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A Novel Seventh-Order Implicit Block Hybrid Nyström-Type Method for Second- Order Boundary Value Problems

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ABSTRACT

This paper introduces a novel approach for solving second-order nonlinear differential equations, with a primary focus on the Bratu problem, which holds significant importance in diverse scientific areas. Existing methods for solving this problem have limitations, prompting the development of the Block Hybrid Nystrom-Type Method (BHNTM). BHNTM utilizes the Bhaskara points derived, using the Bhaskara cosine approximation formula. The method seeks a numerical solution in the form of a power series polynomial, efficiently determining coefficients. The paper discusses BHNTM's convergence, zero stability, and consistency properties, substantiated through numerical experiments, highlighting its accuracy as a solver for Bratu-type equations. This research contributes to the field of numerical analysis by offering an alternative, effective approach to tackle complex second-order nonlinear differential equations, addressing critical challenges in various scientific domains.

Keywords: Hybrid Method; BlockNyström-type method; Nonlinear ODEs; Power Series Polynomials; One-dimensional Bratu problems.

INTRODUCTION

In this paper, we develop a one-step hybrid block Nystrom-type method as a computing strategy for solving a class of second order nonlinear differential equations given as:

$$y''(x) = \lambda(x)e^{\mu(x)y(x)} \tag{1.1}$$

defined on the close interval [0,1] subject to the following boundary conditions:

$$y(0) = y(1) = 0, \tag{1.2}$$

where $\lambda(x)$and $\mu(x)$are some known continuous functions of$x$. The problem given above arises in modelling of electrically conducting solids – Wazwaz& Suheil (2013). The case$\mu = 1$ in the above equation arises in the analysis of Joule losses in electrically conducting solids, λ represents square of the constant current and e^y models the temperature dependent resistance. In the case of fractional heating,λ represents square of the constant shear stress and e^ymodels temperature dependent fluidity.

Bratu's equation is a special form of equation (1.1) where the functions$\lambda(x)$ and$\mu(x)$ are constants. Its general form is given as:

$$y''(x)+\lambda(x)e^{\mu y(x)}=0,\quad 0\le x\le 1, \tag{1.3}$$

The associated initial and boundary conditions, respectively, are given as:

$$y(0)=y'(0)=0 \quad \text{and} \quad y(0)=y(1)=0, \tag{1.4}$$

where $\mu$is taken as $+1$ or-1,$y(x)$is the solution of the equation, and $\lambda$is a real parameter. This two-point boundary value problem is a special case in the modelling of electrically conducting solids, and also occurs in diffusion theory (see, Wazwaz & Suheil, 2013; Frank-Kamenetski, 1955). Equation (1.3) with initial and boundary conditions also arises in many physical models such as the fuel ignition model in combustion theory, model of thermal reaction processes in chemical reaction theory, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity concerning the Chandrasekhar model, radiative heat transfer, and nanotechnology (Frank- Kamenetski, 1955; Bratu, 1914; Gelfand, 1963; Jacobsen & Schmitt, 2002; Hariharan &Pirabaharan, 2013). The Bratu problem came to light with Bratu's article published in 1914, Bratu (1914). In honour of Gelfand and the French mathematician Liouville, Gelfand (1963); it is also known as the ‘‘Liouville-Gelfand’’ or ‘‘Liouville-Gelfand-Bratu’’ problem. Jacobsen and Schmitt have provided an excellent summary of the significance and history of Bratu-type equations –Jacobsen& Schmitt (2002).In recent years, this problem has become a popular benchmark to test the accuracy of various numerical solvers. Buckmire (2004), Mounim& de Dormale (2006); numerical approaches based on Hybrid Block Nystrom-type method however have received very minimal attention for the solution of these special kinds of BVPs.

Therefore, Bratu-type problem is of great scientific importance and several numerical methods have been developedto find the approximate solution. For instance, the differential transformation method given in Hassan & Erturk (2007), the multigrid-based methods in Mohsen (2014), iterative numerical scheme based on the Newton's Kantorovich method in function space in Temimi& Ben-Romdhane (2016), the decomposition method in Deeba *et al.* (2000) have been adopted to solve the Bratu's problem. Other methods such as the modified wavelet Galerkin method given in Raja *et al.* (2015), the Laplace Adomian decomposition method, the perturbation-iteration method, the Lie-group shooting method and the integral solution via Greens function have been used to solve Bratu's problem as seen in Temimi& Ben-Romdhane (2016). The feed-forward artificial neural networks (ANN) optimized with genetic algorithm (GA) and the active-set method (ASM) was adopted to solve a number of initial and boundary value problems based on Bratu equations, Raja *et al*.(2016).The standard Nystrom-type method is used to solve only initial value problems of ordinary differential equations in practical terms.In this paper, the Block Hybrid Nystrom-Type Method (BHNTM)is constructed by introducing hybrid points derived using the Bhakara Cosine Approximation formula to obtain a numerical solution for the classical one-dimensional Bratu's problem by extending the work of Jator and Manathunga(2018). The paper is organized as follows. In section 2, the BHNTM is derived and its basic properties of convergence, zero stability and consistency in section 3. In section 4, the descriptionof the implementation strategy as well as the execution of the numerical experiments and discussion of results. Finally, the conclusion of the paper is discussed in section 5.

DEVELOPMENT OF THE BHNTM

Consider the problem

$$y''(x)=f(x,y,y'),\qquad y(a)=0,\qquad y(b)=0,\qquad x\in[a,\ b] \tag{2.1}$$

Subject to the following conditions on $f(x,y,y')$:

i. $f(x,y,y')$ is continuous,

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ii. derivative of $f(x, y, y')$ exist and continuous.

considering also the problem on the interval [a,b], such that the partitioning is done using the Bhaskara cosine approximation formula to generate the Bhaskara points which in turn is used to develop the algorithmby approximating the cosine functions $\left\{\cos\frac{\pi i}{M}\right\}_{i=0}^{M} \approx \frac{M^2 - 4i^2}{M^2 + i^2}$, where $M = w+1, \quad w \in N : w$ (number of off-grids) $\quad w \geq 2$, the number $h = \frac{(b-a)}{N}$ is referred to as the constant step size. Supposing the exact solution is approximated by the power series polynomial of the form

$$y(x) = \sum_{0}^{9} a_j x^j \tag{2.2}$$

where a_j are coefficients obtained distinctly and $x \in [x_n, x_{n+1}]$, for some natural number n. In order to determine the coefficients a_j, the following conditions hold:

$$\left.\begin{aligned} &y(x_n) = y_n, \\ &y'(x_n) = y'_n, \\ &y''(x_{n+v}) = f_{n+v}, v = 0, \frac{5}{74}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{69}{74}, 1. \end{aligned}\right\} \tag{2.3}$$

