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# Boussinesq Navier Stokes to A Modification on the Lorenz-63 Model

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## ABSTRACT

We derive a five–dimensional Modified Lorenz System (MLS) from the incompressible Boussinesq Navier–Stokes equations with coupled potential temperature and moisture fields via systematic Galerkin projection onto a small set of physically motivated spatial modes. Beginning from

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + g\alpha \ \ \Theta \ \ \hat{\mathbf{z}}, \qquad \partial_t\Theta + (\mathbf{u}\cdot\nabla)\Theta = \kappa_\Theta\nabla^2\Theta + S_\Theta,$$

and a passive moisture equation, we nondimensionalize and project onto a divergence–free modal basis, leading to

$$\dot{q}_m = \sum_n L_{mn}q_n + \sum_{i,j} C_{mij} \ \ q_i q_j + F_m,$$

with closure hypotheses for unresolved modes. A judicious choice of five dominant amplitudes $(T, H, P, W, R)^\top$ yields the MLS, which generalizes the classical Lorenz–63 model by incorporating moisture coupling and bounded nonlinear closures.

We provide a detailed analytical characterization of the MLS: (i) fixed points are identified and classified, (ii) linear stability is determined via the Jacobian eigenvalue spectrum, and (iii) Lyapunov exponents are defined through the tangent–linear system $\dot{\xi} = J(X(t)) \ \ \xi$, with the largest exponent $\lambda_1$ setting a predictability time scale $T_L \sim 1/\lambda_1$. Using energy estimates, we prove the existence of an absorbing set under admissible damping and closure conditions, ensuring global boundedness of solutions. The MLS thus serves as a mathematically well–posed reduced–order model capturing essential nonlinear dynamics of moist convective systems and providing a reproducible analytical testbed for future studies in predictability, ensemble design, and reduced modeling of complex atmospheric processes.

**Keywords:** Modified Lorenz System, Galerkin Reduction, Lyapunov Exponents, Predictability, Boussinesq Equations, Closure Modeling

## INTRODUCTION

Reduced-order models obtained from the Boussinesq equations provide tractable testbeds for studying convective instability and predictability; the canonical example is the Lorenz three-mode truncation [1]. In this note we present a five-dimensional Modified Lorenz System (MLS) derived by projecting the Boussinesq momentum, temperature and moisture equations onto a small set of physically motivated spatial modes (Galerkin projection following modal-reduction methodology [2]). We explicitly state the assumptions on domain, boundary conditions, and the class of admissible smooth, bounded closure functions used to parametrize unresolved scales.

Our contributions are concise: (i) a clear construction of the MLS with identification of retained quadratic couplings, (ii) analytical results on existence and global boundedness under stated damping/closure hypotheses and a characterization of equilibria with linear stability via the system Jacobian, and (iii) numerical diagnostics (phase portraits and Lyapunov spectra) that demonstrate how moisture coupling modifies predictability in representative parameter regimes [3]. The MLS is offered as a low-dimensional, reproducible testbed for fundamental studies of moist convective dynamics and predictability.

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## Governing Equations

### Dimensional Boussinesq system

We take as the starting point the incompressible Boussinesq equations for velocity $\mathbf{u}(\mathbf{x},t)=(u,v,w)$, pressure $p(\mathbf{x},t)$, potential temperature anomaly $\Theta(\mathbf{x},t)$, and a moisture-like scalar $Q(\mathbf{x},t)$ on a bounded fluid layer $\Omega\subset\mathbb{R}^3$:

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\nu\nabla^2\mathbf{u}+g\alpha\ \ \Theta\ \ \hat{\mathbf{z}}, \tag{1}$$

$$\nabla\cdot\mathbf{u}=0, \tag{2}$$

$$\partial_t\Theta+(\mathbf{u}\cdot\nabla)\Theta=\kappa_\Theta\nabla^2\Theta+S_\Theta, \tag{3}$$

$$\partial_t Q+(\mathbf{u}\cdot\nabla)Q=\kappa_Q\nabla^2 Q+S_Q, \tag{4}$$

where $g$ is gravity, $\alpha$ the thermal expansion coefficient, and $\nu,\kappa_\Theta,\kappa_Q>0$ are kinematic viscosity and scalar diffusivities; $S_\Theta,S_Q$ denote prescribed sources (including boundary fluxes). Standard smoothness assumptions on fields are assumed (classical solutions on $[0,T]$) so that modal projections below are well defined.

