite atmosphere.

As shown in Fig. 1, the normalized number density decreases exponentially with height and retains a finite first moment despite the formally infinite vertical domain. The universal collapse of the profile when height is measured in units of the scale height confirms that the classical result is controlled by the competition between thermal agitation and gravitational confinement. This behavior provides the graphical basis for the finite mean height and, consequently, for the result given in Eq. (4).

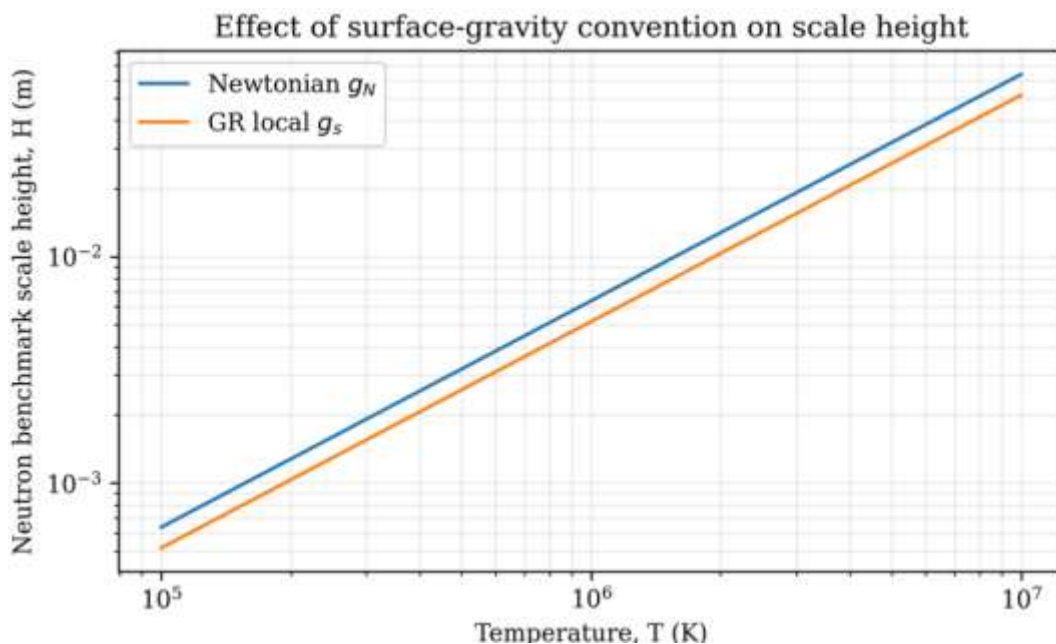


Fig. 2 Newtonian and local general-relativistic scale heights for the canonical neutron-star model. The gravity convention changes the numerical scale height and must be stated explicitly.

Figure 2 demonstrates the quantitative effect of the gravitational prescription on the atmospheric scale height. For the adopted canonical neutron-star parameters, use of the local general-relativistic surface gravity reduces the free-neutron benchmark scale height relative to the Newtonian value. The comparison confirms that the absolute vertical length scale is sensitive to strong-field gravity even though the normalized mean potential energy derived in Eq. (14) is unchanged in the locally linear potential approximation.

