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# Early Coefficient Bounds and Fekete–Szegö Inequality for a Subclass of Analytic Functions Defined by a New Generalized Differential Operator

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## ABSTRACT

In this paper, a new subclass, $\mathcal{E}_{\psi,\kappa,\tau}(\sigma,\lambda,\mu,\alpha,\beta,\delta,\eta,l,t)$ of analytic functions, defined through a new generalized differential operator $D^{m}_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)$ and the Janowski function is introduced and analyzed. The subclass $\mathcal{E}_{\psi,\kappa,\tau}(\sigma,\lambda,\mu,\alpha,\beta,\delta,\eta,l,t)$, is constructed via subordination involving a linear combination of the operator and its derivatives. For this class, sharp bounds for the initial coefficients $|a_2|$ and $|a_3|$ were established and a precise form of the Fekete–Szegö inequality derived.

**Keywords:** Univalent functions, generalized differential operator, Unit disc, Janowski Function, subordination, coefficient bounds, Fekete–Szegö inequality.

## INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

which are analytic in the open unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

where $\mathbb{C}$ is the set of complex numbers and normalized with $f(0) = 0$ and $f'(0) = 1$.

A key concept in this paper is that of subordination. For two functions $g(z)$ and $h(z)$ in $\mathcal{A}$, we say that $g(z)$ is subordinate to $h(z)$, written $g(z) \prec h(z)$, if there exists a Schwarz function $w(z)$ which is analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $g(z) = h(w(z))$ for all $z \in \mathbb{D}$

In recent years, considerable attention has been devoted to the construction of new differential operators in order to generate wider families of analytic functions and to explore their structural and geometric properties. Such operators have led to significant developments, particularly in obtaining coefficient estimates, distortion and growth theorems, neighborhood results, inclusion relationships, and bounds for Hankel determinants.

Motivated by these ongoing developments, a generalized differential operator

$$D^{m}_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)f(z),$$

was introduced [10] which extends and unifies several well-known operators previously studied in the literature.

**Definition 1** [New Generalized Differential Operator] (See [10, 17]) For $f \in \mathcal{A}$, the new generalized differential operator is defined as follows:

Let

$$D^0_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)f(z) = f(z)$$

$$D^1_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)f(z) = \left(\frac{(\mu+\lambda)[l+(1+(n+\eta-\delta-1)t)]-(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right)f(z)$$

$$+\left(\frac{(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right)zf'(z)$$

$$D^m_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)f(z) = D_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)\left(D^{m-1}_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)\right)f(z) \quad (2)$$

for $l,\alpha,\sigma,t \geq 0, \quad \beta,\lambda,\mu > 0, \quad \lambda \neq \alpha, \quad 0 \leq \eta \leq \delta, \quad m \in N_0$

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then from (2),

$$D^m_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)f(z) = z + \sum_{n=2}^{\infty} \Omega_n a_n z^n, \quad m \in N_0 \quad (3)$$

where

$$\Omega_n = \left(\frac{(\mu+\lambda)[l+(1+(n+\eta-\delta-1)t)]+(n-1)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right)^m. \quad (4)$$

When $l = t = 0, \quad \beta = \sigma = 1, \quad D^m_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)f(z) = f(z).$

The flexibility of this operator, arising from its multiple parameters, allows for the recovery of numerous earlier operators as special cases (see [2,3,6,9,11,14,15]) and provides a broader framework for investigating analytic function classes.

**Definition 2** (Janowski Function). (see [8,16,4,7,13]) For real parameters P and Q satisfying

$-1 \leq Q < P \leq 1$, the function

$$\Phi_{P,Q}(z) = \frac{1+Pz}{1+Qz}, \quad z \in \mathbb{D}, \quad (5)$$

is called a Janowski function.

The function $\Phi_{P,Q}(z)$ is analytic and univalent in $\mathbb{D}$. It serves as a dominant function in subordination theory and plays a central role in defining subclasses of analytic functions with prescribed geometric properties. The Janowski function generalizes several well-known functions:

- For $P = 1$ and $Q = -1$, we obtain

$$\Phi_{P,Q}(z) = \frac{1+z}{1-z},$$

(see [5,1,12]).

