Numerical Solutions of the Burgers Equation: A Comparative Study of Finite Difference and Differential Quadrature Methods

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ABSTRACT

The Finite Difference Method (FDM) and Differential Quadrature Method (DQM) are utilized to solve the partial differential equation in Burgers-Huxley equation. While FDM is relatively simple to implement, it tends to yield less accurate results. To address this limitation, DQM is employed, which improves accuracy but requires more computational effort and time. Various numbers of nodes are used in these methods to examine their accuracy. A program developed in C language based on these methods is used to solve Burgers' equation. The results are compared in terms of numerical solution accuracy and convergence. Overall, the numerical results indicate that the Differential Quadrature Method outperforms the Finite Difference Method in both accuracy and convergence.

Keywords: Finite difference method, Differential quadrature method, Burgers-Huxley equation, Partial differential equations, Ordinary differential equations

INTRODUCTION

Partial Differential Equations (PDEs), whether singular or in a system, frequently arise in various science and engineering disciplines, such as heat transfer [1]. These equations are essential for describing fundamental natural laws in physical and chemical phenomena. Nonlinear PDEs, in particular, hold significant importance across many scientific fields. The Burgers–Huxley equation, a notable nonlinear partial differential equation, effectively models the interaction between diffusion transport, convection effects, and reaction mechanisms [2, 3].

Typically, closed-form solutions for such equations are either unavailable or difficult to obtain due to their nonlinear nature. This challenge has driven the development of alternative approaches for approximating solutions to these equations. Over the years, scientific research has demonstrated that approximations can be achieved by applying numerical discretization techniques, which evaluate function values at specific discrete points known as grid points or mesh points.

In engineering and computational fluid dynamics, three numerical methods are widely used for this purpose: the finite difference method, the finite element method, and the finite volume method. Separate paragraphs should be spaced by a single line. The Burgers–Huxley equation is a nonlinear partial differential equation (PDE) with significant applications in modeling various mechanisms across engineering fields, such as nonlinear wave processes in economics, physics, and ecology [4]. According to many researchers, there is no universal numerical approximation method for solving this nonlinear PDE. As a result, different numerical techniques have been employed to approximate its solution. However, Yefimova and Kudryashov (2004) and Gao and Zhao (2009) successfully derived exact solutions for the Burgers–Huxley equation [5,6].

One of the simplest techniques to solve the Burgers–Huxley equation is the Finite Difference Method (FDM). In this method, the value at a specific grid point is replaced with approximate derivatives, calculated as differences between values at neighboring grid points. Partial derivatives in the PDE are estimated based on these neighborhood values at each grid point.

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Another technique discussed in this study is the Differential Quadrature Method (DQM). As noted by [7], DQM can be considered an extension of FDM, applied to higher-order finite difference schemes. This method involves summing up all derivatives of the function at grid points, transforming the PDE into a system of ordinary differential equations (ODEs) or algebraic equations [8]. The resulting ODE system is then solved using numerical methods.

There are several methods to approximate the solution of the Burgers equation, which requires a set of initial and boundary conditions for its solution. The Finite Difference Method (FDM) is straightforward and commonly used for solving examples of the Burgers equation, though it tends to be less accurate. To address this limitation, the Differential Quadrature Method (DQM) is employed, offering improved accuracy but at the cost of higher computational effort and increased time consumption.

During the 1960s and 1970s, FDM dominated computational sciences and was the preferred method of that era. A key characteristic of FDM is its reliance on uniformly spaced grids. At each grid node, derivatives are approximated using algebraic expressions involving adjacent nodes. By applying these approximations to each node, a system of algebraic equations is generated, which is then solved for the dependent variable. FDM approximates the derivative of a function at a point using discrete function values, allowing solutions to partial or ordinary differential equations to be obtained by substituting derivatives with finite difference expressions. Although FDM is easy to apply, its accuracy is inferior compared to DQM.

In 1972, R.E. Bellman and his collaborators introduced DQM to address the limitations of FDM. DQM has since become significant in various fields, including biosciences, system identification, diffusion processes, fluid dynamics [9], chemical engineering, lubrication, acoustics, and contact problems [10]. It has been demonstrated to be equivalent to the general collocation method [11] and, as noted by Shu and Chew (1998), represents the highest-order finite difference scheme, extending lower-order finite difference methods [12].

DQM transforms partial differential equations into ordinary differential equations by replacing the derivatives of a smooth function with a weighted linear combination of function values [13]. Unlike FDM, which relies on local information, DQM utilizes function values across all grid points. While this approach enhances accuracy, it also increases computational expense and time requirements.