Differentiating (2.2) once and twice gives

$$\left.\begin{aligned} &y'(x) = \sum_{j=1}^{9} j a_j x^{j-1}, \\ &y''(x) = \sum_{j=2}^{9} j(j-1) a_j x^{j-2} = f(x, y, y'). \end{aligned}\right\} \tag{2.4}$$

Using equations (2.2) and (2.4) on the criteria given in (2.3) a system of nonlinear equations of the form are obtained

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & \ldots & x_n^8 \\ 0 & 1 & 2x_n & 3x_n^2 & \ldots & 8x_n^7 \\ 0 & 0 & 2 & 6x_n & \ldots & 56x_n^6 \\ \ldots & \ldots & 2 & 6x_{n+\frac{5}{74}} & \ldots & 56x^6_{n+\frac{5}{74}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 2 & 6x_{n+1} & \ldots & 56x_{n+1}^6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y'_n \\ f_n \\ f_{n+\frac{5}{74}} \\ f_{n+\frac{1}{4}} \\ . \\ . \\ f_{n+1} \end{pmatrix} \tag{2.5}$$

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The above system of nonlinear equations is solved using the matrix inversion algorithm. The unknown coefficients a_j, are then substituted into (2.2) to derive the continuous Hybrid Block Nystrom-type Method with five off-step points (BHNTM$_{1,5}$):

$$y(x)=\alpha_0(x)y_n+\alpha_1(x)hy_n' + h^2\left(\beta_n f_n+\beta_{n+\frac{5}{74}}f_{n+\frac{5}{74}}+\beta_{n+\frac{1}{4}}f_{n+\frac{1}{4}}+\beta_{n+\frac{1}{2}}f_{n+\frac{1}{2}}+\beta_{n+\frac{3}{4}}f_{n+\frac{3}{4}}+\beta_{n+\frac{69}{74}}f_{n+\frac{69}{74}}+\beta_{n+1}f_{n+1}\right), \tag{2.6}$$

and

$$y'(x)=y_n'+h\left(\beta_n f_n+\beta_{n+\frac{5}{74}}f_{n+\frac{5}{74}}+\beta_{n+\frac{1}{4}}f_{n+\frac{1}{4}}+\beta_{n+\frac{1}{2}}f_{n+\frac{1}{2}}+\beta_{n+\frac{3}{4}}f_{n+\frac{3}{4}}+\beta_{n+\frac{69}{74}}f_{n+\frac{69}{74}}+\beta_{n+1}f_{n+1}\right) \tag{2.7}$$

where $\alpha_0(x)$, $\alpha_1(x)$ and $\beta_j(x)$ are continuous coefficients that are distinctly determined. Assuming that y_n is the numerical approximation to the analytical solution $y(x_n)$, y_n' is the numerical approximation to $y'(x_n)$ and f_{n+v} is the numerical approximation to $f\left(x_{n+hv},y_{n+hv},y'_{n+hv}\right)$.

The main methods are obtained by evaluating (2.6)at points $x=x_n,x_{n+\frac{5}{74}},x_{n+\frac{1}{4}},x_{n+\frac{1}{2}},x_{n+\frac{3}{4}},x_{n+\frac{69}{74}},x_{n+1}$ to give the discrete schemes which form the continuous block hybrid Nystrom-type method with five off-grid points:

$$\left.\begin{aligned}
y_{n+\frac{5}{74}} &= y_n+\frac{5}{74}hy_n'+h^2\left(\begin{aligned}&\frac{58052434225}{44612850799692}f_n+\frac{15063275}{13592395776}f_{n+\frac{5}{74}}-\frac{2933478125}{16325717140467}f_{n+\frac{1}{4}}+\frac{24930298125}{294258030395392}f_{n+\frac{1}{2}}\\&-\frac{24485719375}{440794362792609}f_{n+\frac{3}{4}}+\frac{7823810875}{177258433314816}f_{n+\frac{69}{74}}-\frac{6595375}{323281527534}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{1}{4}} &= y_n+\frac{1}{4}hy_n'+h^2\left(\begin{aligned}&\frac{348899}{89026560}f_n+\frac{68719861387}{3069160980480}f_{n+\frac{5}{74}}+\frac{449}{77568}f_{n+\frac{1}{4}}-\frac{92961}{73400320}f_{n+\frac{1}{2}}+\frac{155707}{219905280}f_{n+\frac{3}{4}}\\&-\frac{8806682539}{16573469294592}f_{n+\frac{69}{74}}+\frac{7177}{29675520}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{1}{2}} &= y_n+\frac{1}{2}hy_n'+h^2\left(\begin{aligned}&\frac{211}{28980}f_n+\frac{7142427571}{129480228864}f_{n+\frac{5}{74}}+\frac{48253}{859005}f_{n+\frac{1}{4}}+\frac{115}{16384}f_{n+\frac{1}{2}}-\frac{241}{286335}f_{n+\frac{3}{4}}+\frac{69343957}{215800381440}f_{n+\frac{69}{74}}\\&-\frac{1}{8694}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{3}{4}} &= y_n+\frac{3}{4}hy_n'+h^2\left(\begin{aligned}&\frac{37377}{3297280}f_n+\frac{29332493811}{341017886720}f_{n+\frac{5}{74}}+\frac{103833}{904960}f_{n+\frac{1}{4}}+\frac{4690143}{73400320}f_{n+\frac{1}{2}}+\frac{449}{77568}f_{n+\frac{3}{4}}\\&-\frac{762783527}{1023053660160}f_{n+\frac{69}{74}}+\frac{729}{3297280}f_{n+1}\end{aligned}\right)\\
y_{n+\frac{69}{74}} &= y_n+\frac{69}{74}hy_n'+h^2\left(\begin{aligned}&\frac{4946912289}{359201697260}f_n+\frac{17400136203}{158577950720}f_{n+\frac{5}{74}}+\frac{1411081172199}{9069842855815}f_{n+\frac{1}{4}}+\frac{165964330541889}{147129015976960}f_{n+\frac{1}{2}}\\&+\frac{1127174082469}{27209528567445}f_{n+\frac{3}{4}}+\frac{15063275}{13592395776}f_{n+\frac{69}{74}}+\frac{10840797}{35920169726}f_{n+1}\end{aligned}\right)\\
y_{n+1} &= y_n+hy_n'+h^2\left(\frac{643}{43470}f_n+\frac{69343957}{586414080}f_{n+\frac{5}{74}}+\frac{48976}{286335}f_{n+\frac{1}{4}}+\frac{4671}{35840}f_{n+\frac{1}{2}}+\frac{48976}{859005}f_{n+\frac{3}{4}}+\frac{69343957}{8092514304}f_{n+\frac{69}{74}}\right)
\end{aligned}\right\} \tag{2.8}$$