- Different choices of P and Q yield various subclasses of starlike, convex, and close-to-convex functions.

To obtain its Taylor series expansion about $z = 0$, we rewrite $\Phi_{P,Q}(z)$ as

$$\Phi_{P,Q}(z) = (1 + Pz)\frac{1}{1 + Qz}.$$

Using the geometric series expansion

$$\frac{1}{1 + Qz} = \sum_{n=0}^{\infty}(-Q)^n z^n, \quad |z| < 1,$$

we obtain

$$\Phi_{P,Q}(z) = (1 + Pz)\sum_{n=0}^{\infty}(-Q)^n z^n.$$

Expanding and collecting terms, we get

$$\Phi_{P,Q}(z) = \sum_{n=0}^{\infty}(-Q)^n z^n + Pz\sum_{n=0}^{\infty}(-Q)^n z^n.$$

This gives

$$\Phi_{P,Q}(z) = 1 + \sum_{n=1}^{\infty}[(-Q)^n + P(-Q)^{n-1}] z^n.$$

Factoring the coefficient, we obtain

$$\Phi_{P,Q}(z) = 1 + \sum_{n=1}^{\infty}(P - Q)\,(-Q)^{n-1}z^n. \tag{6}$$

Thus, the Taylor series expansion of the Janowski function is given by

$$\Phi_{P,Q}(z) = 1 + (P - Q)z + (P - Q)(-Q)z^2 + (P - Q)(-Q)^2z^3 + \cdots, \tag{7}$$

or, in compact form,

$$\Phi_{P,Q}(z) = 1 + \sum_{n=1}^{\infty}(P - Q)\,(-Q)^{n-1}z^n, \quad |z| < 1.$$

To define the new subclass, an associated linear combination of an operated function and its derivatives is constructed as follows:

Let $F(z) = D^{m}_{\mu,\lambda,\sigma}(\alpha,\beta,\delta,\eta,l,t)f(z)$ and define $G(z)$ using two additional real parameters, $\theta$ and $\Delta$:

$$G(z) := \theta\Delta\, z^2F''(z) + (\theta - \Delta)\, zF'(z) + (1 - \theta + \Delta)\, F(z), \tag{8}$$

where

$$F(z) = z + \sum_{n=2}^{\infty}\Omega_n\, a_n z^n, \quad F'(z) = 1 + \sum_{n=2}^{\infty} n\, \Omega_n a_n z^{n-1}, \quad F''(z) = \sum_{n=2}^{\infty} n\,(n-1)\,\Omega_n a_n z^{n-2}. \tag{9}$$

Plugging (9) into (8):

$$G(z)$$

$$= \theta\Delta\, z^2 \sum_{n=2}^{\infty} n\,(n-1)\Omega_n a_n z^{n-2} \; + \; (\theta-\Delta)\, z\left(1+\sum_{n=2}^{\infty} n\;\Omega_n a_n z^{n-1}\right) + \; (1-\theta+\Delta)\left(z+\sum_{n=2}^{\infty} \Omega_n\, a_n z^n\right).$$

$$G(z) = z + \sum_{n=2}^{\infty}\big(\theta\Delta\, n(n-1) \; + \; (\theta-\Delta)n + \; (1-\theta+\Delta)\big)\,\Omega_n a_n z^n.$$

Let

$$\Sigma_n := \theta\Delta\, n(n-1) + (\theta-\Delta)\, n + (1-\theta+\Delta), \qquad \therefore \;\; G(z) = z + \sum_{n=2}^{\infty} \Sigma_n\;\Omega_n\, a_n\, z^n. \tag{10}$$

It is convenient to name the first two composite coefficients:

$$A := \Sigma_2\, \Omega_2\, a_2 = (2\theta\Delta + \theta - \Delta + 1)\, \Omega_2\, a_2, \quad B := \Sigma_3\, \Omega_3\, a_3 = (6\theta\Delta + 2\theta - 2\Delta + 1)\, \Omega_3\, a_3.$$

**Definition 3**. Let $\psi, \kappa \in \mathbb{R}$ with $0 \le \psi \le 1$ and $\kappa \ge 0$, and let $\tau > 0$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{E}_{\psi,\kappa,\tau}(\sigma,\lambda,\mu,\alpha,\beta,\delta,\eta,l,t)$ if it satisfies the subordination condition:

$$\psi\left(\frac{zG'(z)}{G(z)}\right)^{\tau} + (1-\psi)\left(\frac{zG'(z)}{G(z)}\right)^{\kappa}\left(1+\frac{zG''(z)}{G'(z)}\right)^{1-\kappa} \prec \Phi_{P,Q}(z), \tag{11}$$

where $G(z)$ is given by (10) and $\Phi_{P,Q}(z)$ is the Janowski function.

