The Effect of the Hurst Parameter on Value at Risk Estimation in Fractional Geometric Brownian motion Price Simulation

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ABSTRACT

This study assessed the impact of the Hurst parameter on the accuracy of Value at Risk (VaR) estimation using fractional Geometric Brownian motion (fGBM) for stock price simulation. The fGBM model, known for its ability to capture long-term memory in financial time series, was employed to simulate stock prices with varying Hurst parameters. The accuracy of VaR estimations obtained from these simulations was then assessed using mean absolute error metric. The research findings revealed that the Hurst parameter significantly influences the accuracy of VaR estimation in fGBM models. The study identified 0.7 as the optimal Hurst parameter value that enhances VaR estimation accuracy, highlighting the importance of incorporating long-term memory effects in risk assessment. The insights have practical implications for investors and financial institutions seeking to enhance risk management practices. The researchers recommended further researches using different levels of Hurst parameters and VaR at different significant levels.

Keywords: Fractional Geometric Brownian motion, Hurst parameter, stock price simulation, Value at risk.

INTRODUCTION

Value at risk (VaR) measures the possible loss that an investor or financial institution might experience over a given period, given typical market conditions and a particular likelihood (Happer, 2024; Will Kenton, 2023). It states the company's probability of losing more than a given amount within a specified period. It is a tool used by risk managers to gauge and manage the degree of risk exposure. Investors and financial institutions frequently use VaR to make investment decisions that include determining the amount of assets required to cover potential losses.

VaR is often predicated on the notion that the price of the underlying asset moves in accordance with a Geometric Brownian motion (GBM). However, empirical evidence reveals that asset values often demonstrate long-term memory, which means that they tend to trend either upwards or downwards over time (Ibrahim, Misiran & Laham, 2020). Therefore, there is need for a Hurst parameter, which captures the long-term memory and measures the size of the memory in the price data. The Hurst parameter also increases the precision of VaR estimations (Ouyang, Yang, Zhou, 2018).

However, there is debate in literature on the optimal Hurst parameter for VaR estimation using the fractional Geometric Brownian Motion (fGBM) price simulation model (Andersen & Piterbarg, 2007; Cont & Tankov, 2003; Zumbach, 2003). The debate prompted the current study. The purpose of the study was to assess how the size of the Hurst parameter affects the precision of VaR calculations made using fGBM model and to find the optimal Hurst parameter for estimation using the fGBM model. The following research questions guided the study:

1. What effect does the Hurst parameter have on the fGBM model's ability to estimate VaR with accuracy?
2. How does the accuracy of VaR estimations using the fGBM model vary with different Hurst parameters?
3. What is the optimal Hurst parameter for VaR estimation using the fGBM model?

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The Conceptual Framework

Fig 1 summarizes the structure of enquiry the researchers followed during data modelling and analysis.

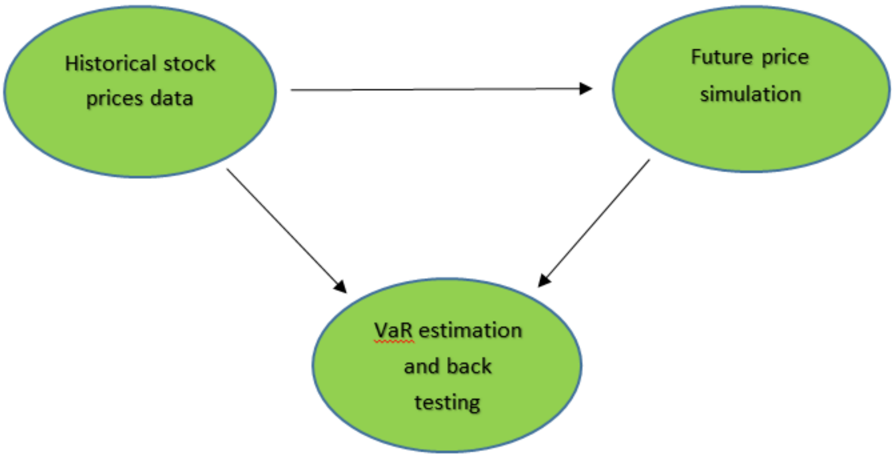

Fig 1 the conceptual framework

The researchers used historical stock prices data in simulating future price behavioural patterns using the fGBM model. VaR was then estimated on the simulated price patterns and the results were back-tested on real data in order to find the VaR estimation accuracy.