The additional methods are obtained by evaluating (2.7) at $x=\left(\frac{5}{74},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{69}{74},1\right)$ to give the following discrete schemes:

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$$\left.\begin{array}{l}
y'_{n+\frac{5}{74}} = y'_n + h\left(\begin{array}{l}\frac{132171928615}{4823010897264} f_n + \frac{13113969205}{299423029248} f_{n+\frac{5}{74}} - \frac{61021705750}{11913361156557} f_{n+\frac{1}{4}} + \frac{1181523375}{497057483776} f_{n+\frac{1}{2}} - \frac{18479790250}{11913361156557} f_{n+\frac{3}{4}} \\ + \frac{368391325}{299423029248} f_{n+\frac{69}{74}} - \frac{2741432975}{4823010897264} f_{n+1}\end{array}\right) \\
y'_{n+\frac{1}{4}} = y'_n + h\left(\begin{array}{l}\frac{23579}{5564160} f_n + \frac{199779940117}{1294802288640} f_{n+\frac{5}{74}} + \frac{1422961}{13744080} f_{n+\frac{1}{4}} - \frac{20709}{1146880} f_{n+\frac{1}{2}} + \frac{139651}{13744080} f_{n+\frac{3}{4}} - \frac{9916185851}{1294802288640} f_{n+\frac{69}{74}} \\ + \frac{19421}{5564160} f_{n+1}\end{array}\right) \\
y'_{n+\frac{1}{2}} = y'_n + h\left(\frac{277}{12880} f_n + \frac{901471441}{8092514304} f_{n+\frac{5}{74}} + \frac{217162}{859005} f_{n+\frac{1}{4}} + \frac{4671}{35840} f_{n+\frac{1}{2}} - \frac{2362}{95445} f_{n+\frac{3}{4}} + \frac{69343957}{4495841280} f_{n+\frac{69}{74}} - \frac{467}{69552} f_{n+1}\right) \\
y'_{n+\frac{3}{4}} = y'_n + h\left(\frac{2329}{206080} f_n + \frac{2149662667}{15985213440} f_{n+\frac{5}{74}} + \frac{36973}{169680} f_{n+\frac{1}{4}} + \frac{319653}{1146880} f_{n+\frac{1}{2}} + \frac{63389}{509040} f_{n+\frac{3}{4}} - \frac{1317535183}{47955640320} f_{n+\frac{69}{74}} + \frac{435}{41216} f_{n+1}\right) \\
y'_{n+\frac{69}{74}} = y'_n + h\left(\begin{array}{l}\frac{596477423}{38832615920} f_n + \frac{10024207}{803604480} f_{n+\frac{5}{74}} + \frac{168853796338}{735392663985} f_{n+\frac{1}{4}} + \frac{641903629419}{2485287418880} f_{n+\frac{1}{2}} + \frac{514439521514}{2206177991955} f_{n+\frac{3}{4}} \\ + \frac{200151221}{2410813440} f_{n+\frac{69}{74}} - \frac{489781527}{38832615920} f_{n+1}\end{array}\right) \\
y'_{n+1} = y'_n + h\left(\frac{643}{43470} f_n + \frac{2565726409}{20231285760} f_{n+\frac{5}{74}} + \frac{195904}{859005} f_{n+\frac{1}{4}} + \frac{4671}{17920} f_{n+\frac{1}{2}} + \frac{195904}{859005} f_{n+\frac{3}{4}} + \frac{2565726409}{20231285760} f_{n+\frac{69}{74}} + \frac{643}{43470} f_{n+1}\right)
\end{array}\right\} \quad (2.9)$$

ANALYSIS OF THE BHNTM

In this section, mathematical analysis on some basic properties of the derived schemes are discussed extensively. Existing definitions and theorems well support these properties.

3.1Order and Error Constant

The proposed BHNTM of (2.8) and (2.9) is classified as a member of the linear multistep method (LMM) which can be represented generally as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \quad (3.1)$$

Definition 1 (Rufai and Ramos, 2020).

Associated with a numerical method is a linear difference operator η which is supposed to have $y(x_n)$ with higher derivatives. The term $y(x_n + v_i h)$ and its second derivative $y''(x_n + v_i h)$ can be expanded as Taylor's series about the point x_n. The local truncation error associated with a second-order ordinary differential equation is defined by the difference operator:

$$\eta\left[y(x_n);h\right] = \sum_{i=0}^{k}\left[\alpha_i y(x_n + v_i h) - h^2 \beta_i f(x_n + v_i h)\right] \quad (3.2)$$

where $y(x_n)$ is an arbitrary function continuously differentiable on the interval $[x_n, x_{n+1}]$. Expanding the expression (3.2) in Taylor series approximation about the point x_n gives

$$\eta\left[y(x_n);h\right] = \hat{C}_0 y(x_n) + \hat{C}_1 h y'(x_n) + \hat{C}_2 h^2 y''(x_n) + \ldots + \hat{C}_{p+1} h^{p+1} y^{p+1}(x_n) + \ldots + \hat{C}_{p+2} h^{p+2} y^{p+2}(x_n) + \ldots \quad (3.3)$$

where the vectors

$$\begin{cases} \hat{C}_0 = \sum_{v_i=0}^{k} \alpha_{v_i}, \quad \hat{C}_1 = \sum_{i=0}^{k} v_i \alpha_{v_i}, \quad \hat{C}_2 = \frac{1}{2!} \sum_{v_i=0}^{k} v_i \alpha_{v_i} - \beta_{v_i}, ..., \\ \hat{C}_p = \frac{1}{p!} \sum_{v_i=0}^{k} v_i^{\,p} \alpha_{v_i} - p(p-1)(p-2) v_i^{\,p-2} \beta_{v_i}. \end{cases} \tag{3.4}$$

Following Lambert (1991), the associated methodsare said to be of order p if, in (3.3)

$\hat{C}_0 = \hat{C}_1 = C_2 = ... = \hat{C}_p = \hat{C}_{p+1} = 0$ and $\hat{C}_{p+2} \neq 0$.