## RESULTS AND DISCUSSION

To prove the main results for the class $\mathcal{E}_{\psi,\kappa,\tau}(\sigma,\lambda,\mu,\alpha,\beta,\delta,\eta,l,t)$, it requires the Taylor series expansions of the terms in (11). Since $\Phi_{P,Q}\big(\omega(z)\big)$ is analytic in $\mathbb{D}$, the subordination implies the existence of a Schwarz function $\omega(z)$ such that the right-hand side of (11) is shown as follows:

$$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + \cdots,$$

where $\omega(0) = 0$ and $|\omega(z)| < 1$.

Then,

$$\Phi_{P,Q}\big(\omega(z)\big) = \frac{1+P\omega(z)}{1+Q\omega(z)} = 1 + (P-Q)\omega(z) - (P-Q)Q\omega(z)^2 + (P-Q)Q^2\omega(z)^3 + \cdots.$$

Substituting the series expansion of $\omega(z)$ and simplifying, we obtain

$$\Phi_{P,Q}\big(\omega(z)\big) = 1 + (P-Q)c_1 z + (P-Q)(c_2 - Qc_1^2)z^2 + (P-Q)(c_3 - 2Qc_1c_2 + Q^2c_1^3)z^3 + \cdots \tag{12}$$

For the left-hand side of (11), first expand the geometric quantities.

$$G(z) = z + Az^2 + Bz^3 + \cdots, \qquad G'(z) = 1 + 2Az + 3Bz^2 + \cdots, \qquad G''(z) = 2A + 6Bz + \cdots.$$

$$G(z) = z\,(1 + Az + Bz^2 + \cdots).$$

Expansions for $[G(z)]^{-1}$, $[G'(z)]^{-1}$, and the two normalized quotients $\frac{zG'(z)}{G(z)}$ and $1+\frac{zG''(z)}{G'(z)}$ are needed. Therefore,

$$(1+u)^{-1} = 1 - u + u^2 + \cdots \quad \Rightarrow \quad (1+Az+Bz^2)^{-1} = 1-(Az+Bz^2)+(Az+Bz^2)^2+\cdots.$$

$$1 - Az + Bz^2 + A^2z^2 + 2ABz^3 + B^2z^4 + \cdots = 1 - Az + (A^2-B)z^2 + \cdots.$$

Hence

$$^{-1} = \frac{1}{z}(1 - Az + (A^2-B)z^2 + \cdots) = \frac{1}{z} - A + (A^2-B)z + \cdots.$$

Similarly, for $G'(z) = 1 + 2Az + 3Bz^2 + \cdots$,

$$(1+2Az+3Bz^2)^{-1} = 1 - 2Az + (4A^2-3B)z^2 + \cdots,$$

so

$$^{-1} = 1 - 2Az + (4A^2-3B)z^2 + \cdots.$$

Now we compute

$$\frac{zG'(z)}{G(z)} = \frac{z(1+2Az+3Bz^2+\cdots)}{z(1+Az+Bz^2+\cdots)} = (1+2Az+3Bz^2)\,(1-Az+(A^2-B)z^2)+\cdots.$$

Multiply term-by-term up to $z^2$, we have:

$$= 1 - Az + A^2z^2 - Bz^2 + 2Az - 2A^2z^2 + 2A^3z^3 - 2ABz^3 + 3Bz^2 - 3ABz^3 + 3A^2Bz^4 - 3B^2z^4 + \cdots$$

Thus

$$1 + (2A-A)z + (A^2 - B - 2A^2 + 3B)z^2 + \cdots = 1 + Az + (2B - A^2)z^2 + \cdots.$$