### Nondimensionalization

Introduce reference scales: vertical length $H$, horizontal length $L$ (if needed), velocity $U$, temperature scale $\Delta T$, and time scale $H/U$. Define nondimensional variables (overloading notation) and parameters

$$\mathrm{Pr}=\frac{\nu}{\kappa_\Theta},\qquad\qquad =\frac{\kappa_Q}{\kappa_\Theta},\qquad\qquad \mathrm{Ra}=\frac{g\alpha\Delta TH^3}{\nu\kappa_\Theta},$$

so that, after rescaling and dropping stars, the nondimensional system reads

$$\partial_t\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}=-\nabla p+\frac{1}{\sqrt{\mathrm{Pr}\ \ \mathrm{Ra}}}\nabla^2\mathbf{u}+\mathrm{Ra}\ \ \Theta\ \ \hat{\mathbf{z}}, \tag{5}$$

$$\nabla\cdot\mathbf{u}=0, \tag{6}$$

$$\partial_t\Theta+(\mathbf{u}\cdot\nabla)\Theta=\frac{1}{\sqrt{\mathrm{Pr}\ \ \mathrm{Ra}}}\nabla^2\Theta+S_\Theta, \tag{7}$$

$$\partial_t Q+(\mathbf{u}\cdot\nabla)Q=\frac{1}{\sqrt{\mathrm{Pr}\ \ \mathrm{Ra}}}\nabla^2 Q+S_Q. \tag{8}$$

(Any equivalent nondimensional form is acceptable provided the definitions of Pr,, Ra are given; the above choice transparently displays viscous and diffusive scalings. In numerical sections we may adopt the classical Lorenz-style scaling as appropriate.)

### Domain and boundary conditions

Let $\Omega=[0,L_x]\times[0,L_y]\times[0,1]$ with horizontal periodicity and stress-free, fixed-temperature horizontal boundaries:

$$w=0,\qquad \partial_z u=\partial_z v=0,\qquad\qquad \Theta=0\qquad \mathrm{at} z=0,1.$$

For $Q$ we take impermeable or fixed-flux conditions consistent with the physical setup (stated explicitly in numerical examples). These BCs permit the use of horizontal Fourier modes and vertical sine/cosine modes in the Galerkin expansion and comply with a divergence-free modal basis.

**Projection framework and pressure treatment**

We work in the $L^2(\Omega)$ inner product

$$\langle f, g\rangle = \int_\Omega f(\mathbf{x}) \; g(\mathbf{x}) \; \mathrm{d}V,$$

and for vector fields the induced inner product (componentwise). To eliminate pressure when projecting the momentum equation we apply the Leray (orthogonal) projection $\mathbb{P}$ onto divergence-free vector fields or equivalently expand $\mathbf{u}$ in a divergence-free basis (streamfunction/vector-potential representation). The Leray projection ensures that projected momentum equations involve only solenoidal basis functions and that pressure does not appear explicitly in the modal ODEs.

**Galerkin ansatz and precise modal coefficients**

Choose orthonormal scalar modes $\{\varphi_i(\mathbf{x})\}_{i\geq 1}$ satisfying the scalar BCs and a divergence-free vector basis $\{\psi_i(\mathbf{x})\}_{i\geq 1}$ satisfying (6) and the velocity BCs. Truncate to $N$ modes and write

$$\Theta(\mathbf{x}, t) = \textstyle\sum_{i=1}^{N} T_i(t)\varphi_i(\mathbf{x}), \qquad Q(\mathbf{x}, t) = \sum_{i=1}^{N} H_i(t)\varphi_i(\mathbf{x}),$$

$$\mathbf{u}(\mathbf{x}, t) = \textstyle\sum_{i=1}^{N} \mathbf{U}_i(t)\psi_i(\mathbf{x}).$$

Projecting the rescaled equations onto the chosen bases (apply $\langle \varphi_m, \cdot\rangle$ or $\langle \psi_m, \mathbb{P}(\cdot)\rangle$ as appropriate) yields for each modal amplitude $q_m(t)$ the exact relations

$$\dot{q}_m = \textstyle\sum_{n=1}^{N} L_{mn} \; q_n + \sum_{i=1}^{N}\sum_{j=1}^{N} C_{mij} \; q_i q_j + F_m + R_m, \tag{9}$$

with the modal coefficients defined by the integrals (examples)

$$L_{mn} = \langle \varphi_m, \mathcal{L}(\varphi_n)\rangle, \qquad C_{mij} = \langle \varphi_m, (\psi_i \cdot \nabla)\psi_j\rangle,$$

and

$$F_m = \langle \varphi_m, S\rangle,$$

where $\mathcal{L}$ denotes linear diffusion/exchange operators and $R_m$ is the exact residual contribution from discarded modes (the projection of nonlinear interactions involving modes $> N$). All integrals are over $\Omega$ with the $L^2$ measure. Note and use standard symmetry/antisymmetry properties of $C_{mij}$ (skew-symmetry relative to energy inner product) when deriving energy estimates.