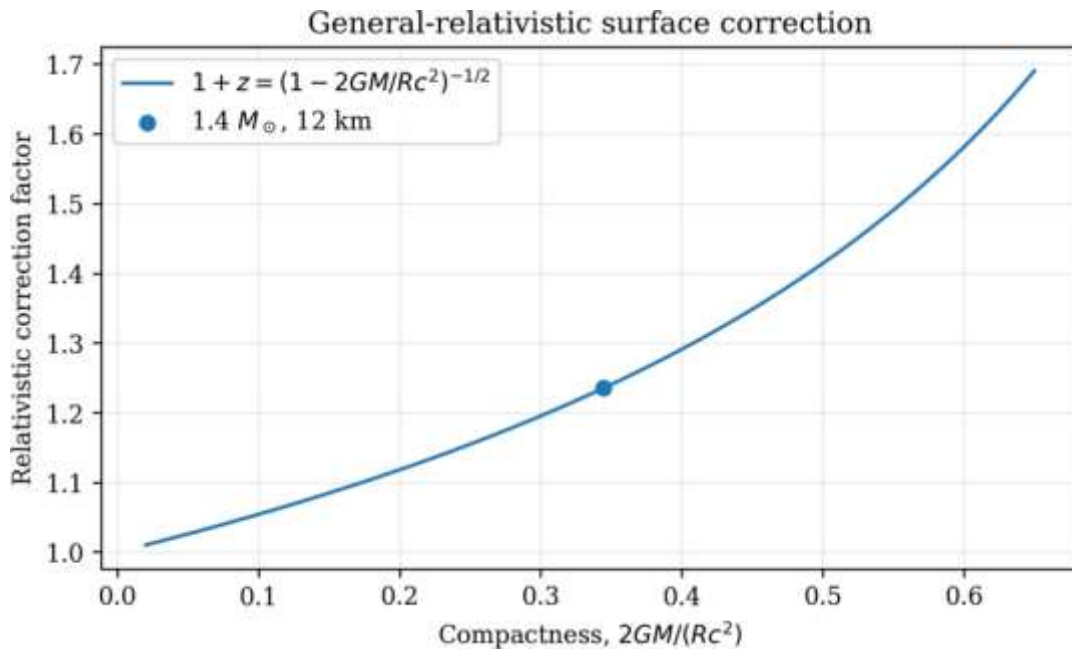


Fig. 3 Relativistic surface-gravity correction factor as a function of compactness. Strong-field corrections become appreciable for canonical neutron-star compactness.

The relativistic enhancement of the local surface gravity is displayed in Fig. 3 as a function of stellar compactness. The correction increases monotonically with compactness, showing that Newtonian gravity becomes progressively less adequate for compact neutron stars. For the canonical model used in this study, the relativistic correction is already substantial, thereby justifying the explicit use of the local proper surface gravity in the neutron-star surface-layer calculation.

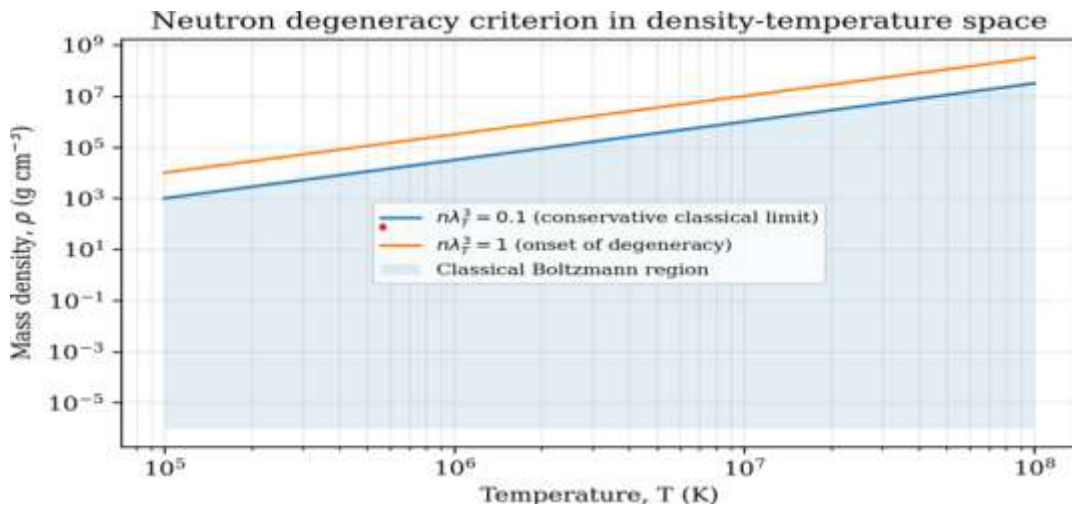


Fig. 4 Density–temperature map of the ideal-neutron degeneracy parameter. The classical Maxwell–Boltzmann region lies below the conservative  $\eta d = 0.1$  boundary.

The density-temperature validity domain of the classical approximation is summarized in Fig. 4. The plotted degeneracy boundaries show that the maximum density compatible with Maxwell-Boltzmann statistics increases with temperature. The conservative degeneracy-parameter threshold therefore provides a quantitative criterion for deciding when the classical atmosphere derivation can be used and when Fermi-Dirac corrections must be introduced, rather than relying on a qualitative statement of non-degeneracy alone.