The interpretation of the Value at Risk (VaR)

The p value at risk is the largest possible loss over time after rejecting all worse events whose combined probability is at most p for a particular portfolio, time horizon, and probability p (CFI, 2019). The statistic presupposes market-to-market pricing and a portfolio free of transactions. For instance, if a portfolio of equities has a one-day 95% VaR of $1 million, it has a 5% likelihood of losing more than $1 million in value over the course of a single day if there is no trading. On average, this portfolio is predicted to lose $1 million or more on 1 out of every 20 days (based on a 5% probability).

Fig 2 illustrates the concept of value at risk.

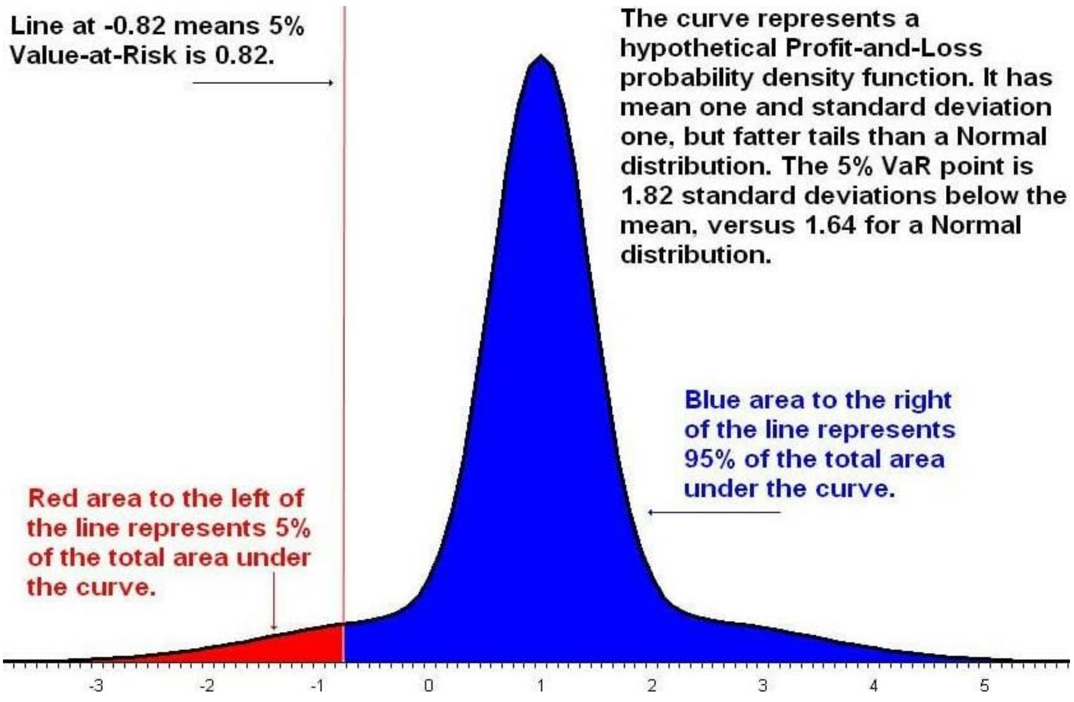

Fig 2 VaR illustration (Adopted form CF1, 2019)

There are a number of ways of calculating VaR. The current researchers employed the Monte Carlo simulation method. The method operates by modelling numerous potential future market movement possibilities, and then computing VaR based on the resulting distribution. The method depends on the assumption that there is a known probability distribution for the risk factors (CFI, 2019). The steps involved include listing the pertinent risk factors, describing the time-frame and level of certainty, creating numerous random scenarios for potential future market movements, determining the value of the portfolio for each scenario and then calculating the value at risk using the distribution of the portfolio values.