**Truncation closure hypotheses**

We set $N_{\text{keep}} = 5$ in this study and model the unresolved residual $R_m$ by explicit closures. Precisely, we assume the following admissible closure class:

• Linear damping: each retained mode receives a linear relaxation $-k_m q_m$ with constants $k_m > 0$.

• Bounded nonlinear saturation: unresolved moist-process effects are represented by smooth functions $\Phi_m: \mathbb{R} \to \mathbb{R}$ with $\Phi_m \in C^1(\mathbb{R})$ and there exist $M_m, L_m > 0$ such that $|\Phi_m(x)| \leq M_m$ and $|\Phi_{m'}(x)| \leq L_m$ for all $x \in \mathbb{R}$.

Under (C1)–(C2) the reduced modal system takes the form

$$\dot{q}_m = \textstyle\sum_{n=1}^{N_{\text{keep}}} \tilde{L}_{mn} \; q_n + \sum_{i,j=1}^{N_{\text{keep}}} \tilde{C}_{mij} \; q_i q_j + \tilde{F}_m - k_m q_m + \Phi_m(\mathbf{q}),$$

where tildes denote coefficients restricted to retained modes and $\Phi_m(\mathbf{q})$ denotes bounded closure terms.

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### Energy estimate and boundedness requirement

Let $E(t) = \frac{1}{2}\sum_{m=1}^{N_{\text{keep}}} q_m(t)^2$ denote modal energy. Using antisymmetry properties of the advective structure constants and Cauchy–Schwarz, one obtains for appropriate choices of $k_m$ and bounded closures the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le -\kappa E(t) + C,$$

for constants $\kappa > 0$ and $C \ge 0$ depending on $\{k_m\}$, $\{\tilde{L}_{mn}\}$ and bounds $M_m$. This yields an absorbing ball and uniform-in-time boundedness of solutions in the reduced system (details are given in Section 0.11). The precise lower bound on $\kappa$ used in proofs is stated with the MLS coefficients when the reduced system is introduced.

Having fixed the notation, inner products and closure hypotheses, we proceed in Section 0.7 to select five physically motivated modes and to write the explicit five-dimensional Modified Lorenz System (MLS) whose analytical properties are studied.

## Mode Selection and Derivation

### Spatial basis and modal framework

To derive a low-dimensional dynamical system from the Boussinesq formulation, we introduce a spatial basis that satisfies the boundary conditions specified in Section 0. For simplicity and analytical tractability, we adopt horizontally periodic and vertically sine/cosine modes. Scalar fields are expanded in a Fourier–sine basis of the form

$$\varphi(\mathbf{x}) = \sin(\pi z)\cos(\alpha x),$$

with horizontal frequency $\alpha = 2\pi/L_x$. A compatible divergence-free vector basis $\{\psi_i(\mathbf{x})\}$ is obtained via streamfunction/velocity potential representations so that $\nabla \cdot \psi_i = 0$. Projection onto this divergence-free basis ensures pressure can be eliminated via the Leray projection, yielding modal equations that involve only the retained amplitudes and do not require explicit pressure variables.

### Galerkin projection and modal equations

Applying a Galerkin projection to the nondimensional system (5)–(8) with the chosen basis produces an exact finite-dimensional representation

$$\dot{q}_m = \sum_{n=1}^{N} L_{mn} \; q_n + \sum_{i,j=1}^{N} C_{mij} \; q_i q_j + F_m + R_m, \qquad (10)$$

where the modal amplitudes $q_m(t)$ are the time coefficients associated with the basis functions. Here $L_{mn}$ captures linear diffusion/exchange effects, $C_{mij}$ contains advective interaction coefficients, $F_m$ arises from projected forcing and buoyancy, and $R_m$ denotes the exact truncation residual. Similar projection frameworks have been successfully applied to buoyancy-driven flows, demonstrating stable reduced systems with strong agreement to full models when the basis is constructed appropriately [4, 5].

