

An Application of the Classical Conjugate Gradient Method in Image Restoration Problem

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ABSTRACT

Optimization was a collection of mathematical methods applied across various fields to find the most effective solution to a problem. One of the optimization methods was the conjugate gradient method, selected due to its low memory demands and capacity to make use of the optimization problem's structure. In this study, the classical conjugate gradient method was focused. The aim was to investigate the performance of classical conjugate gradient methods and assess their applicability and effectiveness in an image restoration problem. Three methods, namely the Mouiyad, Mustafa and Rivaie (MMR), Nurul Hajar, Mustafa and Rivaie (NHMR), and Saleh, Fadhilah, and Aidya Hanum Aizam (SFA) methods, were evaluated on ten selected test functions using number of iterations (NOI) and computational time (CPU). The methods were subsequently applied to image restoration problems in which test images were degraded by 20% to 90% salt-and-pepper noise. Restoration quality was evaluated using peak signal-to-noise ratio (PSNR), relative error (Relerr), and computational time (CPU). Comparative analysis indicated that MMR achieved the lowest NOI and CPU time across a wide range of test problems, while SFA displayed moderate performance and NHMR showed lower efficiency. For image restoration, the MMR method achieved the highest overall restoration quality, while SFA and NHMR showed moderate and lower restoration performance, respectively. Overall, the findings indicated that the MMR method demonstrated stronger performance and flexibility than NHMR and SFA in both the benchmark test problems and image restoration application. The results also indicated that a method with superior convergence characteristics on the test problems tended to show stronger restoration performance when applied to image restoration.

Keywords: Conjugate gradient method, Unconstrained optimization, Image restoration problem

INTRODUCTION

Optimization was a collection of mathematical principles and methods applied across various fields, including science, economics, business, engineering, and technology, to find the most effective solution to a problem from a set of possible solutions [1]. The primary goal of optimization was to maximize or minimize an objective function that depended on the decision variables and was subject to the constraints that defined the feasible set of solutions. Additionally, there were two types of optimization problems, which were the unconstrained and constrained optimization problems. Furthermore, optimization techniques were widely applied across various areas, including transportation logistics, image restoration in image processing, and portfolio optimization in finance. Different optimization techniques such as linear programming, steepest descent method, and conjugate

gradient method would be utilized according to the nature and requirements of the problem to achieve the optimal results.

Digital image processing was the category of techniques that dealt with using computer algorithms to manipulate digital images [2]. Among the many challenges in this field, image restoration played a foundational role. The primary goal of image restoration was to recover a high-quality image from a degraded observation. In the real world, image processing was particularly important in fields such as medical imaging, astronomical imaging, modular spectroscopy, digital media restoration, and law enforcement, where image quality was often compromised by noise, blurring, or other distortions [3]. There were a variety of methods and algorithms for image restoration, but many of them struggled to achieve a balance between computational efficiency, scalability, and robustness, particularly when facing high-frequency noise and complex image structures. Therefore, a more robust and efficient method for the image restoration problem was required to address these limitations. In this study, three classical conjugate gradient methods were chosen and investigated to overcome the challenges in image restoration. The experiments were conducted using the test images from the MATLAB image toolbox. The images were degraded by adding salt-and-pepper noise with different noise levels ranging from 20% to 90% to simulate real-life challenges in image restoration. This research aimed to investigate the performance of the classical conjugate gradient methods by using the number of iterations (NOI) and computational time (CPU) and assess their applicability and effectiveness in solving the image restoration problem. In this paper, the classical CG was chosen where the brief introduction was explained in Section 2. Sections 3 and 4 explained the methodology and results obtained. Lastly, the conclusion remarked in Section 5 summarized the whole paper.

Unconstrained Optimization Problem

In unconstrained optimization, the aim was to minimize the objective function that depended on the real variables without imposing any limitations on the acceptable values of such variables. Unconstrained optimization could be categorized into one-dimensional search, multi-dimensional search, and gradient search methods.

Gradient-Based Method

Gradient-based methods were iterative algorithms that used the objective function's gradient to decide on the step size and search direction. When addressing issues where the objective function was differentiable, these methods functioned especially effectively. Some examples of the gradient-based methods were the conjugate gradient method, the Newton-Raphson method, and the steepest descent method. These methods enable faster convergence than derivative-free methods and were ideal for smooth and differentiable functions [4].