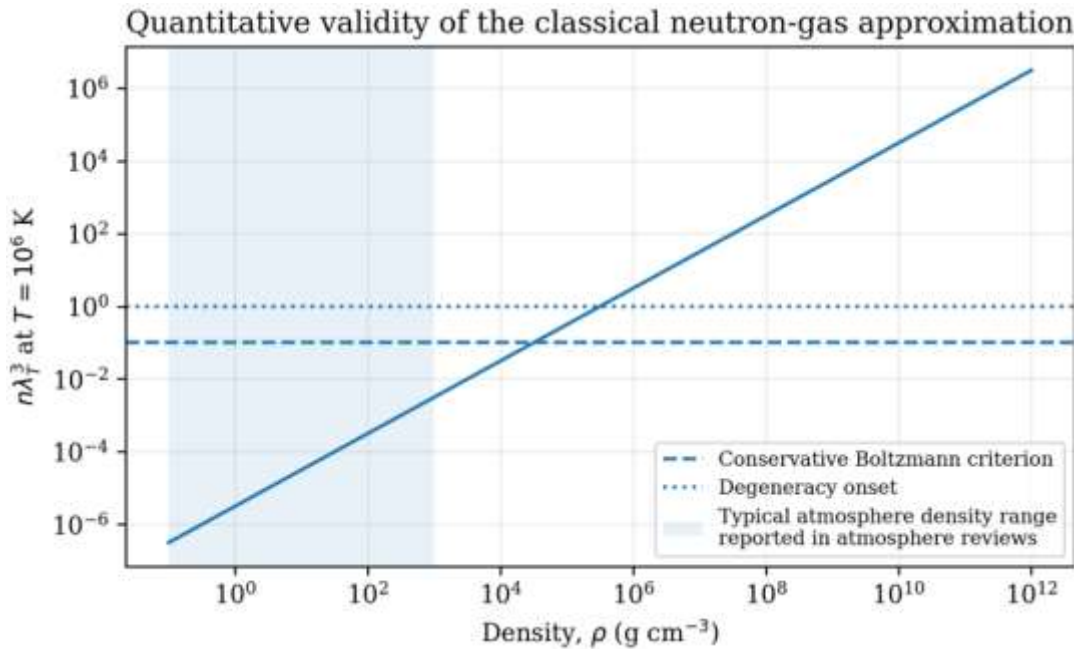


Fig. 5 Neutron degeneracy parameter versus density at  $T = 10^6$  K, illustrating the continuous transition from dilute classical matter toward quantum degeneracy.

As illustrated in Fig. 5, the neutron degeneracy parameter increases continuously with density at fixed temperature. The figure therefore makes explicit that the transition from dilute classical matter to quantum-degenerate matter is gradual and density dependent. This continuous behavior separates the atmospheric benchmark from deeper neutron-star layers without introducing an artificial discontinuity between two representative energy values taken from different physical regions.

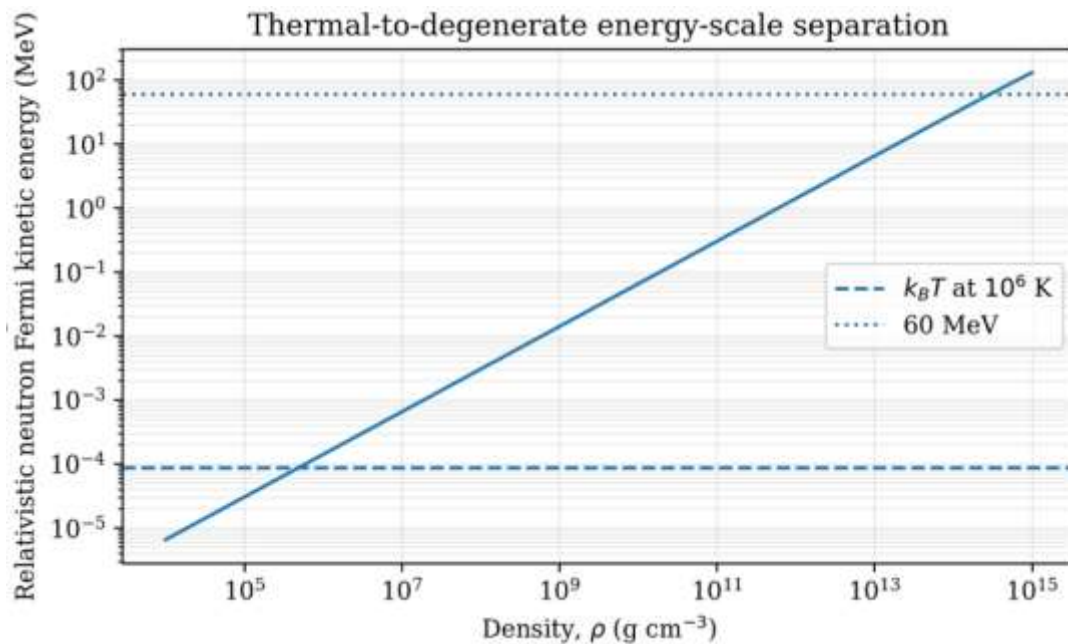


Fig. 6 Relativistic neutron Fermi kinetic energy versus density. The  $10^6$  K thermal scale is separated by many orders of magnitude from tens-of-MeV Fermi energies characteristic of dense neutron-star matter.