Next,

$$1 + \frac{zG''(z)}{G'(z)} = 1 + z(2A+6Bz)\,(1-2Az+(4A^2-3B)z^2)+\cdots.$$

Starting with $\frac{zG''(z)}{G'(z)}$ up to $z^2$:

$$\frac{zG''(z)}{G'(z)} = z(2A+6Bz)\,(1-2Az+\cdots) = 2Az + (6B-4A^2)z^2 + \cdots,$$

hence

$$1 + \frac{zG''(z)}{G'(z)} = 1 + 2Az + (6B-4A^2)z^2 + \cdots.$$

Now expand $\left(\frac{zG'(z)}{G(z)}\right)^{\tau}$:

Let

$$S(z) := \frac{zG'(z)}{G(z)} = 1 + Az + (2B-A^2)z^2 + \cdots.$$

Using $(1+u)^{\tau} = 1 + \tau u + \frac{\tau(\tau-1)}{2}u^2 + \cdots$ with $u = Az + (2B - A^2)z^2$,

$$S(z) = 1 + \tau(Az + (2B - A^2)z^2) + \frac{\tau(\tau-1)}{2}(Az + (2B - A^2)z^2)^2 + \cdots$$

Thus

$$\left(\frac{zG'(z)}{G(z)}\right)^{\tau} = 1 + \tau A\, z + (2\tau B + \tau(\tau-3)/2\, A^2)z^2 + \cdots.$$

and multiplying by $\psi$ we have:

$$\psi\left(\frac{zG'(z)}{G(z)}\right)^{\tau} = \psi + \psi\tau A\, z + (2\psi\tau B + \psi\tau(\tau-3)/2\, A^2)z^2 + \cdots. \qquad (13)$$

Expand $\left(\frac{zG'(z)}{G(z)}\right)^{\kappa}(1 + \frac{zG''(z)}{G'(z)})^{1-\kappa}$

First,

$$\left(\frac{zG'(z)}{G(z)}\right)^{\kappa} = 1 + \kappa Az + (2\kappa B + \kappa(\kappa-3)/2\, A^2)z^2 + \cdots, \qquad (14)$$

by the same binomial computation as above with $\tau \to \kappa$.

Second,

$$\left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\kappa} = (1 + 2Az + (6B - 4A^2)z^2)^{1-\kappa}$$

Let $\gamma := 1 - \kappa$ and $w = 2Az + (6B - 4A^2)z^2$,

$$(1+w)^{\gamma} = 1 + \gamma w + \frac{\gamma(\gamma-1)}{2}w^2 = 1 + 2\gamma Az + (\gamma(6B - 4A^2) + 2\gamma(\gamma-1)A^2)z^2.$$

Therefore,

$$\left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\kappa} = 1 + 2(1-\kappa)Az + (6(1-\kappa)B - (4+2\kappa)(1-\kappa)A^2)z^2 + \cdots, \qquad (15)$$

Now multiplying (14) and (15),

$$\left(\frac{zG'(z)}{G(z)}\right)^{\kappa}\left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\kappa} = 1 + (2-\kappa)A\, z + (2(3-2\kappa)B + \kappa^2 + 5\kappa - 8/2\, A^2)z^2 + \cdots. \qquad (16)$$

Multiply (16) by $(1-\psi)$:

$$\begin{aligned}(1-\psi)\left(\frac{zG'(z)}{G(z)}\right)^{\kappa}\left(1 + \frac{zG''(z)}{G'(z)}\right)^{1-\kappa} &= (1-\psi) + (1-\psi)(2-\kappa)A\, z \\ &\quad + (2(1-\psi)(3-2\kappa)B + 1 - \psi/2\,(\kappa^2 + 5\kappa - 8)A^2)z^2 + \cdots. \qquad (17)\end{aligned}$$