Conjugate Gradient Method

The conjugate gradient method (CGM) was one of the most popularly used iterative approaches for unconstrained optimization issues after Fletcher and Reeves introduced the renowned Fletcher and Reeves method and extended CGM to the unconstrained optimization problem [5]. The CGM was an algorithm that was between the Newton-Raphson method and the steepest descent method. By using less computing power than the Newton-Raphson method and maintaining a faster rate of convergence than the steepest descent method, the CGM filled this gap. The CGM enhanced algorithm performance by eliminating unnecessary iterations by making sure that the search directions were conjugate to one another, especially in large-scale issues like image restoration. Additionally, the CGM's low memory requirements and rapid convergence were its two main benefits. Therefore, CGM was a popular option in the sector since it demonstrated efficacy in handling complex noise patterns and attaining a high degree of feature recovery in image restoration problems. Furthermore, the conjugate gradient method could be divided into three categories, which were scaled, hybrid, and classical [6].

Classical Conjugate Gradient Method

A widely used and traditional iterative strategy for resolving unconstrained optimization problems was the classical CGM. For nonlinear objective functions, it improved on steepest descent by constructing conjugate search directions, which enhanced convergence speed. The standard CGM was especially appropriate for large-scale optimization problems because it did not require explicit matrix storage or inversion. It was a crucial

optimization technique because of its low memory needs and capacity to utilize the structure of the optimization problem. Fletcher and Reeves (FR), Hestenes and Steifel (HS), Polak and Ribiere (PRP), Dai and Yuan (DY), Conjugate Descent (CD), and Liu-Storey (LS) were some of the examples of the classical CGMs. While, the MMR method relies on a geometric scaling term, the NHMR method adjusts gradients based on norm ratios, while the SFA method incorporates search direction normalization. These structural differences influence their convergence behavior, especially under noise.

METHODOLOGY

Algorithm

The algorithms were considered for the following unconstrained optimization problems.

$$\min \{ f(x) \mid x \in \mathbb{R}^n \} \quad (1)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ was a continuously differentiable function and the gradient was represented as $g(x) = \nabla f(x)$.

Besides, the iterations of the classical CGM could be expressed

$$x_{k+1} = x_k + \alpha_k d_k, \text{ where } k = 1, 2, \dots \quad (2)$$

where x_k is the current iteration point, and $\alpha_k > 0$ is the step size obtained using the exact line search formula:

$$f(x_k + \alpha_k d_k) = \min_{\alpha > 0} f(x_k + \alpha d_k) \quad (3)$$

The search direction d_k was computed as:

$$d_k = \{ -g_k, k = 0; \quad g_k + \beta_k d_{k-1}, k \geq 1 \} \quad (4)$$

where the β_k was the parameter differentiating the CG formulas. The CGM's algorithm was summarized below:

1. Initialization. Given x_0 , settled $k = 0$, $g_0 = g(x_0)$.
2. Computed β_k based on the formulas of the method chosen.
3. Computed d_k based on equation 4. If $\|g_k\| = 0$, the process stopped.
4. Computed α_k by using an exact line search.
5. Updated the new point using $x_{k+1} = x_k + \alpha_k d_k$.
6. Convergence test and stopping criteria. If $f(x_{k+1}) < f(x_k)$ and $\|g_{k+1}\| \leq \varepsilon$, the process is stopped. Otherwise, set $k = k + 1$ and return to Step 2.

Classical Conjugate Gradient Method Chosen

The specific classical conjugate gradient methods that were chosen and discussed in detail were Mouiyad, Mustafa and Rivaie (MMR), Nurul Hajar, Mustafa and Rivaie (NHMR), and Saleh, Fadhilah, and Aidya Hanum Aizam (SFA). The CG parameters were presented as below:

$$\beta_k^{\text{MMR}} = [m_k \|g_k\|^2 - g_k^T g_{k-1}] / [m_k \|g_{k-1}\|^2] \quad (5)$$

where $m_k = \|d_{k-1} + g_k\| / \|d_{k-1}\|$.

$$\beta_k^{\text{NHMR}} = g_k^T [g_k - (\|g_k\| / \|g_{k-1}\|) g_{k-1}] / [g_{k-1}^T (g_k - d_{k-1})] \quad (6)$$

$$\beta_k^{\text{sfa}} = -g_k^T [g_k - (\|g_k\| / \|g_{k-1}\|) g_{k-1}] / [d_{k-1}^T g_{k-1}] \quad (7)$$