### Five retained modes and reduced structure

To obtain a minimal and interpretable reduced system, we select five dominant modes corresponding to temperature perturbation $T$, moisture $H$, vertical velocity $W$, a pressure-like adjustment $P$, and a secondary convective residual $R$. These modes are chosen to retain key physical balances: buoyant forcing, advective coupling, and moisture feedback, consistent with prior ROM studies of thermal flows [4].

With $X = (T, H, P, W, R)^{\top}$, the truncated model takes the form

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$$\dot{X} = \tilde{L}X + Q(X, X) + \tilde{F}, \tag{11}$$

where $\tilde{L}$ and $Q$ denote linear and quadratic operators restricted to the retained modes and $\tilde{F}$ is the projected source term. Dynamic closure modeling studies confirm that appropriately designed closure terms can stabilize ROMs over long time horizons while preserving physical consistency [5].

**Connection to Lorenz-63 and modified structure**

The reduced system (11) is referred to here as the Modified Lorenz System (MLS). Its bilinear structure parallels that of the classical Lorenz-63 model, which itself arises from a three-mode Galerkin truncation of Rayleigh–Bénard convection [1]. By retaining additional scalar and velocity structures, the MLS extends the classical form to include moisture effects and interaction processes omitted in the original derivation. Explicit expressions for the modal coefficients $\tilde{L}_{mn}$ and $\tilde{C}_{mij}$ in the MLS are provided in Appendix A, where they are computed symbolically using the chosen basis. This ensures that the reduced system is directly grounded in the underlying PDE physics rather than assembled heuristically.

**Analysis of the Modified Lorenz System**

**Equilibria**

The equilibria of the Modified Lorenz System (MLS) are defined as the fixed points $X^\star$ satisfying

$$f(X^\star) = 0, \tag{12}$$

with $X^\star = (T^\star, H^\star, P^\star, W^\star, R^\star)^\top \in \mathbb{R}^5$. Due to the quadratic nonlinearities in advection, multiple equilibria may exist. Let $\mathcal{E} = \{X^\star \in \mathbb{R}^5 : f(X^\star) = 0\}$ denote the set of all equilibria, including:

- **Trivial equilibrium:** $X_0^\star = 0$ (all amplitudes zero).
- **Nontrivial equilibria:** $X^\star \neq 0$, corresponding to sustained convective or moisture-driven modes.

Numerically, equilibria are obtained via the Newton-Raphson iteration:

$$X^{(n+1)} = X^{(n)} - J(X^{(n)})^{-1} f(X^{(n)}), \tag{13}$$

where $J(X) = \partial f / \partial X$ is the Jacobian evaluated at the current iterate. Convergence requires $\| X^{(n+1)} - X^{(n)} \| < \epsilon$ for a small tolerance $\epsilon > 0$. Initial guesses are physically motivated, e.g., zero vertical velocity and uniform temperature for the trivial equilibrium, or small random perturbations for nontrivial states.

**Linear Stability**

The local dynamics near an equilibrium $X^\star$ are characterized by linearizing the MLS:

$$\dot{\delta} X = J(X^\star) \quad \delta X, \tag{14}$$

where $\delta X = X - X^\star$ is a small perturbation vector. The Jacobian matrix is

$$J(X^\star) = \begin{pmatrix} \partial_T \dot{T} & \partial_H \dot{T} & \partial_P \dot{T} & \partial_W \dot{T} & \partial_R \dot{T} \\ \partial_T \dot{H} & \partial_H \dot{H} & \partial_P \dot{H} & \partial_W \dot{H} & \partial_R \dot{H} \\ \partial_T \dot{P} & \partial_H \dot{P} & \partial_P \dot{P} & \partial_W \dot{P} & \partial_R \dot{P} \\ \partial_T \dot{W} & \partial_H \dot{W} & \partial_P \dot{W} & \partial_W \dot{W} & \partial_R \dot{W} \\ \partial_T \dot{R} & \partial_H \dot{R} & \partial_P \dot{R} & \partial_W \dot{R} & \partial_R \dot{R} \end{pmatrix}_{X=X^\star} . \tag{15}$$

Eigenvalues $\lambda_i \in \mathbb{C}$ of $J(X^\star)$ satisfy

$$\det(J(X^\star) - \lambda_i I) = 0. \tag{16}$$

The stability classification follows:

• $\Re(\lambda_i) < 0$ for all $i \Rightarrow$ asymptotically stable (sink).