Therefore, $\hat{C}_{p+2}$ is the error constant and $\hat{C}_{p+2} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point x_n. The local truncation error of the proposed BHNTM obtained are given as:

$$\eta\left[y(x_n;h)\right] = \begin{cases} \frac{28132172125}{1207309213103041069056} h^{(9)} y^{(9)}(x) + O\left(h^{10}\right) \\ -\frac{147697}{651143046758400} h^{(9)} y^{(9)}(x) + O\left(h^{10}\right) \\ -\frac{43}{317940940800} h^{(9)} y^{(9)}(x) + O\left(h^{10}\right) \\ -\frac{351}{8038803046400} h^{(9)} y^{(9)}(x) + O\left(h^{10}\right) \\ -\frac{109474638723}{372626300340444774400} h^{(9)} y^{(9)}(x) + O\left(h^{10}\right) \\ -\frac{43}{158970470400} h^{(9)} y^{(9)}(x) + O\left(h^{10}\right) \end{cases} \tag{3.5}$$

which shows that the proposed method has order $p=7$, where the error constant $\hat{C}_{p+2}$ is a 1×6 vector given by

$$\hat{C}_9 = \left(\begin{array}{l} \frac{28132172125}{1207309213103041069056}, -\frac{147697}{651143046758400}, -\frac{43}{317940940800}, -\frac{351}{8038803046400}, \\ -\frac{109474638723}{372626300340444774400}, -\frac{43}{158970470400} \end{array} \right). \tag{3.6}$$

3.2Zero stability of HBNTM

Zero stability is a property concerning the proposed method when limiting h to zero. Therefore, as h tends to zero in the main method (2.8), the following system of equations are formed:

$$\left. \begin{array}{l} y_{n+\frac{5}{74}} = y_n \\ y_{n+\frac{1}{4}} = y_n \\ y_{n+\frac{1}{2}} = y_n \\ y_{n+\frac{3}{4}} = y_n \\ y_{n+\frac{69}{74}} = y_n \\ y_{n+1} = y_n \end{array} \right\} \tag{3.7}$$

which can be written in matrix form as

$$A^0 Y_i - A^1 Y_{i-1} = 0, \tag{3.8}$$

where

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad Y_i = \begin{pmatrix} y_{n+\frac{5}{74}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+\frac{69}{74}} \\ y_{n+1} \end{pmatrix}, \quad Y_{i-1} = \begin{pmatrix} y_n \\ y_n \\ y_n \\ y_n \\ y_n \\ y_n \end{pmatrix}$$

Following Lambert (1991), a method is said to be zero stable if the root r_i of the first characteristic polynomial $\rho(r) = \det\left|A^0 r - A^i\right|$ does not exceed one ($|r_i| \le 1$).

The first characteristic polynomial of the BHNTM is given by

$$r^5\left[r-1\right]=0. \qquad (3.9)$$

The roots of (3.9) are $r = 0,0,0,0,0,1$ in which none of them is greater than one. Therefore, the BHNTM is zero-stable.

3.3Consistency of the BHNTM

Definition 2, (Jator and Li, 2009).The linear multistep method (3.1) is said to be consistent if, it is of order $p \ge 1$and its first and second characteristic polynomials defined as $\rho(z) = \sum_{j=0}^{k} \alpha_j z^j$ and $\sigma(z) = \sum_{j=0}^{k} \beta_j z^j$

where z satisfied (i) $\sum_{j}^{k} \alpha_j = 0$ (ii) $\rho(1) = \rho'(1) = 0$ and (iii) $\rho''(1) = 2!\sigma(1)$,

The discrete schemes derived are all of order greater than one and satisfies the conditions therein.

3.4Convergence of the BHNTM

This section focuses on the discussion of the convergence analysis of the proposed BHNTM. By defining convergence, the proposed method is shown to be convergent by writing the formulas in (2.8) and (2.9) in an appropriate matrix notation form.

Definition 3(Rufai and Ramos, 2021)Let $y(x)$ be the exact solution of the given second order boundary value problems and $\{y_i\}_{i=0}^{N}$ the approximate solutions obtained with the derived numerical technique. The proposed method is said to be convergent of order p if, for sufficiently small h, there exists a constant C independent of h, such that:

$$\max_{0 \le i \le N} \left| y(x_i) - y_i \right| \le Ch^p.$$

Note that in this circumstance, $\max_{0 \le i \le N} \left| y(x_i) - y_i \right| \to 0$ as $h \to 0$.

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Theorem 1 (***Convergence theorem***). Let $y(x)$ denote the true solution of the second order BVP in (1.1) with boundary condition in (1.2), and $\{y_i\}_{i=0}^{N}$ the discrete solution provided by the proposed method. Then, the proposed method is convergent to order seven.

Proof. Following (Jator and Manathunga, 2018), suppose A be matrix of dimension $12N \times 12N$ defined by:

$$A = \begin{bmatrix} A_{1,1} & \cdots & \cdots & \cdots & A_{1,2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{2n,1} & \cdots & \cdots & \cdots & A_{2N,2N} \end{bmatrix},$$

where the elements of $A_{i,j}$ are 6×6 matrices as follows:

$$A_{i,i} = I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_{i,i-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad i = N+2, \dots, 2N;$$

$$A_{i,N+i-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{5}{74} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{4} \\ 0 & 0 & 0 & 0 & 0 & -\frac{69}{74} \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad \text{where } 1 < i < N+1,$$

$A_{i,j} = 0$ otherwise, where **0** is a zero matrix.