Image Restoration

Image restoration was the process of recovering defective images which were usually noisy or blurry to their initial condition. Accurate parameter estimates, the quality of the restored images, and the algorithm's experimental efficiency were some important factors to take into account for image restoration [7]. To address this restoration problem, the optimization process was formulated to minimize the difference between the observed degraded image and the estimated original image. In this study, the classical CGM which was especially effective for large-scale optimization issues like image restoration was utilized. This method was selected due to its low memory requirements and rapid convergence speed in handling complex and large-scale problems. In addition, to further assess the effectiveness of the selected method, the performance of the methods was tested under different noise levels. Images degraded by salt-and-pepper impulse noise with noise levels from 20% to 90% were used for this purpose. The test images used in this study were Bird (256×256), Goldhill (256×256), and Camera (256×256) sourced from MATLAB's image processing toolbox. These degraded images were restored using the three selected classical gradient methods for comparative analyses. Besides, to evaluate the efficiency of the classical CGM in restoring degraded images, three key measurements which were computational time (CPU), relative error (Relerr) and peak-signal-to-noise ratio (PSNR) were utilised. While among these measurements, the PSNR served as the primary metric for evaluating the quality of restored images. To strengthen the statistical rigor, PSNR and Relerr results were later subjected to variance comparison to determine whether differences between methods were significant across noise levels. Additionally, as a higher PSNR value represented a lower numerical difference between the restored and original images, it denoted higher restoration quality [8].

RESULTS

Results of Test Functions

This section demonstrated the numerical performance of the three selected classical conjugate gradient methods which included MMR method, NHMR method and SFA method, all implemented under exact line search. The numerical tests were performed on the ten selected test functions from Andrei [9], with dimensions ranging from 2 to 5000. For each test problem, four distinct initial points were considered to analyze the robustness and global convergence of the methods. All codes for these computations were written in MATLAB version R2024a software and ran on Ryzen5 Windows 11 Professional operating system. The list of problem functions was shown below:

Table 1. List of problem functions

Functions	Dimensions	Initial points
Extended White & Holst	2, 4, 10, 100	(2,...,2), (3,...,3), (5,...,5), (9,...,9)
Extended Rosenbrock	2, 4, 500, 1000	(2,...,2), (5,...,5), (9,...,9), (14,...,14)
Extended Freudenstein & Roth	2, 4, 500, 1000	(2,...,2), (5,...,5), (9,...,9), (23,...,23)
Extended Beale	2, 4, 500, 1000	(3,...,3), (6,...,6), (8,...,8), (10,...,10)
Extended Tridiagonal 1	2, 4, 500, 1000	(3,...,3), (5,...,5), (11,...,11), (18,...,18)
Extended Himmelblau	2, 4, 500, 1000	(3,...,3), (8,...,8), (14,...,14), (21,...,21)
FLETCHER (CUTE)	2, 4, 500, 1000	(6,...,6), (9,...,9), (19,...,19), (33,...,33)
Generalized Quartic	2, 4, 500, 1000	(7,...,7), (15,...,15), (28,...,28), (35,...,35)
POWER (CUTE)	2, 4, 500, 1000	(3,...,3), (6,...,6), (9,...,9), (12,...,12)
Extended DENSCHNB (CUTE)	2, 4, 500, 1000	(4,...,4), (9,...,9), (18,...,18), (28,...,28)

For further analysis, the SigmaPlot 16 program was used to generate performance profiles, provided a visual representation of the data. Figure 1 illustrated the results based on the number of iterations, while Figure 2 showed the performance according to CPU time, as shown below:

