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Super solvability and Freeness for ψ Graphical Arrangements

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ABSTRACT

We present a self-contained exposition of super solvability and freeness for ψ-graphical arrangements, including full statements of the main characterization theorems, proof sketches, worked examples, and a short discussion of methods used (modular chains, vertex-weighted graphs, and the addition–deletion techniques for free arrangements). The paper collects and arranges the results that answer Stanley's conjectures on ψ-graphical arrangements: the characterization of super solvability due to Mu & Stanley and the subsequent resolution of the freeness conjecture by Suyama & Tsujie. We highlight the key combinatorial and algebraic tools (chordality, modular chains in intersection lattices, and Terao's addition–deletion framework), indicate how they are applied in this context, and point out directions for further research.

Keywords: solvability, Stanley's conjectures, freeness conjecture, lattices.

INTRODUCTION

Hyperplane arrangements arising from graphs — *graphical arrangements,* form an important family that ties combinatorics, algebraic geometry and topology. The classical graphical arrangement A_G encodes edges of a simple graph (G) as hyperplanes (x_i-x_j = 0). Two central structural properties of arrangements are **supersolvability** (a lattice-theoretic condition introduced by Stanley) and **freeness** (an algebraic condition of the module of logarithmic derivations introduced by Terao). For (ordinary) graphical arrangements these properties are tightly linked to the graph-theoretic notion of *chordality*; in particular, a graphical arrangement A_G is supersolvable (and free) iff (G) is chordal.

Stanley later introduced a natural generalization, the **ψ-graphical arrangement** obtained by adding hyperplanes of the form $x_i = \alpha y$ (one extra coordinate (y) serves as a parameter) indexed by vertex-weights $\psi(v_i)$ This generalization prompted two conjectures: (1) a characterization of when a ψ -graphical arrangement is supersolvable, and (2) whether supersolvability and freeness remain equivalent in this generalized setting. Mu & Stanley proved the first conjecture and provided groundwork toward the second; Suyama & Tsujie subsequently proved the freeness conjectu re under the appropriate formulation. In what follows we collect definitions, state the main theorems, sketch proofs, and give examples suitable for journal publication.

Preliminaries

Notation and basic definitions

- Let G=(V,E) be a finite simple graph with vertex set V=$\{v_1, v_2, \ldots, v_n\}$.
- For a field (K), of characteristic 0 or large enough, the **graphical arrangement** A_G in K^n is the set of hyperplanes $A_G = \{x_i - x_j = 0: (i, j) \in E\}$.
- A **ψ-graphical arrangement** $A_{G,\psi}$ is a subarrangement of the extended braid arrangement in K^{n+1} with coordinates $\{x_1, x_2, \ldots, x_n, y\}$ consisting of:

○ the usual hyperplanes $\{x_i - x_j = 0: (i, j) \in E\}$, and ○ for each vertex v_i and each $\alpha \in \psi(v_i)$, a finite multiset or list of scalars in K, the hyperplane $x_i -$

$\alpha y = 0$ ○ This follows the construction and notation of Mu & Stanley.

Supersolvable lattices and free arrangements

- An arrangement $\mathcal{A}$ is supersolvable if the intersection lattice $L(c(A))$ the lattice of intersections of hyperplanes of the cone over $\mathcal{A}$contains a maximal chain of modular elements.
- An arrangement $\mathcal{A}$ is **free** (in the sense of Terao/Saito) if its module (D($\mathcal{A}$)) of logarithmic derivations is a free (S)-module (where (S) is the coordinate polynomial ring). Terao's addition–deletion theorem is the central practical tool for constructing and verifying freeness.

Main results (statements)

We collect the principal theorems relevant to ψ-graphical arrangements.

Theorem A: Supersolvability characterization — Mu & Stanley

Let $A_{G,\psi}$ be a ψ −graphical arrangement as above. Then$A_{G,\psi}$ is supersolvable **if and only if** the underlying graph G is chordal and the vertex-labeling ψ satisfies a compatibility condition described as there exists a perfect elimination ordering of (G) compatible with the ψ -labels so that the chain of modular elements arises from this ordering.

For the ordinary graphical arrangement $\psi(v) = \phi$, this recovers the classical result as A_G is supersolvable iff G is chordal.

Theorem B: Freeness equivalence — Suyama & Tsujie

Under the same hypotheses and set-up for ψ-graphical arrangements, freeness of $A_{G,\psi}$ in Terao's sense is equivalent to the supersolvability characterization in Theorem A. In particular, the combinatorial conditions that characterize supersolvability also characterize freeness. Suyama & Tsujie, who prove the conjecture originally proposed by Stanley.

Sketches of proofs and methods

Below we outline the ideas behind the proofs of Theorems A and B full, rigorous proofs appear in the cited literature.

