

Existence of Weak Solutions to a Class of Nonlinear Elliptic Boundary Value Problems

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ABSTRACT

The existence of weak solutions to a class of elliptic boundary value problems is investigated. *A priori* estimates are obtained by considering the auxiliary linear problem which is an approximation to the given problem. The existence and uniqueness of the weak solution is established by Banach's fixed point theorem. An illustrative example is given to demonstrate applicability

Keywords: Elliptic boundary value problems, A priori estimates, Weak solutions, Schaefer's fixed point theorem

MSC: 35B45 35J66 35D30.

INTRODUCTION

We consider a class of nonlinear elliptic boundary value problems

$$-\alpha \Delta u = f(u) + g(x) \quad \text{in } \Omega \quad (1)$$

$$u = 0 \quad \text{on } \partial\Omega \quad (2)$$

where $\alpha > 0$ is a constant, Ω is open and bounded subset in $\mathbb{R}^n$, $g \in L^2(\Omega)$ and $f: \mathbb{R} \rightarrow \mathbb{R}$ a differentiable function satisfying $0 \neq |f'| \leq \sigma = \text{constant}$. Density of some physical quantities is denoted by an unknown quantity u (e.g. temperature) in equilibrium within the region Ω [3]. Examples of the fields where the governing equations are described by singular Boundary Value Problems (BVP) are gas dynamics, boundary layer theory, combustion, nonlinear optics and so on. One can further check for more examples in [1-5]. and the literature cited in them. Sanni [5] considered a non-linear elliptic boundary value problem with the convective term $\mu \frac{x}{|x|^k}$.

∇u . The current work is motivated by seeking for existence of solution for a linear elliptic equation (1) and the space where the solutions belong. Thus, we prove the existence and uniqueness of the solutions to (1) – (2) in Sobolev space using the techniques found in [6-9].

The arrangement of this paper is done as follows: In section 2, some function spaces used in the work are defined and proof of a useful embedding lemma done. In section 3, Lax-Milgram Theorem is applied to prove the existence and uniqueness of solutions of the auxiliary problem. Section 4 is the main result where the existence and uniqueness of the given problem using Banach's fixed point theorem is established. This is followed by an illustrative example to enhance clarity.

Preliminaries

The following Sobolev's spaces are used in this paper;

Let $L^2(\Omega) := \{w: \Omega \rightarrow \mathbb{R} \mid |w|_{L^2(\Omega)} < \infty\}$

with the norm

$$|w|_{L^2(\Omega)} := \sqrt{\int_{\Omega} w^2 dx}$$

$$\text{Let } H^1(\Omega) := \{w: \Omega \rightarrow \mathbb{R} \mid |w|_{H^1(\Omega)} < \infty\}$$

with the norm

$$|w|_{H^1(\Omega)} := \sqrt{\int_{\Omega} w^2 dx + \int_{\Omega} |\nabla w|^2 dx}$$

$$\text{Let } H_0^1(\Omega) := \text{The closure of the } C_c^\infty(\Omega) \text{ in } H^1(\Omega)$$

with the norm

$$|w|_{H_0^1(\Omega)} := \sqrt{\int_{\Omega} |\nabla w|^2 dx}$$

$$\text{Thus we define } X := H_0^1(\Omega)$$

Definition 2.1. A solution $u \in X$ is called a weak solution of the boundary value problem (1) – (2) provided

$$-\alpha \int_{\Omega} v \Delta u dx = \int_{\Omega} f(u) v dx + \int_{\Omega} g(x) v dx \text{ for } u, v \in X.$$

Applying integration by parts in (3) and using the boundary condition of (1) – (2), we have

$$\alpha \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f(u) v dx + \int_{\Omega} g(x) v dx,$$

where α is a strictly positive constant, Ω is an open and bounded subset in $\mathbb{R}^n$ and $u \in H_0^1(\Omega)$.

Auxiliary Linear Problem

Consider the linear boundary value problem

$$Lu: -\alpha \Delta u = f(s) + g(x) \text{ in } \Omega \in X \tag{4}$$

$$u = 0 \text{ on } \partial\Omega \tag{5}$$

where $s \in X$, $g \in L^2(\Omega)$ and $f(s)$ is a function of s only.

Theorem 3.1. (Apriori Estimate) Let u be a solution of (4) – (5) and $u \in H_0^1(\Omega)$. Then we have the estimate

$$|u|_X^2 \leq C \left[|s|_{L^2(\Omega)}^2 + |g|_{L^2(\Omega)}^2 + 1 \right] \tag{6}$$

Proof. The proof is done in two steps.