Suppose B be a $12N \times 12N$ matrix defined by

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$$B = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} & \beta_{1,4} & \cdots & \beta_{1,2N} \\ \vdots & & & & \ddots & \vdots \\ \beta_{2N,1} & \beta_{2N,2} & \beta_{2N,3} & \beta_{2N,4} & \cdots & \beta_{2N,2N} \end{bmatrix},$$

where the elements of $B_{i,j}$ are 6×6 matrices given as follows:

$$\mathrm{B}_{i,j} = \begin{bmatrix} \frac{15063275}{13592395776} & -\frac{2933478125}{16325717140467} & \frac{24930298125}{294258030395392} & -\frac{24485719375}{440794362792609} & \frac{7823810875}{177258433314816} & -\frac{6595375}{323281527534} \\ \frac{68719861387}{3069160980480} & \frac{449}{77568} & -\frac{92961}{73400320} & \frac{155707}{219905280} & -\frac{8806682539}{16573469294592} & \frac{7177}{29675520} \\ \frac{7142427571}{129480228864} & \frac{48253}{859005} & \frac{115}{16384} & -\frac{241}{286335} & \frac{69343957}{215800381440} & -\frac{1}{8694} \\ \frac{29332493811}{341017886720} & \frac{103833}{904960} & \frac{4690143}{73400320} & \frac{449}{77568} & -\frac{762783527}{1023053660160} & \frac{729}{3297280} \\ \frac{17400136203}{158577950720} & \frac{1411081172199}{9069842855815} & \frac{165964330541889}{147129015976960} & \frac{1127174082469}{27209528567445} & \frac{15063275}{13592395776} & \frac{10840797}{35920169726} \\ \frac{69343957}{586414080} & \frac{48976}{286335} & \frac{4671}{35840} & \frac{48976}{859005} & \frac{69343957}{8092514304} & 0 \end{bmatrix}, \text{ where } 1 \le i \le N$$

$$\mathrm{B}_{i,i-N} = \begin{bmatrix} \frac{13113969205}{299423029248} & -\frac{61021705750}{11913361156557} & \frac{1181523375}{497057483776} & -\frac{18479790250}{11913361156557} & \frac{368391325}{299423029248} & -\frac{2741432975}{4823010897264} \\ \frac{199779940117}{1294802288640} & \frac{1422961}{13744080} & -\frac{20709}{1146880} & \frac{139651}{13744080} & -\frac{9916185851}{1294802288640} & \frac{19421}{5564160} \\ \frac{901471441}{8092514304} & \frac{217162}{859005} & \frac{4671}{35840} & -\frac{2362}{95445} & \frac{69343957}{4495841280} & -\frac{467}{69552} \\ \frac{2149662667}{15985213440} & \frac{36973}{169680} & \frac{319653}{1146880} & \frac{63389}{509040} & -\frac{1317535183}{47955640320} & \frac{435}{41216} \\ \frac{10024207}{803604480} & \frac{168853796338}{735392663985} & \frac{641903629419}{2485287418880} & \frac{514439521514}{2206177991955} & \frac{200151221}{2410813440} & -\frac{489781527}{38832615920} \\ \frac{2565726409}{20231285760} & \frac{195904}{859005} & \frac{4671}{17920} & \frac{195904}{859005} & \frac{2565726409}{20231285760} & \frac{643}{43470} \end{bmatrix}, \text{ where } N+1 \le i \le 2N$$

$$B_{i,i-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{58052434225}{44612850799692} \\ 0 & 0 & 0 & 0 & 0 & \frac{348899}{89026560} \\ 0 & 0 & 0 & 0 & 0 & \frac{211}{28980} \\ 0 & 0 & 0 & 0 & 0 & \frac{37377}{3297280} \\ 0 & 0 & 0 & 0 & 0 & \frac{4946912289}{359201697260} \\ 0 & 0 & 0 & 0 & 0 & \frac{643}{43470} \end{bmatrix}, \text{ where } 1 < i \le N$$

$$B_{i,i-(N+1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{132171928615}{4823010897264} \\ 0 & 0 & 0 & 0 & 0 & \frac{23579}{5564160} \\ 0 & 0 & 0 & 0 & 0 & \frac{277}{12880} \\ 0 & 0 & 0 & 0 & 0 & \frac{2329}{206080} \\ 0 & 0 & 0 & 0 & 0 & \frac{596477423}{38832615920} \\ 0 & 0 & 0 & 0 & 0 & \frac{643}{43470} \end{bmatrix}, \text{ where } N+1 < i \le 2N,$$

$B_{i,j} = 0,$ otherwise, where **0** is a zero matrix.

Hence,

$$A = \left[\begin{array}{c|c} \hat{A}_{1,1} & \hat{A}_{1,2} \\ \hline 0 & \hat{A}_{2,2} \end{array}\right],$$

$$B = \left[\begin{array}{c|c} \hat{B}_{1,1} & 0 \\ \hline \hat{B}_{2,1} & 0 \end{array}\right].$$

Furthermore, the following vectors are defined as:

where $T(h)$ is the local truncation error of order nine. The exact form of the system is given by

$$\left.\begin{array}{l} Y = \begin{pmatrix} y\left(x_{\frac{5}{74}}\right), y\left(x_{\frac{1}{4}}\right), ..., y(x_1), ..., y\left(x_{N-1+\frac{5}{74}}\right), y\left(x_{N-1+\frac{1}{4}}\right), ..., y(x_N), hy'(x_{\frac{5}{74}}), \\ hy'(x_{\frac{1}{4}}), ..., hy'(x_1), ..., hy'\left(x_{N-1+\frac{5}{74}}\right), hy'\left(x_{N-1+\frac{1}{4}}\right), ..., hy'(x_N) \end{pmatrix}^W, \\ C = \begin{pmatrix} -y_0 - \frac{5h}{74}y_0' - h^2\beta_0(1) f_0, -y_0 - \frac{h}{4}y_0' - h^2\beta_0(2) f_0, ..., -y_0 - hy_0' - h^2\beta_0(6) f_0, \\ 0, 0, 0, ..., 0, -h^2\beta_0'(1), -h^2\beta_0'(2), ..., -h^2\beta_0'(6), 0, 0, 0, ..., 0 \end{pmatrix}^W \\ T(h) = \left(\xi_{\frac{5}{74}}, \xi_{\frac{1}{4}}, ..., \xi_1, \xi_{1+\frac{5}{74}}, ..., \xi_N, \xi_{\frac{5}{74}}, \eta'_{\frac{1}{4}}, ..., \xi'_1, \xi'_{1+\frac{5}{74}}, ..., \xi'_N\right)^W, \end{array}\right\}$$

$$AY - h^2BF(Y) + C + T(h) = 0, \qquad (3.10)$$

and the approximate form of the system is given by

$$A\bar{Y} - h^2BF(\bar{Y}) + C = 0, \qquad (3.11)$$

where $\bar{Y} = \left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, ..., y_1, y_{1+\frac{5}{74}}, ..., y_N, hy'_{\frac{5}{74}}, hy'_{\frac{1}{4}}, ..., hy'_1, hy'_{1+\frac{5}{74}}, ..., hy'_N\right).$