METHODOLOGY

Burgers-Huxley Equation

M. Sari and G. Gurarslan (2009) asserted that the generalized Burgers-Huxley equation serves as a useful model for representing the interplay between reaction mechanisms, convection effects, and diffusion transport processes. The equation of the Burgers-Huxley Equation is:

$$\frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} + \alpha u^{\delta} \frac{\partial u}{\partial x} = \beta\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right)u$$

where where α, β, γ, and δ are parameters that $\beta \geq 0$, $\delta > 0$,

with initial condition,

$$u(x,0) \rightleftarrows \rightleftarrows = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh(A_1 x)\right)^{\frac{1}{\delta}}$$

and the boundary conditions,

$$u(0,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh(-A_1 A_2 x)\right)^{\frac{1}{\delta}} t \geq 0$$

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$$u(1,t) = \left(\frac{\gamma}{2} + \frac{\gamma}{2} tanh\big(A_1(1 - A_2 t)\big)\right)^{\frac{1}{\delta}} t \geq 0$$

where

$$A_1 = \frac{-\alpha\delta + \delta\sqrt{\alpha^2 + 4\beta(1+\delta)}}{4(1+\delta)}\gamma,$$

$$A_2 = \frac{\gamma\alpha}{(1+\delta)} + \frac{(1+\delta-\gamma)\left(-\alpha + \sqrt{\alpha^2 + 4\beta(1+\delta)}\right)}{2(1+\delta)}$$

For this study we focus on the equation as follows:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = e^{-t}(1 + x^2) \qquad 0 \leq x \leq 1, \qquad t \geq 0$$

with initial condition:

$$u(x,0) = 1 + x^2 \qquad 0 < x < 1$$

and the boundary equation:

$$u(0,t) = e^{-t}, \qquad u(1,t) = 2e^{-t} \qquad t > 0$$

Finite Difference Method

The finite difference method (FDM) is a numerical technique used to solve differential equations by approximating derivatives with finite differences. It is commonly applied to partial differential equations (PDEs) and ordinary differential equations (ODEs) in engineering, physics, and mathematics. Instead of solving a differential equation analytically (which is often difficult or impossible), the FDM replaces derivatives with algebraic approximations based on discrete points in the domain of the problem. The continuous domain is divided into a grid or mesh, and the solution is computed at these discrete points.The equation for the first derivative and second derivative are as follow:

$$u^{'}(x) \approx \lim_{h\to 0} \frac{u(x+h) - u(x-h)}{2h}$$

$$u^{(2)}(x) \approx \lim_{h\to 0} \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

Differential Quadrature Method

The Differential Quadrature Method (DQM) is a numerical technique for solving differential equations. It approximates the derivatives of a function at a set of discrete points by expressing them as weighted sums of the function values at those points. DQM is an efficient and accurate method often used for solving problems in structural mechanics, fluid dynamics, and heat transfer. The equation for the derivative is given by

$$u_x^{(1)}(x_i,t) \cong \sum_{j=1}^{N} a_{ij} u(x_j,t), \qquad \text{for } i = 1,2,\ldots,N$$

$$u_x^{(2)}(x_i,t) \cong \sum_{j=1}^{N} b_{ij} u(x_j,t), \qquad \text{for } i = 1,2,\ldots,N$$

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where

$$b_{ij} = r\left[a_{ij}w_{ii}^{(1)} - \frac{w_{ij}^{(1)}}{x_i - x_j}\right], \qquad \text{for } i \neq j$$

$$b_{ii} = -\sum_{j=1, j\neq i}^{N} w_{ij}^{(2)} \qquad \text{for } i = j$$

This approximation of derivatives, are used in the differential equation which replace the original derivatives. The differential equation can be reduced to a set of algebraic system or ordinary differential equation. Then, the solutions can be obtained by using standard numerical method such as the Runge-Kutta method.

RESULT AND DISCUSSION

The example above has been solved by using C++ programming with different methods.

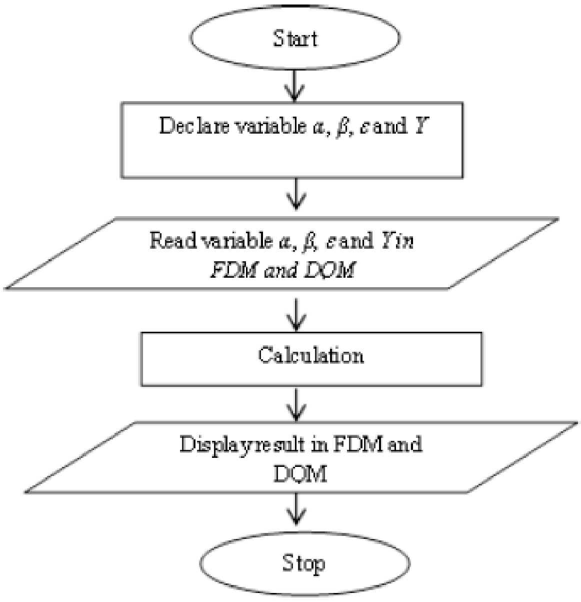

Fig. 1 Flowchart of C++ programming for solving Burger-Huxley equation.

