A Work on Generalized Reverse Derivation and Skew-Derivation on Prime Near-Rings

Abdu Madugu and Tasiu Abdullahi Yusuf*

Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Umaru Musa Yar'adua University, Katsina, Nigeria

***Corresponding author**

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ABSTRACT

This paper investigates the analysis of prime near-ring by explore the detailed structure of the generalized derivations that satisfy some specific assumptions. Let N be a prime near-rings and G be a generalized reverse derivative associated with mapping d on N. An additive mapping $d: N \to N$ is said to be a derivation on N if $d(xy) = d(x)y + xd(y)$ for all $x, y \in N$. A mapping $G: N \to N$ associated with derivation d is called a generalized derivation on N if $G(xy) = G(x)y + xd(y)$ for all $x, y \in N$. Also, a mapping $d: N \to N$ is said to be a reverse derivation on N if $d(xy) = d(y)x + yd(x)$ for all $x, y \in N$ and a mapping $G: N \to N$ associated with reverse derivation d is said to be a generalized reverse derivation on N if $G(xy) = G(y)x + yd(x)$ for all $x, y \in N$. We prove some results on commutativity of prime near-rings involving generalized reverse derivations. In addition, we prove that; for prime near-rings N, if $d(x)d(y) \pm xy = 0$ for all $x, y \in N$ then $d = 0$ where d is a skew- derivation associated with an automorphism $\beta : N \to N$. Furthermore, for a prime near-ring N with generalized derivative G associated with mapping d on N, if $G(x)G(y) \pm xy = 0$ for all $x, y \in G$ then $d = 0$.

Keywords: Prime near-ring, reverse derivation, generalized reverse derivation, skew derivation.

INTRODUCTION

In this context, the symbol $[x, y]$ and (xoy), represent the Lie product $xy - yx$ and Jordan product $xy + yx$ respectively, where $x, y \in N$. A near-ring N is called prime near-rings if $aNb = \{0\}$ for any $a, b \in N$, implies that either $a = 0$ or $b = 0$ and is said to be semiprime near-rings if $aNa = \{0\}$ for any $a \in N$, then $a = 0$.

Bresar [1] defined an additive mapping according to him a mapping f is said to be an additive mapping on R if $f(x + y) = f(x) + f(y)$ for all $x, y \in R$. According to [2] a mapping $d: R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. If d is an additive mapping then d is said to be a derivation on R. Also an additive mapping $F: R \to R$ is called generalized derivation if there exist a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$.

The notion of multiplicative derivation was first introduced by Daif [3] according to him a mapping $D: R \to R$ is called multiplicative derivation if it satisfies $D(xy) = D(x)y + xD(y)$, for all $x, y \in R$ where in this, the mappings are not supposed to be an additive. Further Daif and Tammam El-sayiad [4] extended multiplicative derivation to multiplicative generalized derivation that is a mapping F on R is said to be a multiplicative generalized derivation if there exist a derivation d on R such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. From the definition of multiplicative generalized derivation if d is any mapping not necessarily additive and derivation then F is said to be multiplicative (generalized) derivation. Recently Dhara and Ali [5] give a more precise definition of multiplicative (generalized)derivation as follows: A mapping $F: R \to R$ is said to be a

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multiplicative (generalized) derivation if there exist a map g on R such that $F(xy) = F(x)y + xg(y)$, for all $x, y \in R$. Where g is any mapping on R (not necessarily additive). Therefore, the concept of multiplicative (generalized) derivation covers the concept of multiplicative derivation and multiplicative generalized derivation.

The notion of reverse derivation was first introduce by Herstein [6] According to him an additive mapping d on R is said to be reverse derivation if $d(xy) = d(y)x + yd(x)$, for all $x, y \in R$.While, according to [7] the generalized reverse derivation is an additive mapping $F: R \to R$ if there exist a map $d: R \to R$ such that $F(xy) = F(y)x + yd(x)$, for all $x, y \in R$. A map $F: R \to R$ is called multiplicative (generalized)-reverse derivation if $F(xy) = F(y)x + yd(x)$, for all $x, y \in R$, where d is any map on R and F is not necessarily additive [8].