VaR is applicable to all classes of assets, including derivatives, stocks, bonds and currencies (CFI, 2019). It is easy to grasp and interpret. However, it has its own portion of weaknesses. One of its weaknesses is that it is difficult to calculate for large and diverse portfolios. In addition to that, it relies on some assumptions on the distribution of the assets, which may not be correct. It also disregards the potential of severe events, such as financial crises as it relies on historical data. This may result in losses that are much higher than the VaR.

The fractional Geometric Brownian Motion Stochastic Process

The stochastic process known as fractional Geometric Brownian motion (fGBM) mimics stock prices. Mandelbrot and Van Ness (1998) advocated for the model. It is a generalization of the frequently employed Geometric Brownian motion (GBM) model for stock price modeling used in determining future stock prices.

The GBM model presupposes that stock prices move in a random manner, which means that price fluctuations are unrelated to one another. The model also works under the assumption that stock returns are normally distributed. The assumptions, however, are not always true because stock prices frequently display long-term memory, which means that they frequently trend in one direction or the other over time (Djauhari, Li, & Salleh, 2016).

By incorporating a Hurst parameter, the fGBM model is able to capture the long-term memory of stock prices. The Hurst parameter gauges the time series long-term memory. When the Hurst parameter is set to 0, the time series has no memory, indicating that its future values are unrelated to its prior ones. When the Hurst parameter is greater than 0.5, the time series is said to have positive long-term memory, which means that its future values will typically resemble its prior ones. If the Hurst parameter is less than 0.5, the time series has negative long-term memory, which means that its future values are more likely to diverge from its prior ones.

Mandelbrot and Van Ness (1968) proclaimed that fGBM is a more accurate model of stock prices than more conventional models, like the geometric Brownian motion (GBM) model. Naldi (2018) supported the proclamation. Empirical research by Baillie et al. (1996) showed that fGBM models outperformed GBM models in fitting stock price data.

Research indicates that a crucial element in fGBM models is the Hurst parameter (Naldi, 2018). The level of long-range dependence in an fGBM process is set by the Hurst parameter. Increased Hurst parameter values indicates robust long-range dependence. According to Naldi, the ideal Hurst parameter for VaR estimate using the fGBM model is contingent upon several elements, such as the particular asset under consideration, the VaR estimation time horizon, the required degree of precision, and the available processing capacity.

Diverse opinions were voiced regarding how the accuracy of VaR estimation is affected by the fGBM price simulation model using various Hurst parameters. According to some studies, the fGBM model provides a good representation of the behavior of financial returns, and VaR estimations can be more accurate when various Hurst parameters are used (Huang & Zhou, 2009; Li and Zhang, 2010; Wang et al., 2017).

However, some researchers argued that the fGBM model is not a reliable representation of the behavior of financial returns and that VaR estimations may actually be less accurate when various Hurst parameters are used (Bratian, et al., 2021; Cont, 2013; Mandelbrot & Taleb, 2010). Cont stated that the complexity of financial markets is not sufficiently captured by the fGBM model because it is overly simplistic. Mandelbrot and Taleb were of the view that employing alternative Hurst parameters can actually lower the accuracy of VaR estimations hence the fGBM model is not a reasonable representation of the behavior of financial returns.

Andersen and Piterbarg (2007) revealed that the ideal Hurst parameter for calculating the value at risk (VaR) of daily returns is roughly 0.5 whilst Cont and Tankov (2003) argued that it is 0.7. Zumbach, (2003) conducted a study which revealed that employing a Hurst parameter of 0.5 in the fGBM price simulation model yields precise VaR estimates, but for longer time horizons, a higher Hurst parameter of 0.7 is more appropriate. Gatheral, Schied and Slynko (2012) obtained the same result.

MATERIALS AND METHODS

The research design

The study followed a quasi-experimental research design. The approach aimed to establish cause-and-effect links between variables but could not randomly assign individuals to groups. The study's investigation of the impact of a variable (the Hurst parameter) on an outcome (VaR estimation) in a practical context justified the approach. Quantitative methods were used to respond to each study question. The researcher used stocks from a company in the manufacturing sector in Zimbabwe. The researchers chose the company due to data availability and accessibility.