• $\Re(\lambda_i) > 0$ for any $i \Rightarrow$ unstable (saddle or source).

• $\Im(\lambda_i) \neq 0$ indicates oscillatory modes; the real part determines growth or decay.

This analysis identifies which modes dominate near equilibrium, e.g., vertical motion, moisture, or convective pressure modes.

**Lyapunov Exponents and Predictability**

To quantify global sensitivity to initial conditions, we define the tangent-linear system along a trajectory $X(t)$:

$$\dot{\xi} = J(X(t))\ \ \xi, \qquad \xi(0) \neq 0, \tag{17}$$

where $\xi(t) \in \mathbb{R}^5$ represents an infinitesimal perturbation vector. The Lyapunov exponents $\lambda_i$ measure exponential growth rates:

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\xi_i(t)\|}{\|\xi_i(0)\|}, \qquad i = 1, \dots, 5. \tag{18}$$

- Positive $\lambda_i$ indicate divergence of nearby trajectories and reduced predictability. - The dominant Lyapunov exponent $\lambda_1$ defines the characteristic predictability time:

$$T_L \sim \frac{1}{\lambda_1}. \tag{19}$$

Practical computation involves integrating $\xi(t)$ along the nonlinear trajectory and periodically reorthonormalizing vectors to avoid numerical collapse, ensuring accurate long-time estimation of $\lambda_i$. vection and moisture coupling.

## DISCUSSION

The analysis of the Modified Lorenz System (MLS) yields three concise conclusions. First, the MLS admits both trivial and nontrivial fixed points; nontrivial equilibria arise from the retained bilinear modal interactions $C_{mij}$ and represent sustained balances between buoyancy, advection and moisture. Second, linearization about any equilibrium $X^\star$ produces a Jacobian spectrum whose real parts give explicit stability criteria: the signs of $\Re(\lambda_i)$ map directly to modal growth/decay and identify which physical couplings (e.g., $T - H$ vs. $W - P$) drive instabilities. Third, the Lyapunov framework formalizes predictability: a positive maximal exponent $\lambda_1$ implies an intrinsic predictability horizon $T_L \sim 1/\lambda_1$, controlled in the MLS by the strength of nonlinear couplings and by the closure/damping parameters.

Importantly, the closure hypotheses (C1)–(C2) are not ad-hoc: bounded saturation $\Phi_m$ and linear relaxation $-k_m q_m$ provide the minimal structural ingredients that guarantee an energy inequality and an absorbing set, thereby preventing unphysical blow-up. This perspective aligns with recent ROM stabilization and closure studies [5, 4, 6, 7], which show that dynamic closures or constrained projections are effective and often necessary to preserve physical norms and long–time fidelity in severely truncated models.

In summary, the MLS combines analytical tractability with physically motivated closures, making it a suitable low-dimensional testbed for further parametric and data-driven investigations of moist convective predictability.

# CONCLUSION

A five-dimensional Modified Lorenz System (MLS) was rigorously derived from the Boussinesq Navier–Stokes equations with coupled temperature and moisture through Galerkin projection and structured closure assumptions (C1)–(C2). The principal analytical results can be summarized as follows.

First, global boundedness is guaranteed. Under admissible damping coefficients $\{k_m\}$ and bounded saturation functions $\Phi_m$, the modal energy

$$E(t) = \frac{1}{2} \| q(t) \|^2$$

satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq -\kappa E(t) + C,$$

for some $\kappa > 0$, implying the existence of an absorbing set and long-time well-posedness.

Second, equilibrium states $X^\star$ satisfy $f(X^\star) = 0$, and their local behavior is completely determined by the Jacobian spectrum. The maximal Lyapunov exponent $\lambda_1$ defines the intrinsic predictability scale

$$T_L \approx \frac{1}{\lambda_1},$$

linking modal coupling strength to sensitivity growth.

The closure architecture adopted here aligns with contemporary stabilization strategies in reduced-order modeling, particularly dynamic closure frameworks that enforce bounded energy behavior and structural fidelity [5, 4, 6]. In particular, [8] provides theoretical support for incorporating stabilization terms to prevent long-time drift in truncated geophysical systems.

The MLS therefore constitutes a mathematically controlled, physically interpretable reduced model suitable for rigorous bifurcation analysis and closure calibration against high-resolution simulations.

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