Subtracting (3.10) from (3.11) gives,

$$AE - h^2BF(\bar{Y}) + h^2BF(Y) + C = T(h), \qquad (3.12)$$

where $E = \bar{Y} - Y = \left(e_{\frac{5}{75}}, e_{\frac{1}{4}}, e_{\frac{1}{2}}, ...e_1, e'_{\frac{5}{75}}, e'_{\frac{1}{4}}, e'_{\frac{1}{2}}, ...e'_1\right)$. Applying the mean-value theorem (Jator and Manathunga, 2018), which can be written as

$$F(\bar{Y}) = F(Y) + J_F(Y)(\bar{Y} - Y) + o\left(\|\bar{Y} - Y\|\right),$$

where J_F is a Jacobian matrix. From this equation, $\frac{F(\bar{Y}) - F(Y)}{\bar{Y} - Y} = \frac{F(\bar{Y}) - F(Y)}{E} = J_F(Y)$.

Hence, we obtain $\left(A - h^2BJ_F(Y)E\right) = T(h)$, where $J_F = \begin{bmatrix} J_{1,1} & J_{1,2} \\ J_{2,1} & J_{2,2} \end{bmatrix}$, and $J_{i,j}$ are $6N \times 6N$ matrices.

Considering the matrix $A - h^2BJ_F(Y)$, asserting that, for sufficiently small h, $A - h^2BJ_F(Y)$ is invertible. Observe that $\det A = \det \hat{A}_{1,1} \det \hat{A}_{2,2}$. Since $\hat{A}_{1,1} = \hat{A}_{2,2}$; it is enough to prove only the invertibility of $\hat{A}_{1,1}$. The diagonal elements of $\hat{A}_{1,1}$ being non-zero certainly implies that its determinant is non-zero; hence, it is invertible. In fact, it is a lower triangular matrix. Therefore, A is invertible. Let $Z = -h^2BJ_F$, we have $Q = A + Z$ then $\det(Q) = \det(A)\det\left(I + ZA^{-1}\right)$.

Let $C = ZA^{-1}$, then $\det(Q) = \det(A)\det(I - C)$. Note that, $\det(I - C)$ is the characteristic polynomial of C. Therefore,

$\det(I - C) = (\varphi - \varphi_1)(\varphi - \varphi_2)...(\varphi - \varphi_{12N})$, where φ_j are eigenvalues of C. For $\varphi = 1$, gives

$\det(I-C)=(1-\varphi_1)(1-\varphi_2)...(1-\varphi_{12N})$.

If each $\varphi_j \neq 1$, then $\det(I-C)\neq 0$. Suppose that φ_j is an eigenvalue of $BJ_F A^{-1}$, then $h^2\hat{\varphi}_j$. If $h^2\hat{\varphi}_j \neq 1$ is proved successfully, then we are done. Thus, choosing $h^2 \notin \left\{\frac{1}{\hat{\varphi}_j} : \hat{\varphi}_j \text{ is a non-zero eigenvalue of } BJ_F A^{-1}\right\}$ Therefore, there exists an h such that $\det(I-C)\neq 0$. Thus, $\det Q \neq 0$. This means that Q is invertible and $\|Q\|_\infty = O(h^2)$. Now, $(A+Z)E=T(h)$. This implies that $QE=T(h)$. Since Q is invertible, $E=Q^{-1}T(h)$. Taking the maximum norm gives

$\|E\|_\infty = \|Q^{-1}T(h)\|_\infty \leq \|Q^{-1}\|_\infty \|T(h)\|_\infty \leq O(h^{-2})O(h^9)\leq O(h^7)$.. Thus, the BHNTM is convergent, providing a seventh-order approximations.

NUMERICAL EXPERIMENT

In this section, a range of numerical examples to demonstrate the accuracy of BHNTM are provided. The BHNTM was implemented using a code written in MAPLE 2015and can be obtained from the authors. Moreover, the maximum absolute error of the approximate solution on $[x_0, x_N]$is calculated as $Error = \max|y(x)-y|$. The rate of convergence (ROC) is calculated using the formula $ROC = Log_2\left(\frac{Err^{2h}}{Err^h}\right)$, where $Err^h$is the error derived through the step size h.

4.1Implementation of the Proposed BHNTM

Step 1: choose N, $h=\frac{(x_N-x_0)}{N}$, on the interval $[x_0, x_N]$.

Step 2: using main method and additional method, for $n=0$, generate a system of equations $\left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_{\frac{69}{74}}, y_1\right)$ and $\left(y'_{\frac{5}{74}}, y'_{\frac{1}{4}}, y'_{\frac{1}{2}}, y'_{\frac{3}{4}}, y'_{\frac{69}{74}}, y'_1\right)$on the sub-interval $[x_0, x_1]$and do not solve yet.

Step 3: next, for $n=1$, generate a system of equations $\left(y_{\frac{5}{74}}, y_{\frac{1}{4}}, y_{\frac{1}{2}}, y_{\frac{3}{4}}, y_{\frac{69}{74}}, y_1\right)$ and $\left(y'_{\frac{5}{74}}, y'_{\frac{1}{4}}, y'_{\frac{1}{2}}, y'_{\frac{3}{4}}, y'_{\frac{69}{74}}, y'_1\right)$ on the subinterval$[x_1, x_2]$, and do not solve yet.

Step 4: the process is continued for $n=2,...,N-1$until all the variables on the sub-interval $[x_0,x_1],[x_2,x_3],...,[x_{N-1},x_N]$are obtained.

Step 5: create a single block matrix equation to simultaneously obtain all solutions for (1.1) on the entire interval$[x_0, x_N]$obtained in step 3 and step 4.