Summing (13) and (17):

$$\psi(\frac{zG'(z)}{G(z)})^{\tau} + (1-\psi)(\frac{zG'(z)}{G(z)})^{\kappa}(1+\frac{zG''(z)}{G'(z)})^{1-\kappa} = 1 + \underbrace{[\psi\tau + (1-\psi)(2-\kappa)]}_{=:B_1} A\, z$$

$$+\left\{ \underbrace{2[\psi\tau + (1-\psi)(3-2\kappa)]}_{=:K_1} B + \underbrace{\frac{1}{2}[\psi\tau(\tau-3) + (1-\psi)(\kappa^2+5\kappa-8)]}_{=:R_1} A^2 \right\} z^2 + \cdots .$$

$$\psi(\frac{zG'(z)}{G(z)})^{\tau} + (1-\psi)(\frac{zG'(z)}{G(z)})^{\kappa}(1+\frac{zG''(z)}{G'(z)})^{1-\kappa} = 1 + B_1\, A\, z + \{\, K_1\, B + R_1\, A^2\} z^2 + \cdots. \quad (19)$$

**Theorem 1** (Coefficient estimates). Let $f \in \mathcal{E}_{\psi,\kappa,\tau}(\sigma, \lambda, \mu, \alpha, \beta, \delta, \eta, l, t)$ with parameters such that $\Omega_2\Omega_3\, \Sigma_2\Sigma_3\, B_1K_1 \neq 0$. Then the Taylor coefficients $a_2$ and $a_3$ of $f$ satisfy

$$|a_2| \leq \frac{|P-Q|}{|B_1|\,|\Sigma_2|\,|\Omega_2|}.$$

$$|a_3| \leq \frac{|P-Q|}{|K_1|\,|\Sigma_3|\,|\Omega_3|} max\{1, |V|\}.$$

where $\Sigma_n := \theta\Delta\, n(n-1) + (\theta - \Delta)\, n + (1 - \theta + \Delta)$, $\Omega_n$ is defined in (4),

$B_1 = \psi\tau + (1-\psi)(2-\kappa)$,

$K_1 = 2[\psi\tau + (1-\psi)(3-2\kappa)]$,

$R_1 = \frac{1}{2}[\psi\tau(\tau-3) + (1-\psi)(\kappa^2+5\kappa-8)]$, and

$V = Q + \frac{R_1}{B_1^2}(P-Q)$.

Proof. Equating the coefficients of $z$ and $z^2$ in (12) and (19), we have

$$B_1\, A = (P-Q)c_1, \qquad K_1\, B + R_1\, A^2 = (P-Q)(c_2 - Qc_1^2)$$

Recall that

$$A := \Sigma_2\, \Omega_2\, a_2, \quad B := \Sigma_3\, \Omega_3\, a_3.$$

Hence,

$$B_1\, \Sigma_2\, \Omega_2\, a_2 = (P-Q)c_1$$

$$a_2 = \frac{(P-Q)c_1}{B_1\, \Sigma_2\, \Omega_2} \quad (20)$$

$$K_1\, \Sigma_3\, \Omega_3\, a_3 + R_1\, (\Sigma_2\, \Omega_2\, a_2)^2 = (P-Q)(c_2 - Qc_1^2)$$

$$K_1\, \Sigma_3\, \Omega_3\, a_3 = (P-Q)(c_2 - Qc_1^2) - R_1\, (\Sigma_2\, \Omega_2\, a_2)^2$$

$$K_1\, \Sigma_3\, \Omega_3\, a_3 = (P-Q)(c_2 - Qc_1^2) - R_1 \left( \Sigma_2\, \Omega_2 \cdot \frac{(P-Q)c_1}{B_1\, \Sigma_2\, \Omega_2} \right)^2$$

$$K_1\,\Sigma_3\,\Omega_3\,a_3 = (P-Q)(c_2 - Qc_1^2) - \left(\frac{R_1}{B_1^2}(P-Q)^2\right)c_1^2$$

$$a_3 = \frac{(P-Q)}{K_1\,\Sigma_3\,\Omega_3}\left[c_2 - Qc_1^2 - \left(\frac{R_1}{B_1^2}(P-Q)\right)c_1^2\right]$$

$$a_3 = \frac{(P-Q)}{K_1\,\Sigma_3\,\Omega_3}\left[c_2 - \left(\underbrace{Q + \frac{R_1}{B_1^2}(P-Q)}_{=:V}\right)c_1^2\right] \qquad (21)$$