Figure 6 shows the rapid increase of the relativistic neutron Fermi energy with density and provides the clearest quantitative distinction between the dilute thermal atmosphere and strongly degenerate matter. At the reference temperature, the thermal energy scale is many orders of magnitude smaller than the tens-of-MeV Fermi energies reached in dense neutron-star material. Thus, the atmospheric thermal energy and the interior Fermi energy characterize physically distinct regimes and should not be interpreted as competing energy scales of the same particle population.

## Integrated discussion

Taken together, Figs. 1–6 show that geometrical thinness, relativistic gravity, and quantum degeneracy are separate validity conditions. The scale-height results justify a local plane-parallel treatment; the relativistic correction identifies the appropriate surface acceleration; and the degeneracy diagnostics establish the density–temperature domain in which the Maxwell–Boltzmann result  $\langle U \rangle = kBT$  is applicable. At higher density the Fermi energy becomes the relevant microscopic scale, while global stellar structure requires the TOV equations coupled to a physically consistent equation of state.

## Summary of Assumptions and Applicability

| Regime                         | Primary equations                                                | Valid when                                                 | Principal limitation                                                       |
|--------------------------------|------------------------------------------------------------------|------------------------------------------------------------|----------------------------------------------------------------------------|
| Classical isothermal benchmark | Boltzmann profile;<br>$H = kBT/(mg)$ ; $\langle U \rangle = kBT$ | $\eta d \ll 1$ ; isothermal; ideal gas; local constant $g$ | No radiation transport, interactions, degeneracy, or realistic composition |
| Relativistic surface layer     | $g_s = GM/[R^2\sqrt{(1 - 2GM/Rc^2)}]$                            | $H/R \ll 1$ ; static spherical background                  | $M$ and $R$ are inputs, not predicted                                      |
| Degeneracy transition          | $n\lambda T^3$ ; $pF$ ; $EF$ ; $TF$                              | Ideal-gas diagnostic of quantum occupation                 | Nuclear/Coulomb interactions omitted                                       |
| Stellar interior               | $TOV + EOS$                                                      | Static, spherical, non-rotating perfect fluid              | Requires realistic EOS; rotation and magnetic stresses omitted             |

## CONCLUSIONS

The mean gravitational potential energy per particle in an infinitely high classical isothermal atmosphere is exactly  $kBT$  when the potential is locally linear and the gas obeys Maxwell–Boltzmann statistics. For neutron-star applications, this result should be used as a local benchmark rather than a literal description of the photosphere or interior. General relativity changes the local surface gravity and therefore the scale height, but the normalized mean energy remains  $kBT$  in the thin-layer linear-potential limit.

A quantitative degeneracy analysis based on  $n\lambda T^3$  and  $EF/kBT$  cleanly separates the classical atmosphere from dense quantum matter. The TOV equations then provide the appropriate global framework for the stellar interior once a physically consistent EOS is specified. This hierarchy—classical benchmark, relativistic surface layer, degeneracy transition, and TOV interior—removes the conceptual inconsistency that arises when atmospheric thermal energies and dense-matter Fermi energies are treated as though they refer to the same region.

Future work should replace the free-neutron benchmark by composition-dependent H/He/C or heavy-element atmosphere models, solve radiative transfer and hydrostatic equilibrium simultaneously, and couple the surface boundary condition to modern finite-temperature nuclear equations of state. Magnetic-field and rotational corrections can then be incorporated for applications to strongly magnetized or rapidly rotating neutron stars.

## Declarations

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**Conflict of interest:** The author declares no conflict of interest.

**Data availability:** All numerical values and figures in this analytical study are generated from the equations and physical constants stated in the manuscript; no proprietary dataset is required.

**Code availability:** Numerical plots were generated from direct evaluation of the equations presented in the manuscript. Code can be made available by the author on reasonable request.

Author contributions: J.K.M. conceived the study, developed the analytical formulation, performed the numerical calculations, interpreted the results, and prepared the manuscript.

Use of AI-assisted tools: AI-assisted tools were used for language refinement, equation-formatting support, and document preparation. The author reviewed and takes responsibility for the scientific content, calculations, interpretations, and final manuscript.

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