The following notations are used:

BHNTM$_{1,5}$: One-step Block Hybrid Nystrom-Type Method with five off-grid points

BNM: Block Nyström Method

Problem 1: Boundary value problem of Bratu-type equation
Consider the nonlinear Bratu equation of type (1.1) for $\mu = 1$, with boundary conditions as:

$$y''(x) + \lambda e^{y(x)} = 0, \quad y(0) = y(1) = 0, \qquad 0 \le x \le 1, \qquad (4.1)$$

where λ is taken as zero. The analytical solution to the above equation is:

$$y(x) = -2\ln\left[\frac{c\operatorname{osh}\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{c\operatorname{osh}\left(\frac{\theta}{2}\right)}\right], \quad x \in [0,1] \qquad (4.2)$$

whereθ is the solution of the expression:$\theta = \sqrt{2\lambda}cosh\left(\frac{\theta}{4}\right)$. Such a Bratu Problem has zero, one, or two solutions for$\lambda > \lambda_c$,$\lambda = \lambda_c$ and $\lambda < \lambda_c$respectively. The critical valueλ_c satisfies the relation:

$$1 = \frac{1}{4}\sqrt{2\lambda}\sinh\left(\frac{\theta}{4}\right). \qquad (4.3)$$

Zheng *et al.* (2014) reportedthat the critical value $\lambda_c = 3.513830719$The results for $\lambda = 1, 2,$and $3.51$are presented for Bratu BVP in Tables 2,3, 4 respectively. We solved the example using the step size $h = 0.1$ to compare the absolute errors obtained with the ones given by the methods in Jator &Manathunga (2018); Jalilian (2010); and Liao & Tan (2007). It is obvious that our method (HBNTM) when compared with most of those reported in the literature. A very rapid change in the accuracy of the proposed method is observed by increasing the off-grid points in the system.

Table 3:Comparison of Absolute Error for Problem 1, when $\lambda = 1$

x	McGough (1998)	Liao & Tan (2007)	Jalilian (2010)	Caglar et al. (2010)	Jator&Manathunga (2018)	$HBNTM_{1,5}$
0.1	1.98×10^{-6}	2.68×10^{-3}	5.77×10^{-10}	2.98×10^{-6}	1.03×10^{-15}	4.81×10^{-17}
0.2	3.94×10^{-6}	2.02×10^{-3}	2.47×10^{-10}	5.46×10^{-6}	1.68×10^{-15}	1.02×10^{-16}
0.3	5.85×10^{-6}	1.52×10^{-4}	4.56×10^{-11}	7.33×10^{-6}	3.04×10^{-15}	1.19×10^{-17}
0.4	7.70×10^{-6}	2.20×10^{-3}	9.64×10^{-11}	8.50×10^{-6}	3.77×10^{-15}	1.86×10^{-17}
0.5	9.47×10^{-6}	3.01×10^{-3}	1.46×10^{-10}	8.89×10^{-6}	3.83×10^{-15}	1.77×10^{-18}
0.6	1.11×10^{-5}	2.20×10^{-3}	9.64×10^{-11}	8.50×10^{-6}	3.77×10^{-15}	2.74×10^{-17}
0.7	1.26×10^{-5}	1.52×10^{-4}	4.56×10^{-11}	7.33×10^{-6}	3.03×10^{-15}	8.07×10^{-18}

0.8	1.35×10^{-5}	2.02×10^{-3}	2.47×10^{-10}	5.46×10^{-6}	1.67×10^{-15}	1.97×10^{-18}
0.9	1.20×10^{-5}	2.68×10^{-3}	5.77×10^{-10}	2.98×10^{-6}	1.01×10^{-15}	4.81×10^{-17}

Table 4: Comparison of Absolute Error for Problem 1 when $\lambda = 2$

x	McGough (1998)	Liao & Tan (2007)	Jalilian (2010)	Caglar *et al*. (2010)	Jator &Manathunga (2018)	$BHNTM_{1,5}$
0.1	2.13×10^{-3}	1.52×10^{-2}	9.71×10^{-9}	1.72×10^{-5}	1.91×10^{-14}	1.01×10^{-15}
0.2	4.21×10^{-3}	1.47×10^{-2}	1.41×10^{-8}	3.26×10^{-5}	6.00×10^{-14}	3.20×10^{-15}
0.3	6.19×10^{-3}	5.89×10^{-3}	1.98×10^{-8}	4.49×10^{-5}	1.17×10^{-13}	6.17×10^{-15}
0.4	8.00×10^{-3}	3.25×10^{-3}	2.42×10^{-8}	5.28×10^{-5}	1.67×10^{-13}	8.84×10^{-15}
0.5	9.60×10^{-3}	6.98×10^{-3}	2.60×10^{-8}	5.56×10^{-5}	1.88×10^{-13}	9.92×10^{-15}
0.6	1.09×10^{-3}	3.25×10^{-3}	2.42×10^{-8}	5.28×10^{-5}	1.67×10^{-13}	8.84×10^{-15}
0.7	1.19×10^{-2}	5.89×10^{-3}	1.98×10^{-8}	4.49×10^{-5}	1.16×10^{-13}	6.17×10^{-15}
0.8	1.24×10^{-2}	1.47×10^{-2}	1.41×10^{-8}	3.26×10^{-5}	5.99×10^{-14}	3.20×10^{-15}
0.9	1.09×10^{-3}	1.52×10^{-2}	9.71×10^{-9}	1.72×10^{-5}	1.90×10^{-14}	1.01×10^{-15}

Table 5: Comparison of Absolute Error for Problem 1 when $\lambda = 3.51$

x	Caglar *et al.* (2010)	Jalilian (2010)	Nasab *et al.* (2013)	Jator and Manathunga (2018)	$BHNTM_{1,5}$
0.1	3.84×10^{-2}	6.61×10^{-6}	2.34×10^{-10}	2.75×10^{-12}	1.42×10^{-14}
0.2	7.48×10^{-2}	5.83×10^{-6}	3.20×10^{-10}	1.37×10^{-12}	7.07×10^{-14}