Since $|c_1| \le 1$, from (20) we have

$$|a_2| \le \frac{|P-Q|}{|B_1|\,|\Sigma_2|\,|\Omega_2|}$$

Now applying the Schwarz-function sharp inequality

$$|c_2 - vc_1^2| \le max\{1, |v|\} \quad \text{with } v := V,$$

we obtain

$$|a_3| \le \frac{|P-Q|}{|K_1|\,|\Sigma_3|\,|\Omega_3|}max\{1, |V|\}$$

**Corollary 1**. For $P = 1$ and $Q = -1$

$$|a_2| \le \frac{2}{|B_1|\,|\Sigma_2|\,|\Omega_2|}$$

$$|a_3| \le \frac{2}{|K_1|\,|\Sigma_3|\,|\Omega_3|}max\left\{1, \left|\frac{2R_1}{B_1^2} - 1\right|\right\}$$

**Remark 1**. The estimates $|a_2|$ and $|a_3|$ in corollary 1 are the coefficient estimates that will be obtained for functions with positive real parts instead of the Janowski Function.

**Theorem 2** (Fekete-Szegö Inequality). Let $f \in \mathcal{E}_{\psi,\kappa,\tau}(\sigma, \lambda, \mu, \alpha, \beta, \delta, \eta, l, t)$ with parameters satisfying $K_1\Sigma_3\Omega_3 \ne 0$ and $B_1\Sigma_2\Omega_2 \ne 0$. Then, for any real number $\xi$, the functional $a_3 - \xi a_2^2$ satisfies

$$|a_3 - \xi a_2^2| \le \frac{|P-Q|}{|K_1|\,|\Sigma_3|\,|\Omega_3|} \times \begin{cases} 1, & \text{if } \xi \in [\xi_1, \xi_2], \\ |V + \xi T|, & \text{if } \xi \notin [\xi_1, \xi_2], \end{cases}$$

where

$$T = \frac{(P-Q)K_1\,\Sigma_3\,\Omega_3}{B_1^2\,\Sigma_2^2\,\Omega_2^2},$$

$$\xi_1 := min\left\{\frac{-1-V}{T}, \frac{1-V}{T}\right\}, \qquad \xi_2 := max\left\{\frac{-1-V}{T}, \frac{1-V}{T}\right\},$$

and

$$V = Q + \frac{R_1}{B_1^2}(P - Q).$$

When $T = 0$ the piecewise description collapses to a constant bound.

Proof. From (20) and (21),

$$a_2 = \frac{(P - Q)c_1}{B_1\,\Sigma_2\,\Omega_2} \qquad a_3 = \frac{(P - Q)}{K_1\,\Sigma_3\,\Omega_3}[c_2 - Vc_1^2]$$

$$a_3 - \xi a_2^2 = \frac{(P - Q)}{K_1\,\Sigma_3\,\Omega_3}[c_2 - Vc_1^2] - \xi\,\frac{(P - Q)^2}{B_1^2\,\Sigma_2^2\,\Omega_2^2}c_1^2$$

$$a_3 - \xi a_2^2 = \frac{P - Q}{K_1\,\Sigma_3\,\Omega_3}\left[c_2 - \left(V + \xi\ \underbrace{\frac{(P - Q)K_1\,\Sigma_3\,\Omega_3}{B_1^2\,\Sigma_2^2\,\Omega_2^2}}_{=:T}\right)c_1^2\right]$$

where the constant, $T$ is defined as

$$T = \frac{(P - Q)K_1\,\Sigma_3\,\Omega_3}{B_1^2\,\Sigma_2^2\,\Omega_2^2}.$$

Let $U = V + \xi T$, then

$$a_3 - \xi a_2^2 = \frac{P - Q}{K_1\,\Sigma_3\,\Omega_3}(c_2 - Uc_1^2)$$

Now applying the Schwarz-function sharp inequality

$$|c_2 - vc_1^2| \le max\{1, |v|\} \quad \text{with } v := U,$$

we have

$$|a_3 - \xi a_2^2| \le \frac{|P - Q|}{|K_1|\,|\Sigma_3|\,|\Omega_3|}\,max\{1, |U|\}$$

- If $|U| = |V + \xi T| \le 1$, then $max\{1, |U|\} = 1$. In this case the bound reduces to the constant

$$|a_3 - \xi a_2^2| \le \frac{|P - Q|}{|K_1|\,|\Sigma_3|\,|\Omega_3|},$$

which is independent of $\xi$.