Data collection and cleaning

Quantitative historical financial data on stock prices for two years (2021 and 2022) were collected. The data was sourced from the local bourse, Zimbabwe Stock Exchange. The data was loaded into the Python notebook as a data-frame (in tabular format) for simple presentation and faster processing. Missing values were dropped using the dropna method that deletes every row or columns with N/A entries. The researchers checked the data to ensure that all the stock prices were in the right number format (floating figures). All the data had 197 records indicated by 197 non-nulls. The data was normalized by dividing the variables by their standard deviation. The researchers perused the data for trends, seasonality, or other patterns in order to spot prospective issues that could have had an impact on stock prices.

Model specification for the fGBM stochastic process

The researchers calculated VaR using the Monte Carlo simulation approach while simulating stock prices with the fractional Geometric Brownian Motion stochastic process with varied Hurst parameters ($0.2, 0.4, 0.5, 0.7$ and 0.9). The fGBM model was as follows:

$$d\boldsymbol{S}(t) = \mu\boldsymbol{S}(t)dt + \sigma\boldsymbol{S}(t)\boldsymbol{H}d\boldsymbol{W}(t) \quad (1)$$

Where:

$\boldsymbol{S}(t)$ is the stock price at time t , μ is the expected return of the asset (constant drift), σ is the volatility, H is the Hurst parameter($H\epsilon(0,1)$), **W**(t) is a standard Wiener process.

$$\mu = \frac{r-\delta}{\boldsymbol{H}} \quad (2)$$

$$\sigma = \frac{2\boldsymbol{H}r-\delta}{\boldsymbol{H}}, \quad (3)$$

With r as the expected return of the asset and δ as the dividend yield of the asset. The solution of the fGBM model with an arbitrary initial value S (0) was given by:

$$S(t) = S(0)e^{\left(\mu t-\frac{1}{2}\sigma^2 t^H+\sigma W_H(t)\right)} \quad (4)$$

Drift and volatility (sigma) parameters were calculated from the training data. These parameters together with the initial price $S(0)$ were used to scale the simulated prices paths using the fGBM function. Through varying

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the Hurst parameter for the fGBM function, five different price simulation models were developed. The fGBM models were used to simulate stock prices for different time horizons. Future prices were predicted for a time horizon of 197 days.

The parameters were estimated using historical data of at least one year. The other data were reserved for VaR model back testing in the evaluating stage. After simulating the prices with varying Hurst Parameters, the researchers calculated VaRs using the returns distributions of the simulated prices. VaR was determined at the 95% confidence level for consistency throughout the study.

Data analysis

Python and a Jupyter notebook were used to analyze the data. Python was ideal for data analysis and an interactive setting that made it simple to run code and visualize data was the Jupyter notebook. The initial data preparation was carried out in Microsoft Excel.

The estimated VaRs with varying Hurst parameters were evaluated using real stock market data of the actual losses that were incurred assuming no trading and cost-free financial markets. The data that the researcher used for evaluation was unknown to the simulation models discussed in the prior strategy. The researchers determined the effect of the Hurst parameter on VaR estimation accuracy by analyzing the mean absolute error (MAE). The MAE was the average absolute difference between the actual and anticipated values. The lower the MAE the more accurate the prediction was.

From the graphical analysis of varying Hurst parameters with varying VaR estimation accuracy, the researchers were able to determine the optimal Hurst parameter at the point of the lowest error metric.

RESULTS

The characteristics of the available data

An assessment of the stock prices data indicated that all the data were in the correct number format. However, the distribution of the stock price data as returns shows some significant level of right skewness. The modal returns were typically around 0. Ignoring the extreme returns, the distribution was typically bell shaped (normal distribution).The stock prices data plot had some price variation and volatility clustering, which was typical of time series financial data. These were desired data characteristics for the study hence, the researchers proceeded to split the data into training (price simulation models development data).

Fig 3 shows the Time series plot of the stock prices.