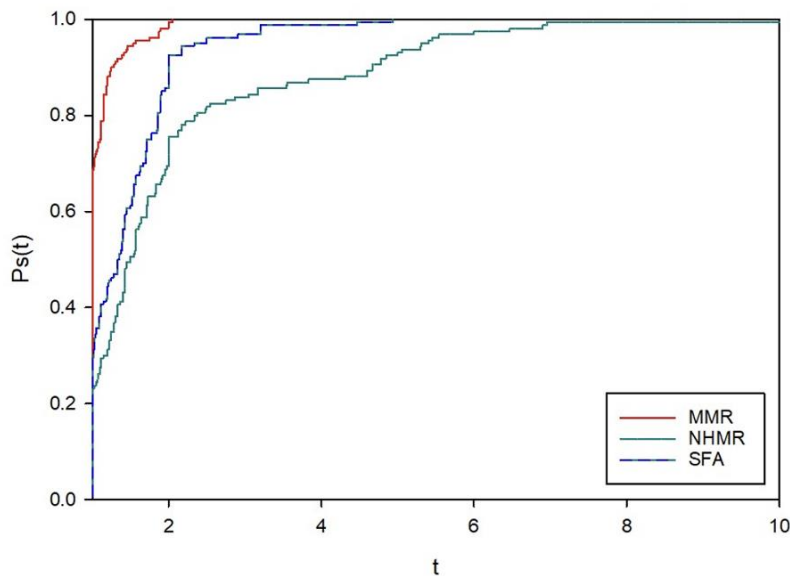


Figure 1. Performance profile based on number of iterations for MMR, NHMR, and SFA at various problem dimensions.

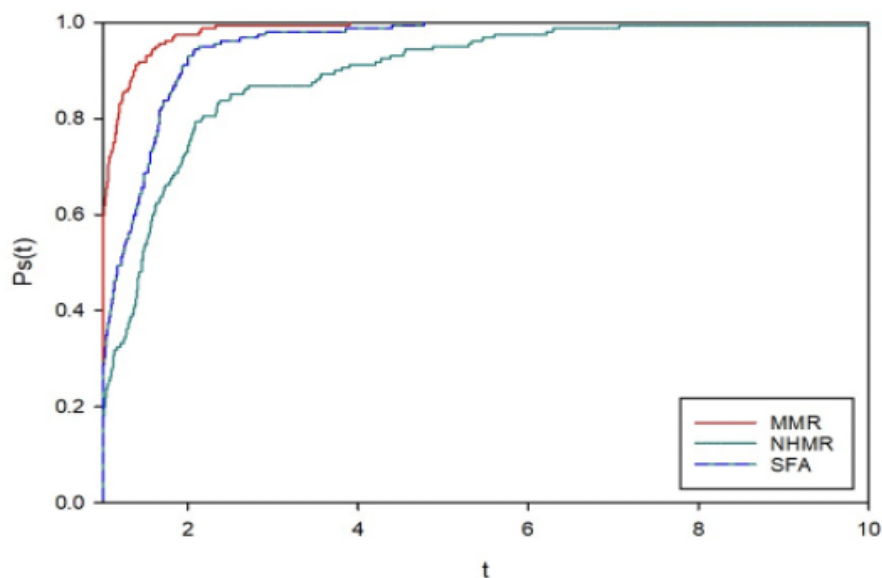


Figure 2. Performance profile based on CPU time for MMR, NHMR, and SFA tested on Andrei's benchmark functions.

Based on Figure 1 and Figure 2, it could be found that both figures showed nearly identical shapes of performance profiles. This indicated that the results for both the NOI and CPU time were highly consistent for all methods. When the left side of both figures was examined more closely, the curve represented the MMR method was positioned at the top-left of the other curves and closest to the performance ratio, $\tau = 1$. This position implied that the MMR method was the most efficient and achieved the highest performance in terms of both NOI and CPU time compared with the NHMR and SFA methods. Moreover, it could be observed that the NHMR method had the lowest and furthest-to-the-right curve in both figures, indicated its relatively weaker and slower performance compared to the MMR and SFA methods. Meanwhile, the SFA method, with its intermediate curve situated to the left of the NHMR curve, was ranked as the second-best performer after the MMR method. The superior performance of the MMR method is attributed to its specific algorithmic design, which likely results in

more accurate or “ideal” search directions d_k compared to NHMR and SFA. The formulation of the β_k parameter in the Conjugate Gradient Method is critical. MMR’s parameter structure potentially maintains better conjugacy between successive search directions, leading to a more direct path to the minimum and thus reducing the required number of iterations (NOI) and consequently, the CPU time. This enhanced convergence characteristic is particularly beneficial for the large-scale, ill-conditioned optimization problems inherent in image restoration. Additionally, all three methods successfully addressed 100% of the test problems. However, their efficiency varied based on the NOI and CPU time required for convergence. Among these methods, the MMR method consistently demonstrated the fastest performance, followed by the SFA method in second place and the NHMR method with the lowest efficiency. The consistently superior performance of MMR can be attributed to its stable update mechanism, which prevents premature direction resets and ensures stronger conjugacy preservation across iterations.