Step 1: multiply (4) by u , integrate by parts and apply (5) to obtain

$$\alpha \int_{\Omega} -u \Delta u dx = \int_{\Omega} u f(s) dx + \int_{\Omega} u g(x) dx$$

$$\Rightarrow \int_{\Omega} |\nabla u|^2 dx = \frac{1}{\alpha} \left[\int_{\Omega} u f(s) dx + \int_{\Omega} u g(x) dx \right]. \tag{7}$$

Applying Holder's inequality in (7), we have

$$\int_{\Omega} |\nabla u|^2 dx = \frac{1}{\alpha} \left[\int_{\Omega} u f(s) dx + \int_{\Omega} u g(x) dx \right] \leq \frac{1}{\alpha} \left[\left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} f^2(s) dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} g^2(x) dx \right)^{\frac{1}{2}} \right].$$

Applying Cauchy's inequality with ε on the terms on the right hand side, we obtain

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx &= \frac{1}{\alpha} \left[\int_{\Omega} u f(s) dx + \int_{\Omega} u g(x) dx \right] \leq \frac{1}{\alpha} \left[\left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} f^2(s) dx \right)^{\frac{1}{2}} + \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} g^2(x) dx \right)^{\frac{1}{2}} \right] \\ &\leq C(\alpha, \Omega) \left[\varepsilon \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\varepsilon} \int_{\Omega} f^2(s) dx + \varepsilon \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\varepsilon} \int_{\Omega} g^2(x) dx \right] \\ &\leq C(\alpha, \Omega) \left[\left(\varepsilon \int_{\Omega} |\nabla u|^2 dx \right) + \left\{ \frac{1}{4\varepsilon} \left(\int_{\Omega} f^2(s) dx + \int_{\Omega} g^2(x) dx \right) \right\} \right] \\ &\leq C(\alpha, \Omega) \left[\left(\int_{\Omega} |\nabla u|^2 dx \right) \left(\int_{\Omega} f^2(s) dx + \int_{\Omega} g^2(x) dx \right) \right] \end{aligned}$$

$$\Rightarrow \int_{\Omega} |\nabla u|^2 dx \leq C \left[\left(\int_{\Omega} |\nabla u|^2 dx \right) \left(\int_{\Omega} f^2(s) dx + \int_{\Omega} g^2(x) dx \right) \right], \quad (8)$$

For some $C = C(\alpha, \Omega)$. Applying Young's inequality in (8) we obtain

$$\begin{aligned} \int_{\Omega} |\nabla u|^2 dx &\leq \frac{C}{2} \left(\int_{\Omega} |\nabla u|^2 dx \right) + \frac{C}{2} \left[\int_{\Omega} f^2(s) dx + \int_{\Omega} g^2(x) dx \right] \\ \Rightarrow \frac{1}{2} \left(\int_{\Omega} |\nabla u|^2 dx \right) &\leq \frac{C}{2} \left[\int_{\Omega} f^2(s) dx + \int_{\Omega} g^2(x) dx \right] \end{aligned} \quad (9)$$

Simplifying (9), we have

$$\int_{\Omega} |\nabla u|^2 dx \leq C \left[\int_{\Omega} f^2(s) dx + \int_{\Omega} g^2(x) dx \right] \quad (10)$$

$$\Rightarrow |u|_X^2 \leq C \int_{\Omega} f^2(s) dx + |g|_{L^2(\Omega)}^2 \quad (11)$$

for some constant $C = C(\alpha, \Omega, C_1)$.

step 2: using $0 \neq |f'| \leq \sigma = \text{constant.}$, we estimate

$$\begin{aligned} \int_{\Omega} f^2(s) dx &= \int_0^s (f'(\varepsilon) d(\varepsilon) + f(0)^2) dx \\ &\leq \int_{\Omega} (\sigma |s| + |f(0)|)^2 dx \\ &\leq 2\sigma \int_{\Omega} \sigma |s|^2 dx + |f(0)|^2 dx \\ &\leq C(|s|_{L^2(\Omega)}^2 + 1) \end{aligned} \quad (12)$$

for some constant $C = C(\sigma, |f(0)|, |\Omega|)$. Putting (12) in (11) yields

$$|u|_X^2 \leq C(|s|_{L^2(\Omega)}^2 + |g|_{L^2(\Omega)}^2 + 1) \quad (13)$$

which deduces (6), completing the proof.

Definition 3.1.

(i) The bilinear form $B[.,.]$ associated with elliptic operator L is

$$B[u, v] := \alpha \int_{\Omega} \nabla u \cdot \nabla v dx + \mu \int_{\Omega} uv dx \quad (14)$$

for $u, v \in X$,

(ii) We call $u \in X$ a weak solution of the boundary value problem (1) – (2) provided

$$B[u, v] := (-\mu u + f(s) + g, v)$$

for all $v \in X$, where $(.,.)$ denotes the inner product in $L^2(\Omega)$.