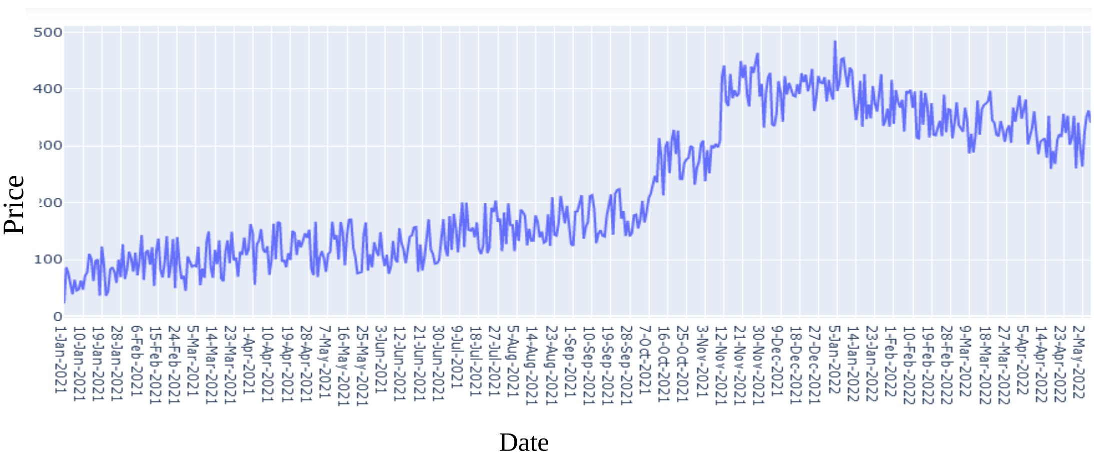

Fig 3 Time series plot of the stock prices

The effect of Hurst parameter levels on the price simulations

This section analyses the effect of varying the Hurst parameter on the accuracy of the price simulations.

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Fig 4 shows the actual prices against the forecasted prices at Hurst parameter 0.2.

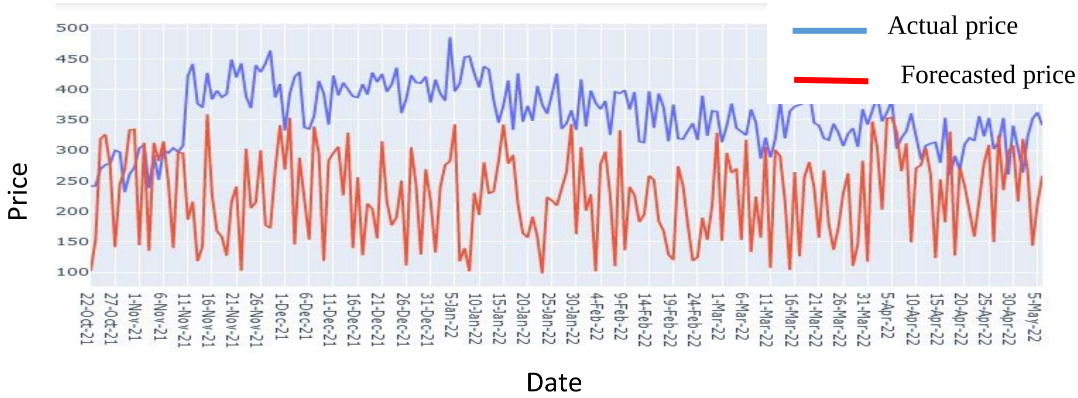

Fig 4 Forecast with Hurst parameter 0.2

The forecasted series in red was significantly far away from the actual price series, but through taking a close look, the shapes of the two series were more or less the same.

Fig 5 shows the actual prices against the forecasted prices at Hurst parameter 0.4.