- If $|U| = |V + \xi T| > 1$, then $max\{1, |U|\} = |V + \xi T|$, and the bound becomes

$$|a_3 - \xi a_2^2| \le \frac{|P - Q|}{K_1\,|\Sigma_3|\,|\Omega_3|}\,|V + \xi T|.$$

Assume $T \neq 0$. Then

$$-1 \le V + \xi T \le 1 \quad \Leftrightarrow \quad \frac{-1 - V}{T} \le \xi \le \frac{1 - V}{T} \quad \text{if } T > 0,$$

and the inequalities reverse if $T < 0$. The clean way to state the interval is

$$\xi_1 := min\left\{\frac{-1-V}{T}, \frac{1-V}{T}\right\}, \qquad \xi_2 := max\left\{\frac{-1-V}{T}, \frac{1-V}{T}\right\}.$$

Hence, the sharp Fekete–Szegö inequality is

$$|a_3 - \xi a_2^2| \le \begin{cases} \dfrac{|P-Q|}{K_1\,|\Sigma_3|\,|\Omega_3|}, & \text{if } \xi \in [\xi_1, \xi_2], \\ \dfrac{|P-Q|}{K_1\,|\Sigma_3|\,|\Omega_3|}\,|V + \xi T\,|, & \text{if } \xi \notin [\xi_1, \xi_2]. \end{cases}$$

Under the hypotheses and notation above, assume $P - Q \neq 0$ and that $B_1\Sigma_2\Omega_2 \neq 0$, $K_1\Sigma_3\Omega_3 \neq 0$. Then, for any $\xi \in \mathbb{R}$,

$$|a_3 - \xi a_2^2| \le \begin{cases} \dfrac{|P-Q|}{K_1\,|\Sigma_3|\,|\Omega_3|}, & \text{if } \xi \in [\xi_1, \xi_2], \\ \dfrac{|P-Q|}{K_1\,|\Sigma_3|\,|\Omega_3|}\left|Q + \dfrac{R_1}{B_1^2}(P-Q) \;-\; \xi \cdot \dfrac{(P-Q)K_1\,\Sigma_3\,\Omega_3}{B_1^2\,\Sigma_2^2\,\Omega_2^2}\right|, & \text{if } \xi \notin [\xi_1, \xi_2], \end{cases}$$

where $\xi_1, \xi_2$ are given above, and

$$\begin{aligned} B_1 &= \psi\tau + (1-\psi)(2-\kappa), \\ K_1 &= 2[\psi\tau + (1-\psi)(3-2\kappa)], \\ R_1 &= \psi\tau(\tau-3) + (1-\psi)(\kappa^2 + 5\kappa - 8), \\ \Sigma_2 &= 2\theta\Delta + \theta - \Delta + 1, \qquad \Sigma_3 = 6\theta\Delta + 2\theta - 2\Delta + 1, \\ \Omega_n &= \frac{(\mu+\lambda)(\zeta + 1 + (n + \eta - \delta - 1)\,t) + (n-1)(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}. \end{aligned}$$

**Corollary 2**. For $\mathrm{P} = 1$ and $\mathrm{Q} = -1$

$$|a_3 - \xi a_2^2| \le \begin{cases} \dfrac{2}{K_1\,|\Sigma_3|\,|\Omega_3|}, & \text{if } \xi \in [\xi_1, \xi_2], \\ \dfrac{2}{K_1\,|\Sigma_3|\,|\Omega_3|}\left|\dfrac{2R_1}{B_1^2} \;-\; 1 \;-\; \xi \cdot \dfrac{2K_1\,\Sigma_3\,\Omega_3}{B_1^2\,\Sigma_2^2\,\Omega_2^2}\right|, & \text{if } \xi \notin [\xi_1, \xi_2], \end{cases}$$

**Remark 2**. The estimates $|a_3 - \xi a_2^2|$ in corollary 2 is the Fekete-Szegö estimates that will be obtained for functions with positive real parts instead of the Janowski Function.

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