Theorem 3.2. The bilinear form $B[u, v]$ satisfies the hypothesis of Lax-Milgram theorem. That is, there exists constants $\beta, \varrho > 0$ such that

$$(i) |B[u, v]| \leq \beta |u|_X |v|_X$$

$$(ii) \varrho |u|_X^2 \leq B[u, u]$$

for all $u, v \in X$.

Proof. (i)

$$\begin{aligned} |B[u, v]| &:= \left| \alpha \int_{\Omega} \nabla u \cdot \nabla v dx + \mu \int_{\Omega} uv dx \right| \\ &\leq \alpha \int_{\Omega} |\nabla u \cdot \nabla v| dx + \mu \int_{\Omega} |uv| dx \end{aligned}$$

applying Holder's inequality, one obtain

$$\begin{aligned} |B[u, v]| &:= \left| \alpha \int_{\Omega} \nabla u \cdot \nabla v dx + \mu \int_{\Omega} uv dx \right| \\ &\leq \alpha \int_{\Omega} |\nabla u \cdot \nabla v| dx + \mu \int_{\Omega} |uv| dx \\ &\leq \alpha \left(\int_{\Omega} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}} + \mu \left(\int_{\Omega} |u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v|^2 dx \right)^{\frac{1}{2}} \\ &\leq \alpha (|u|_X |v|_X) + c(\mu, \Omega) (|u|_X |v|_X) \\ &\leq (\alpha + c(\mu, \Omega)) (|u|_X |v|_X) \\ &\leq \beta (|u|_X |v|_X) \end{aligned}$$

for some constant $\beta = \beta(\alpha, \mu, \Omega) > 0$

$$(ii) B[u, u] = \int_{\Omega} |\nabla u|^2 dx + \mu \int_{\Omega} |u|^2 dx$$

$$\geq \int_{\Omega} |\nabla u|^2 dx = \varrho |u|_X^2$$

for some constant $0 \leq \varrho \leq 1$.

Theorem. There exists a unique weak solution $u \in X$ to the linear boundary value problem (1) – (2).

Proof. By the estimate (13) and the hypothesis on g , we have that $f(s) + g \in L^2(\Omega)$ for fixed $f(s) + g \in L^2(\Omega)$, set $\langle -\mu u + f(s) + g, v \rangle = (-\mu u + f(s) + g, v)_{L^2(\Omega)}$ for all $v \in X$; where $\langle \dots \rangle$ is the pairing of X and its dual X^* and $(\dots)$ is the inner product on $L^2(\Omega)$. This is a bounded linear functional on $L^2(\Omega)$, and thus on X . By applying Lax-Milgram theorem, we obtain a unique function $u \in X$ satisfying

$$B[u, v] = \langle -\mu u + f(s) + g, v \rangle$$

for all $v \in X$, consequently u is the unique weak solution of (1) – (2).

Main Result

Theorem 4.1. There exists a unique weak solution $u \in X$ to the boundary value problem (1) – (2).

Proof. We split the proof in seven steps.

Step 1:

Consider the fixed point argument to (1) – (2) given as

$$-\alpha \Delta w = f(u) + g(x) \text{ in } \Omega \subset \mathbb{R}^n \quad (15)$$

$$w = 0 \text{ on } \partial\Omega \quad (16)$$

Define a mapping

$$A: X \mapsto X$$

by setting $A[u] = w$, whenever w is derived from u using (15) – (16). We claim that A is a continuous and compact mapping.

Step 2

Defining $A[u] = w$, $A[\bar{u}] = \bar{w}$,

and choosing $u, \bar{u} \in X$. Thus for two solutions $w, \bar{w} \in X$ of (15) – (16), we have

$$-\alpha \Delta(w - \bar{w}) = f(u) - f(\bar{u}) \quad (17)$$

$$(w - \bar{w}) = 0 \text{ on } \partial\Omega \quad (18)$$

Notice that equation (17) – (18) is analogous to (15) – (16). Thus we obtain an analogous estimate to (13). This implies that

$$|w - \bar{w}|_X^2 \leq C [\int_{\Omega} (f(u) - f(\bar{u}))^2 dx]. \quad (19)$$

But

$$\begin{aligned} \int_{\Omega} f^2(u) dx &= \int_{\Omega} \sigma^2 (1 + |u|)^2 dx \leq \int_{\Omega} 2\sigma^2 (1 + |u|^2) dx \leq \\ &\leq \sigma^2 \int_{\Omega} (1 + |u|^2) dx; \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega} (f(u) - f(\bar{u}))^2 dx &\leq \int_{\Omega} (\sigma(1 + |u|) - \sigma(1 + |\bar{u}|))^2 dx \\ &\leq \sigma^2 \int_{\Omega} |u - \bar{u}|^2 dx \end{aligned} \quad (20)$$

Using (19) in (20), we deduce that

$$|A[u] - A[\bar{u}]|_X^2 = |w - \bar{w}|_X^2 \leq C|u - \bar{u}|_X \quad (21)$$

for some constant $C = C(\sigma, \alpha, n, |f(0)|)$.