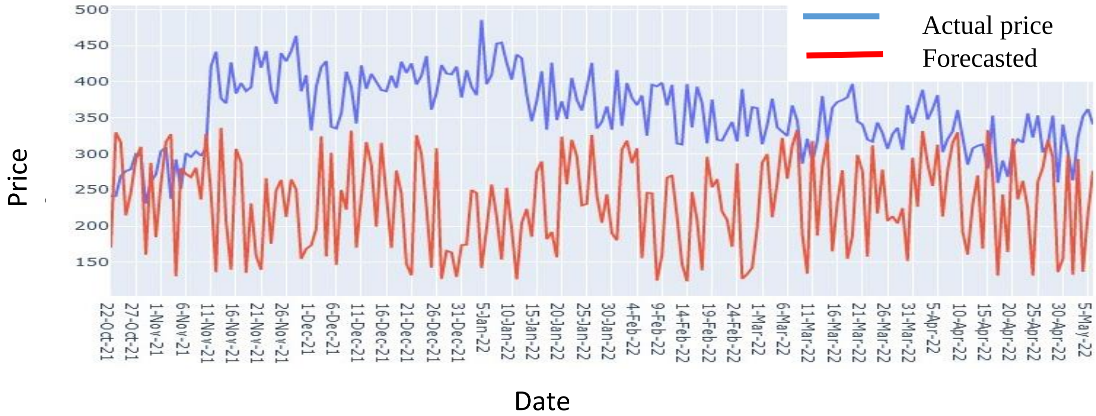

Fig 5 Forecast with Hurst parameter 0.4

The forecasted series in red moved further away from the actual price series compared to the case with Hurst parameter 0.2. However, the shapes of the two series were more or less the same.

Fig 6 shows the actual prices against the forecasted prices at Hurst parameter 0.5.

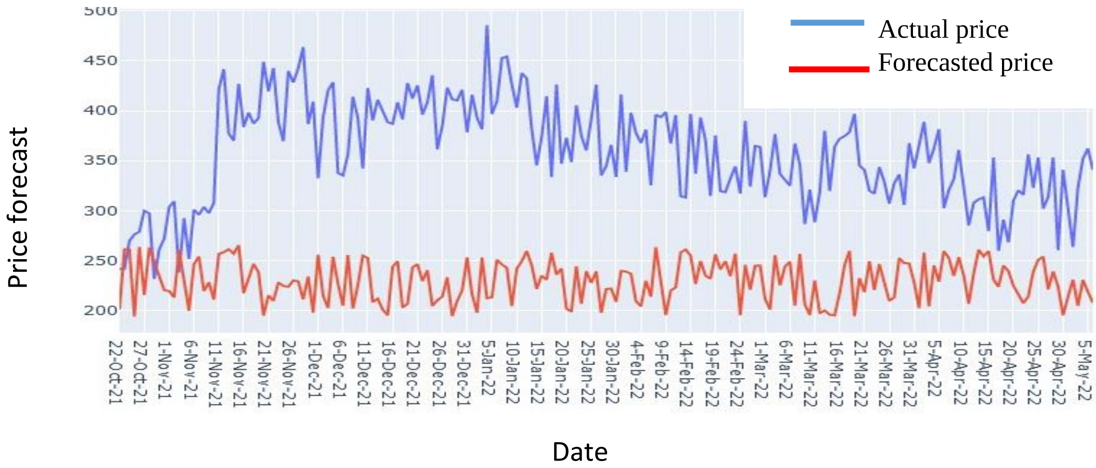

Fig 6 Forecast with hurts parameter 0.5

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In the pure Brownian motion ($H = 0.5$), the forecasts were relatively far away from the actual price series, but the two series exhibited some similarities in shape.

Fig 7 shows the actual prices against the forecasted prices at Hurst parameter 0.7.