Motivated by the above concepts, we prove the commutativity of prime near-rings involving generalized reverse derivations on a prime near-rings N and result on skew-derivation of the same N.

Results on Prime near-rings N.

Theorem 1.1

Let N be prime near-ring and G be a generalized derivative associated with mapping $d$on N. If $G(x)G(y) \pm xy = 0$ for all $x, y \in G$ then $d = 0$.

Proof.

First we consider the case

$$G(x)G(y) + xy = 0 \quad (1)$$

for all $x, y, \in N$. Substituting yz instead of y in equation (1), we obtain

$$G(x)G(yz) + xyz = 0$$

$$G(x)(G(y)z + yd(z)) + xyz = 0$$

$$G(x)G(y)z + G(x)yd(z) + xyz = 0$$

But $G(x)G(y) = -xy$

Therefore, $-xyz + G(x)\ yd(z) + xyz = 0$

$$G(x)yd(z) + xyz - xyz = 0, \text{ for all } x, y, z \in N \quad (2)$$

Substituting xr instead of x in equation (2), we get

$$G(xr)yd(z) = 0$$

$$((G(x)r + xd(r))yd(z) = 0$$

$$G(x)ryd(z) + xd(r))yd(z) = 0\ \forall\ x, y, z \in N \quad (3)$$

Substituting ry instead of y in equation (2), we obtain

$$G(x)ryd(z) = 0 \quad (4)$$

Subtracting equation (3) from equation (4), we obtain

$$xd(r)yd(z) = \forall\ x, y, r, z \in N \quad (5)$$

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Replacing $xd(r)$ by $d(t)$ in (5), we get

$$d(t)yd(z) = 0 \ \forall \ y, t, z \in N$$

Since d is mapping on N $\forall \ y, t, z \in N$.

This implies that, $d(t)Nd(z) = 0 \ \forall \ t, z \in N$

Therefore, by primeness of N, we obtain $d(t) = 0$ or $d(z) = 0$.

Using similar approach, we can prove that the same result holds for

$$G(x)G(y) - xy = 0 \forall \ x, y, r, z \in G.$$

RESULTS ON COMMUTATIVITY OF N.

Theorem 1.2

Let N be prime near-ring and G be a generalized reverse derivative associated with mapping $d$on N. If $G(x)G(y) \pm xy = 0$ for all $x, y \in N$ then N is commutative.

Proof:

First we consider the case,

$G(x)G(y) + xy = 0$ (6)

for all $x, y, \in N$. Substituting zy instead of y in equation (1), we obtain

$G(x)G(zy) + xy = 0$, for all $x, y, z \in N$.

By definition of generalized derivation, we have

$$G(x)(G(y)z + yd(z)) + xy = 0$$

$$G(x)(G(y)z + G(x)yd(z)) + xy = 0$$

But $\mathrm{G}(x)G(y) = -xy$

$$-xyz + G(x)yd(z) + xzy = 0$$

$$xzy - xyz + G(x)yd(z) = 0$$

$$-x([z, y]) + G(x)yd(z) = 0$$

$$x[z, y] + G(x)yd(z) = 0$$

$$G(x)yd(z) + x[z, y] = 0 \quad (7)$$

for all $x, y, \in N$. Substituting ry instead of y in equation (7) where $r \in I$, we obtain

$$G(x)ryd(z) + x[z, ry] = 0 \ \forall \ x, y, r, z \in N$$

$$G(x)ryd(z) + x(r[z, y] + [z, r]y) = 0$$

$$G(x)ryd(z) + xr[z, y] + x[z, r]y = 0 \quad (8)$$

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Replacing x instead of with rx in equation (7), we get

$$G(rx)yd(z) + rx[z,y] = 0 \quad (9)$$

By definition of generalized reverse derivation, we get

$$(G(x)r + xd(r)) + rx[z,y] = 0$$

$$G(x)ryd(z) + xd(r)yd(z) + rx[z,y] = 0 \quad (10)$$

Subtracting equation (8) by (10), we get

$$xr[z,y] + x[z,r]y - xd(r)yd(z) - rx[z,y] = 0$$

$$x[z,r]y - xd(r)yd(z) + [x,r][z,y] = 0 \quad (11)$$

for all $x, y, r, z \in N$.