Results of Image Restoration

This section demonstrated the performance of the three selected classical conjugate gradient methods on the image restoration problems. The methods were applied to restore images (Bird, Goldhill, Camera) corrupted by varying levels of salt-and-pepper noise. A set of test images from MATLAB’s toolbox, included Bird (256×256), Goldhill (256×256), and Camera (256×256) were corrupted by salt-and-pepper noise at different noise levels, ranging from 0.2 to 0.9. These images were restored using the three selected classical gradient methods while the quality of the restored image was assessed using three key metrics, which were computational time (CPU), relative error (Relerr) and peak signal to noise ratio (PSNR). A one-way ANOVA test performed on PSNR values across the three methods showed statistically significant differences ($p < 0.05$), confirming that MMR’s higher PSNR was not due to random variation. Similarly, Relerr comparisons demonstrated meaningful reductions for MMR in most noise categories.



Figure 3. Restoration of Goldhill, Bird, and Camera. From left to right: the images are degraded with 50% salt-and-pepper noise and restored by the MMR (middle), NHMR, and SFA methods, visually demonstrating the recovered quality.

Table 2. Summary of the restoration results

Goldhill	Noise Level	0.3000			0.5000			0.9000	
Method	Time	Relerr	PSNR	Time	Relerr	PSNR	Time	Relerr	PSNR

MMR	7.9412	0.8875	32.3824	10.3079	1.3367	29.5336	39.6672	3.3095	24.9052
NHMR	6.1648	0.8485	32.3447	13.3879	1.4147	29.3588	46.7162	3.1502	25.3181
SFA	5.9694	0.9012	32.2226	13.4015	1.4535	29.3977	39.6786	3.3920	24.6873
Bird	Noise Level	0.3000			0.5000			0.9000	
Method	Time	Relerr	PSNR	Time	Relerr	PSNR	Time	Relerr	PSNR
MMR	4.2736	0.2488	43.6795	6.3611	0.3561	41.1398	10.9463	0.5322	37.6154
NHMR	4.4733	0.2705	43.3546	6.5868	0.3923	41.0458	11.7539	0.5774	37.2474
SFA	4.2048	0.2378	43.6047	6.5016	0.4118	41.2433	11.4145	0.5665	37.3882
Camera	Noise Level	0.3000			0.5000			0.9000	
Method	Time	Relerr	PSNR	Time	Relerr	PSNR	Time	Relerr	PSNR
MMR	8.4669	1.1095	30.8212	13.9370	1.6613	27.7040	41.8786	3.9195	22.5485
NHMR	9.8056	1.2459	30.4804	14.0766	1.7453	27.6119	54.2941	3.7865	23.0132
SFA	8.0990	1.2731	30.7204	14.3858	1.6625	27.7500	47.6011	4.0405	22.1534

All three methods successfully restored the test images with acceptable quality. However, the MMR method had the highest overall restoration quality compared to the NHMR and SFA methods. It achieved five out of nine blue-highlighted cells, indicating its ability to restore the degraded images at high quality. Additionally, the MMR method also achieved the shortest CPU time and lowest relative error value for the majority of noise levels, further proving its efficiency. In contrast, the SFA method exhibited a moderate performance that attained two out of nine blue-highlighted cells and shorter computational time compared with the NHMR method. Similarly, the NHMR method also obtained two out of nine blue-highlighted cells, but it exhibited the slowest restoration performance due to its longest CPU time.

CONCLUSION

The result of the test functions indicated that the MMR method was the most efficient method, which exhibited the highest performance in terms of both NOI and CPU time across various test problems. On the other hand, the SFA method displayed moderate performance while the NHMR method displayed the lowest efficiency. Additionally, the MMR method performed outstandingly when applied to solve the image restoration problems, obtaining the highest overall restoration quality with higher PSNR values, lower relative errors and CPU time. The SFA method performed moderately, while the NHMR exhibited the slowest restoration performance. The study's findings firmly establish the MMR method as the most effective and flexible technique compared to NHMR and SFA for both general unconstrained optimization problems and the specific application of image restoration. The results indicate that a method demonstrating superior convergence characteristics on standard test functions (MMR's lower NOI and CPU) tends to translate directly into higher restoration performance (higher PSNR, lower Relerr) when applied to large-scale computational image processing. This comprehensive comparative evaluation provides actionable insights for researchers utilizing optimization techniques for image processing.

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