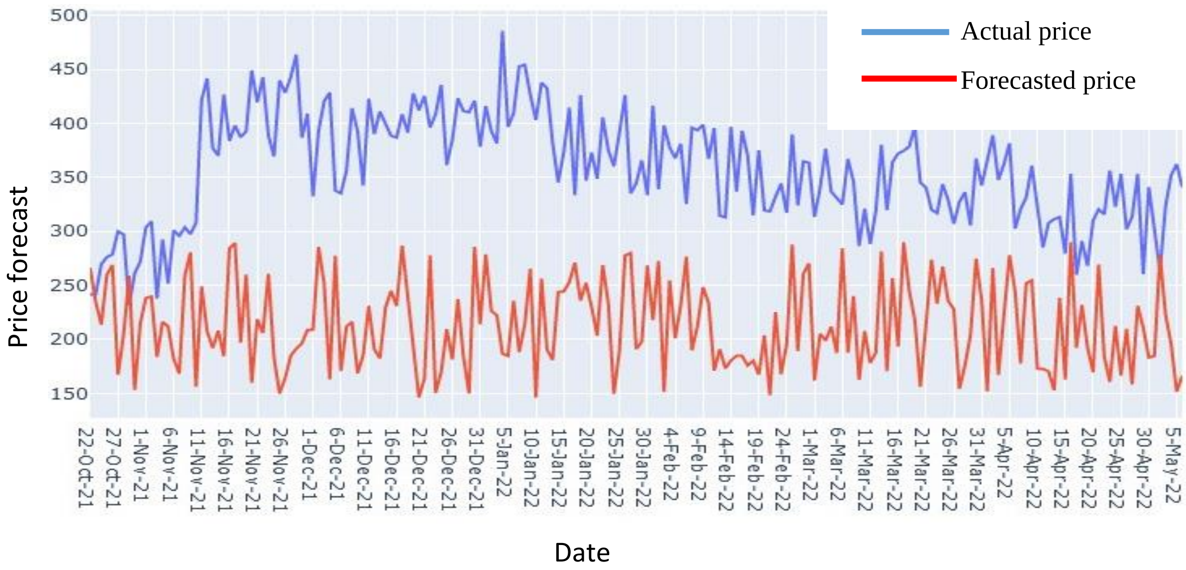

Fig 7 Forecast with Hurst parameter 0.7

Using the model with Hurst parameter 0.7, the forecasts were far away from the actual price series, but the shapes of the two series were closely related.

Fig 8 shows the actual prices against the forecasted prices at Hurst parameter 0.9.

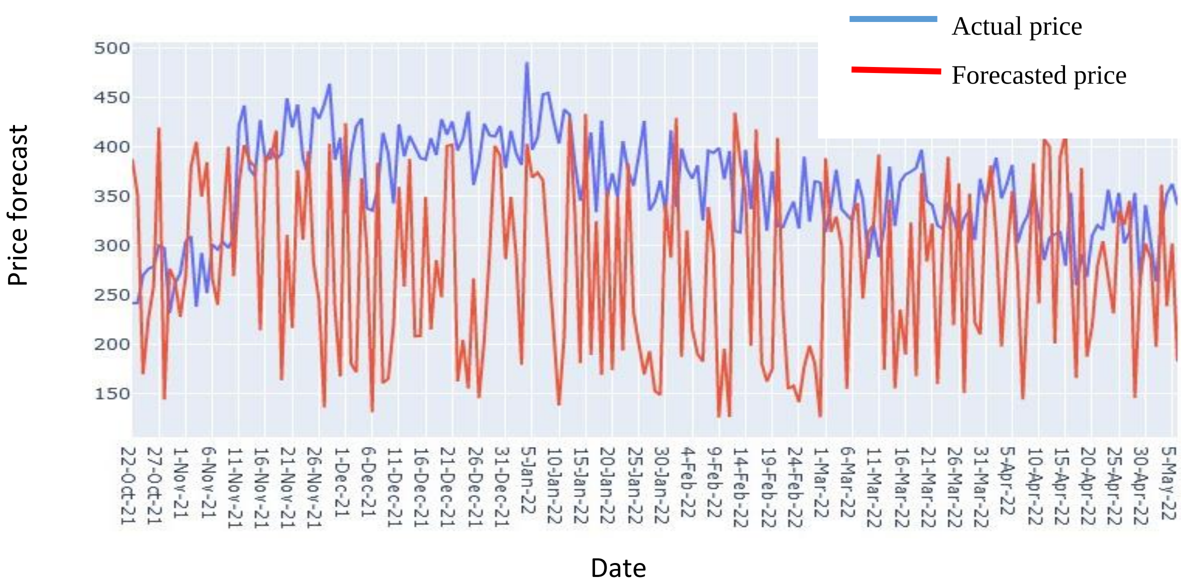

Fig 8 Forecast with Hurst parameter 0.9

The model with Hurst parameter 0.9 produced the forecasts that were quite close to the actual price series, but they were extremely volatile in nature.

VaR estimation accuracy and determination of the optimal Hurst parameter

Table 1 shows the absolute errors in calculating 95% VaR produced by the fGBM models with the varying Hurst parameters.