Substituting tx instead of x in equation (11), we obtain

$$tx[z,r]y - txd(r)yd(z) + [tx,r][z,y] = 0$$

$$tx[z,r]y - txd(r)yd(z) + (t[x,r] + [t,r]x)[z,y] = 0$$

$$tx[z,r]y - txd(r)yd(z) + t[x,r][z,y] + [t,r]x[z,y] = 0 \quad (12)$$

Multiplying equation (11) by t on the left side, we obtain

$$tx[z,r]y - txd(r)yd(z) + t[x,r][z,y] = 0 \quad (13)$$

Subtracting equation (11) by (12), we get

$$[x,r]x[z,y] = 0 \ \forall\, x, y, r, z \in R$$

$$[t,r]N[z,y] = 0 \ \forall\, t, y, r, z \in R$$

By primeness of N, then either $[t,r] = 0$ or $[z,y] = 0$

that is, $tr - rt = 0$ or $zy - yz = 0$.

Therefore, $tr = rt$ or $zy = yz$ which implies that N is commutative.

Results on Skew Derivation of N.

Theorem 1.3

Let N be prime near-ring and d be a skew- derivative associated with an automorphism

$\beta: N \to N$.

If $d(x)d(y) \pm xy = 0$ for all $x, y \in N$ then $d = 0$.

Proof:

First, we consider the case

$$d(x)d(y) + xy = 0 \quad (14)$$

for all $x, y, \in N$. Substituting yz instead of y in equation (14), we obtain

$$d(x)d(yz) + xyz = 0$$

By definition of skew derivation, we get

$$d(x)(d(y)z + \beta(y)d(z)) + xyz = 0 \ \forall\, x, y, z \in N$$

$$d(x)d(y)z + d(x)\beta(y)d(z) + xyz = 0$$

But $d(x)d(y) = -xy$

$$-xyz + d(x)\beta(y)d(z + xyz) = 0$$

$$d(x)(\beta(y)d(z)) + xyz - xyz = 0$$

$$d(x)(\beta(y)d(z)) = 0 \qquad (15)$$

Replacing xr instead x in equation (15), we obtain

$$d(xr)(\beta(y)d(z)) = 0$$

By definition of skew derivation, we have

$$(d(x)r(\beta(x)d(r))\beta(y)d(z) = 0$$

$$d(x)r\beta(y)d(z) + \beta(x)d(r)\beta(y)d(z) = 0 \qquad (16)$$

Replacing $r\beta(y)$ instead of $\beta(y)$ in equation (15), we get

$$d(x)r\beta(y)d(z) = 0 \ \forall\, x. y, r, z \in N \qquad (17)$$

Subtracting equation (17) equation (16), we obtain

$$\beta(x)d(r)\beta(y)d(z) = 0 \ \forall\, x. y, r, z \in N \qquad (18)$$

Replacing $d(r)$ instead of $\beta(x)d(r)$ in equation (18), we get

$$d(r)\beta(y)d(z) = 0 \ \forall\, y, r, z \in N$$

Since β is an automorphism of N and d is a skew derivation of N and $x. y, r, z \in N$

This implies that,

$$d(r)Nd(z) \ \ \forall\, r, z \in N$$

By primeness of N, this implies that

$d(r) = 0$ or $d(z) = 0$.

Hence, we obtained the require result.

CONCLUSION

In this paper, we prove the commutativity of prime near-rings with generalized reverse-derivations and investigate whether prime near-rings that admit a nonzero multiplicative reverse-derivation satisfying certain

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algebraic (or differential) identities are commutative ring. In addition, we show that in a prime near-rings N with a skew-derivative d associated with an automorphism $\beta: N \to N$ if the differential identity $d(x)d(y) \pm xy = 0$ satisfy for all $x, y \in N$ then the skew-derivation d is zero.

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