Proof sketch for Theorem A: Mu & Stanley

1. **Intersection-lattice description:** The cone $C(A_{G,\psi})$has an intersection lattice that can be described combinatorially in terms of *connected partitions* of (V) together with label-equalities involving the extra coordinate (y). Mu & Stanley identify the relevant interval of the partition lattice and determine when a maximal chain of modular elements exists.
2. **Perfect elimination and modular chains:** For ordinary graphical arrangements, a perfect elimination ordering (PEO) of a chordal graph gives rise to the maximal modular chain. Mu & Stanley generalize

this by showing that if G is chordal and the label-sets $\psi(v)$ admit choices compatible with a PEO, the same construction yields a maximal chain of modular elements in L($C(A_{G,\psi})$. Conversely, existence of such a chain forces chordality and the label compatibility. The argument is lattice-theoretic and combinatorial.

Proof sketch for Theorem B: Suyama & Tsujie

1. **Vertex-weighted graphs viewpoint.** Suyama & Tsujie rephrase ψ -graphical arrangements as arrangements associated to vertex-weighted graphs; this allows graph-theoretic manipulations and use of deletion–restriction sequences reflecting edge/vertex removals.

2. **Application of addition–deletion and induction.** Using Terao's addition–deletion theorem (and related inductive/free-path techniques), they show that when the combinatorial conditions of Theorem A hold one may build the arrangement inductively via additions of hyperplanes whose restrictions and deletions maintain freeness — concluding freeness. The reverse implication (freeness ⇒ those combinatorial conditions) uses localizations and factorization properties of the characteristic polynomial together with combinatorial constraints that force chordality/label-compatibility. The result is a complete equivalence. **5. Examples**

Example 1: Ordinary chordal graph

Let (G) be a chordal graph and take $\psi(v) = \phi$for all (v). Then $A_{G,\psi} = A_G$ is supersolvable and free.

Example 2: ψ-labels on a chordal graph

Let G be a 4-vertex chordal graph with a PEO $\{v_1, v_2, v_3, v_4\}$. Assign $\psi(v_1) = \alpha, \psi(v_2) = \alpha, \psi(v_3) = \beta, \psi(v_4) = \phi$ and $\alpha \neq \beta$. If the label-equalities are compatible with the PEO (i.e., whenever a later vertex has a label that could force a non-modular intersection one avoids those label collisions), then Mu & Stanley's criterion gives supersolvability, Suyama & Tsujie's theorem gives freeness. Explicit coordinate hyperplanes and a concrete modular chain can be written downsee.

Techniques and connections

- **Lattice methods:** The characterization of supersolvability is fundamentally lattice-theoretic. Identifying maximal chains of modular elements in the intersection lattice and relating them to perfect elimination orderings (PEOs) of chordal graphs is the key combinatorial link.
- **Addition–Deletion Terao and inductive freeness:** The addition–deletion theorem (Terao) and variants (Abe and collaborators' work on addition-type constructions) give the inductive machinery to prove freeness once the combinatorial structure is right. Suyama & Tsujie implement this via vertexweighted/deletion-restriction arguments.
- **Vertex-weighted graphs perspective:** Representing label-hyperplane $x_i = \alpha y$ as vertex-weights and using graph-theoretic operations deleting vertices/edges consistent with weight-handling, makes the combinatorial verification approachable.

Open problems and further directions

1. **Broader classes of labelings:** The Mu–Stanley result requires a certain finiteness and compatibility of ψ-labels. One can ask for extensions to multiset labels with multiplicities or labels drawn from rings with special structure (e.g., finite fields) and whether the equivalence with freeness persists in those generalities.
2. **Matroidal and graphic generalizations:** Analogues for other Coxeter-type subarrangements (beyond type A) or for arrangements associated to signed/ gain graphs (signed graphic arrangements) are active areas; recent work on signed-graphic arrangements explores freeness under chordality-like conditions.
3. **Algorithmic aspects:** Given a vertex-weighted graph, design efficient algorithms to test the Mu–Stanley combinatorial condition (label-compatibility with a PEO) and thus decide supersolvability/freeness. Connections to chordal graph recognition algorithms suggest polynomial-time approaches.

CONCLUSION

ψ-graphical arrangements provide a natural and fruitful generalization of graphical arrangements. The combined work of Mu & Stanley (supersolvability characterization) and Suyama & Tsujie (freeness equivalence) shows that the deep interplay between graph chordality, lattice modularity, and algebraic freeness survives this generalization under natural hypotheses. The techniques used (lattice combinatorics, vertex-weighted graph interpretations, and addition–deletion freeness machinery) are broadly applicable and suggest a number of promising further directions.

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