Table 1. Absolute errors for the different fGBM models

	Hurst parameter (H)				
	0.2	**0.4**	**0.5**	**0.7**	**0.9**
Actual VaR	0.19	0.19	0.19	0.19	0.19
Forecasted VaR	0.57	0.5	0.37	0.18	0.56
Absolute error	0.38	0.31	0.18	0.01	0.37

The accuracy measured by absolute errors, improved from $H = 0.2$ to $H = 0.7$, and thereafter it deteriorated.

Fig 9 shows a bar graph of the mean absolute errors (MAE) on VaR estimation from the different models.

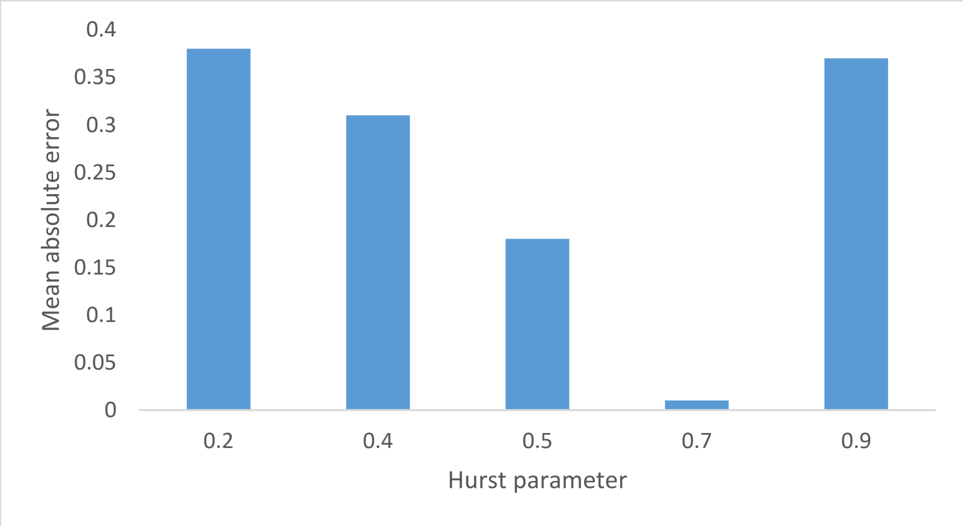

Fig 9 Hurst parameters and the associated Mean absolute errors (MAE) in VaR estimation

An analysis of the Mean Absolute Errors as shown in **Figure** 10 indicates that as the Hurst parameter increases from 0 to 0.7, VaR estimation accuracy improves, and thereafter it deteriorates.

CONCLUSION

The results of the study revealed that VaR estimation accuracies were significantly different as the Hurst parameter levels were varied. The Hurst parameter levels had a significant performance impact on the stock price simulations. The VaR estimation accuracy from the fGBM simulations improved as the Hurst parameters increase from 0 to 0.7 and thereafter it deteriorated. As the Hurst parameters increase from 0.2 to 0.5, the simulated prices moved away from the actual prices but the series of the simulated prices maintained a similar shape to that of the actual prices. The time series for the forecasted prices with Hurst parameter 0.9 was closer to the actual prices series but it was heavily volatile. This suggested that 0.9 could not be an optimal Hurst parameter for the fGBM model for the prices simulation. The study revealed that the Hurst parameter level in fractional Geometric Brownian motion price simulation is only effective for simulation accuracy to a fixed level, beyond, a significant under fit can be noticed. An analysis of the absolute errors and the mean absolute errors (MAE) pointed to the conclusion that the best estimation accuracy is obtained when using a Hurst parameter of 0.7. As a result, the researchers concluded that the optimal Hurst parameter for the stock used in the study was 0.7.

RECOMMENDATIONS

Based on the results of the study, the researchers recommend that for VaR estimation using fGBM models, a Hurst parameter around 0.7 can be used for price simulations. When conducting the study, some research gaps

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were identified that could lead to a more comprehensive study. One of the gaps was that the current study considered only five levels of the Hurst parameter. The researchers suggest further research that replicates the study using more levels of the Hurst parameter in order to get evidence that is more elaborate on how the Hurst parameter affects stock price simulation accuracy and VaR estimation. Studies with varied VaR significance levels and different time horizons are also suggested.

Declaration of competing interest

The researchers declare that they had no competing interests

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