The results from FDM and DQM are compared in term of accuracy of the exact solution for the time t = 0.001 and 0.002.

Table 1 The Result for the Time T = 0.001

u(x,t)\x		**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
FDM		0.999	1.040272	1.159956	1.359504	1.638869	1.998001
DQM		0.999	1.038961	1.158841	1.358641	1.638361	1.998001
EXACT		0.999	1.038961	1.158841	1.358641	1.638361	1.998001
ERROR	**FDM**	0	0.001311	0.001115	0.000863	0.000508	0
	DQM	0	0	0	0	0	0

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Table 2 The Result for the Time T = 0.002

u(x,t)\x		**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
FDM		0.998002	1.040536	1.159911	1.359008	1.637739	1.996004
DQM		0.998002	1.037922	1.157682	1.357283	1.636723	1.996004
EXACT		0.998002	1.037922	1.157682	1.357283	1.636723	1.996004
ERROR	**FDM**	0	0.002614	0.002229	0.001725	0.001016	0
	DQM	0	0	0	0	0	0

From the Table 1 and 2, the errors between the result from FDM and DQM with the exact solution have been plotted in the graph that is depicted in Figure 2 and 3.

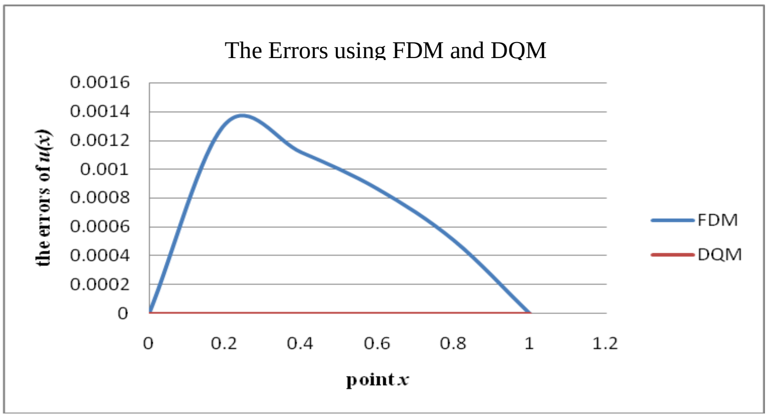

Fig.2 The errors using FDM and DQM when t = 0.001

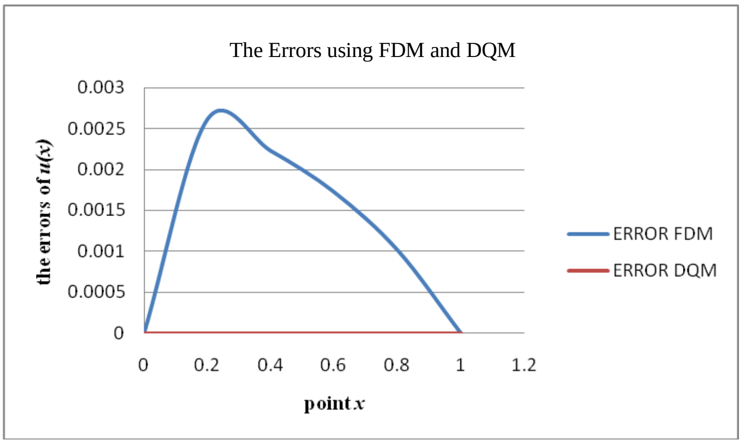

Fig.3 The errors using FDM and DQM when t = 0.002

The blue line represents the error profile for the FDM and the red line represents the error profile for the DQM. DQM exhibits significantly lower errors than FDM across the entire domain. This aligns with the general understanding that DQM often achieves higher accuracy with fewer grid points. FDM error is not uniform across the domain; it peaks in the middle and diminishes near the boundaries. This behavior is typical due to the nature of finite difference schemes, which can have lower accuracy near regions with higher curvature in the solution. DQM maintains a consistent, near-zero error profile, indicating a more accurate approximation.

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CONCLUSION

In conclusion, based on the tables and figures presented above, the results obtained using the Differential Quadrature Method (DQM) consistently outperformed those derived from the Finite Difference Method (FDM) at all tested time intervals. Regarding error analysis, the errors associated with the DQM results were significantly smaller compared to those from the FDM, demonstrating the superior accuracy of the DQM. Additionally, in terms of convergence analysis, the results from the DQM exhibited a noticeably faster convergence to the actual solution when compared to the FDM. These findings collectively highlight the advantages of DQM over FDM in terms of both accuracy and convergence efficiency.

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There is no finding to this article.

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