Thus the mapping A is Lipschitz continuous, hence continuous.

Step 3

Let $\{u_m\}_{m=1}^{\infty}$ be a bounded sequence in X . By Balzano Weierstrass, it has a convergent subsequence, say $\{u_{m_j}\}_{j=1}^{\infty}$. Define

$$u = \lim_{m_j \rightarrow \infty} u_{m_j} \quad (22)$$

using (22) in (21), we have

$$\lim_{m_j \rightarrow \infty} |A[u_{m_j}] - A[u]|_X \leq \lim_{m_j \rightarrow \infty} C|u_{m_j} - u|_X = 0$$

Thus, $A[u_{m_j}] \rightarrow A[u]$ in X . Therefore, A is compact.

Step 4

Let $u_n = u|_{\partial\Omega} = 0$ By induction, define $w = u_{m+1} \in X$ ($m = 0, 1, 2, \dots$) as a unique weak solution of

$$-\alpha \Delta u_{m+1} = f(u_m) + g(x) \quad \text{in } \Omega \subset \mathbb{R}^n \quad (23)$$

$$u_{m+1} = 0 \quad \text{on } \partial\Omega \quad (24)$$

By the definition of the mapping A , we have

$$A[u_m] = u_{m+1}$$

since A has a fixed point in X , there exist $u \in X$ such that

$$\lim_{m \rightarrow \infty} u_{m+1} = \lim_{m \rightarrow \infty} A[u_m] = A[u] = u \quad (25)$$

Step 5

Using (12), we deduce that

$$|f(u_m)|_{L^2(\Omega)} \leq C(1 + |u_m|_{L^2(\Omega)})$$

for some constant $C > 0$.

Taking the limit on the right of (26), we deduce

$$\sup_m |f(u_m)|_{L^2(\Omega)} < C(1 + |u_m|_{L^2(\Omega)}) < \infty$$

$$\Rightarrow \sup_m |f(u_m)|_{L^2(\Omega)} < \infty$$

Thus (27) implies the existence of a subsequence $\{f(u_{m_j})\}_{j=1}^\infty$ converging weakly to $f(u)$ in $L^2(\Omega)$.

Step 6

We now verify that u is a weak solution of (1) - (2). We take the subsequence $\{u_{m_j}\}_{j=1}^\infty$ in step 5 as the sequence $\{u_m\}_{m=1}^\infty$. Thus we fix $v \in X$ and multiply (23) by v , integrate by parts and apply (24)

$$\alpha \int_\Omega \nabla u_{m+1} \cdot \nabla v dx = \int_\Omega v f(u_m) dx + \int_\Omega v g(x) dx \quad (28)$$

Step 7

Hence, letting $m \rightarrow \infty$ in (28), gives

$$\alpha \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega v f(u) dx + \int_\Omega v g(x) dx$$

Illustrative Example

Consider the boundary value problem

$$-\alpha \Delta u = (1 + \sigma u)^{\frac{1}{2}} + \sin x \text{ in } \Omega \subset \mathbb{R}^n \quad (29)$$

$$u = 0 \text{ on } \partial\Omega \quad (30)$$

where $\alpha > 0$.

Thus, we have

$$\int_\Omega |\sin x|^2 dx \leq \Omega < \infty$$

so that $g(x) \in L^2(\Omega)$.

Notice that

$$0 \leq |f'(u)| = \left| \frac{\sigma}{2(1 + \sigma u)} \right| \leq \sigma.$$

Hence, by Theorem (4.1), there exists a unique weak solution to the BVP (29) – (30).

CONCLUSION

In this paper, we established the existence and uniqueness of weak solutions to a class of nonlinear elliptic boundary value problems subject to homogeneous Dirichlet boundary conditions. By constructing an auxiliary linear approximation, we derived essential *a priori* estimates in the Sobolev space $H_0^1(\Omega)$ using the Lax–Milgram theorem. Applying Banach's fixed-point theorem allowed us to prove that the solution operator is a contraction mapping, thereby guaranteeing a unique weak solution that can be constructed iteratively. Finally, an illustrative numerical example was presented to demonstrate the practical applicability of the theoretical framework. The methodology developed here provides a foundation that can be extended to more complex non-autonomous systems or problems involving convective and singular potential terms.

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