0.3	1.06×10^{-1}	6.19×10^{-6}	7.88×10^{-10}	6.91×10^{-12}	4.41×10^{-14}
0.4	1.27×10^{-1}	6.89×10^{-6}	1.11×10^{-9}	3.13×10^{-11}	2.87×10^{-14}
0.5	1.35×10^{-1}	7.31×10^{-6}	1.22×10^{-9}	4.92×10^{-11}	8.44×10^{-14}
0.6	1.27×10^{-1}	6.89×10^{-6}	1.11×10^{-9}	3.13×10^{-11}	2.66×10^{-14}
0.7	1.06×10^{-1}	6.19×10^{-6}	7.88×10^{-10}	6.91×10^{-12}	1.55×10^{-14}
0.8	7.48×10^{-2}	5.83×10^{-6}	3.20×10^{-10}	1.37×10^{-12}	5.70×10^{-14}
0.9	3.84×10^{-2}	6.61×10^{-6}	2.34×10^{-10}	2.75×10^{-12}	4.30×10^{-16}

Problem 2: IVPs of Bratu-type equations

Consider the initial value problem of Bratu-type

$$y''(x) - 2e^{y(x)} = 0, \quad y(0) = y'(0) = 0, \quad 0 \le x \le 1. \tag{4.4}$$

with exact solution $y(x) = -2\ln(\cos x)$

Source: Jator and Manathunga (2018)

Table 6: Maximum absolute error and rate of convergence for problem 2

N	**Error in BNM (Jator, 2018)**	**ROC**	**Error in BHNTM**	**ROC**
2	4.97×10^{-6}		3.13×10^{-7}	
4	4.05×10^{-8}	6.94	2.31×10^{-9}	7.08
8	2.15×10^{-10}	7.56	1.17×10^{-11}	7.63
16	9.26×10^{-13}	7.86	4.95×10^{-14}	7.88
32	1.33×10^{-15}	9.44	1.98×10^{-16}	7.96

Table 6 at different values of N, displays the maximum absolute errors obtained, the proposed BHNTM in comparison with the BNM in Jator and Manathunga (2018). This shows the superiority of the BHNTM. It is observed that the ROCs shows that the BHNTM behaves like an order seven (7) method, which is consistent to the theoretical order of the method $p = 7$.

Table 7: Comparison of Absolute Error for problem 2

x	**Jalilian (2010) (GA)**	**Raja *et al.* (2016) (ASM)**	**Raja *et al.* (2016) (RKM)**	**Raja *et al.* (2016) (GA-ASM)**	**Jator &Manathunga (2018)**	$\mathbf{BHNTM_{1,5}}$
0.0	6.13×10^{-4}	4.66×10^{-8}	0.00	6.88×10^{-8}	0.00	0.00
0.1	5.85×10^{-4}	1.05×10^{-8}	7.45×10^{-9}	5.71×10^{-8}	3.50×10^{-15}	1.92×10^{-16}
0.2	6.19×10^{-4}	5.22×10^{-8}	1.09×10^{-8}	8.43×10^{-8}	1.63×10^{-14}	8.65×10^{-16}
0.3	7.08×10^{-4}	2.16×10^{-7}	1.36×10^{-8}	7.18×10^{-8}	4.46×10^{-14}	2.37×10^{-15}
0.4	8.03×10^{-4}	9.09×10^{-8}	1.75×10^{-8}	4.33×10^{-8}	1.05×10^{-13}	5.58×10^{-15}
0.5	8.73×10^{-4}	1.80×10^{-7}	2.41×10^{-8}	1.30×10^{-7}	2.37×10^{-13}	1.26×10^{-14}
0.6	9.42×10^{-4}	8.97×10^{-8}	3.56×10^{-8}	2.59×10^{-7}	5.47×10^{-13}	2.91×10^{-14}
0.7	1.05×10^{-3}	1.68×10^{-7}	5.43×10^{-8}	5.89×10^{-8}	1.33×10^{-12}	7.10×10^{-14}
0.8	1.21×10^{-3}	3.31×10^{-8}	7.92×10^{-8}	3.13×10^{-7}	3.54×10^{-12}	1.89×10^{-13}
0.9	1.39×10^{-3}	5.53×10^{-8}	1.14×10^{-7}	1.43×10^{-7}	1.06×10^{-11}	5.70×10^{-13}
1.0	1.73×10^{-3}	6.22×10^{-7}	1.66×10^{-7}	1.51×10^{-6}	3.78×10^{-11}	2.03×10^{-12}

Table 7 at different values of x when $N = 10$, displays the absolute errors obtained by different methods. By comparison, the BHNTM outperforms the methods found in literature.

RESULTS AND DISCUSSION

This paper describes the step-by-step implementation of the BHNTM, where a system of equations is generated on sub-intervals using block method approach of a continuous linear multistep method. These equations are not solved immediately, but instead, a block matrix equation is formed to obtain solutions simultaneously for the entire interval. This methodological approach ensures the accurate approximation of solutions to the Bratu-type equation.

The results of the numerical examples presented in the article demonstrate the effectiveness and accuracy of the proposed Block Hybrid Nystrom-Type Method (BHNTM) for solving the Bratu-type equation (4.1). The key findings are discussed in detail for each aspect:

A specific Bratu-type equation with boundary conditions, where λ is set to zero is considered. The analytical solution for this equation is given by equation (4.2), and it has zero, one, or two solutions based on certain conditions (as described in equation 4.3). The critical value of the parameterλ is also discussed in relation to the equation's solutions.

Table 3 presents a comparison of the absolute errors for Problem 1 (Bratu-type equation) when $\lambda = 1$. The results are compared with errors obtained from various other methods, including McGough (1998), Liao & Tan (2007), Jalilian (2010), Caglar *et al*. (2010), and Jator &Manathunga (2018). The $\text{BHNTM}_{1,5}$(Block

Hybrid Nystrom-Type Method with five off-grid points) consistently demonstrates significantly lower absolute errors compared to other methods, indicating its superior accuracy.

Table 4 presents a similar comparison of absolute errors for Problem 1 when $\lambda = 2$. Again, the $BHNTM_{1,5}$outperforms other methods in terms of accuracy, showcasing its robustness and efficiency.

Table 5 extends the comparison of absolute errors for Problem 1 to the case where $\lambda = 3.51$. The $BHNTM_{1,5}$ continues to exhibit remarkable accuracy, with absolute errors consistently lower than those obtained by other methods, including Caglar *et al.* (2010), Jalilian (2010), Nasab *et al.* (2013), and Jator &Manathunga (2018). The results confirm that the BHNTM is highly accurate even for different values of λ.

Overall, the numerical examples validate the accuracy and efficiency of the BHNTM in solving the Bratu-type equation. It consistently outperforms other methods across various scenarios, demonstrating its suitability for solving similar second-order nonlinear differential equations with boundary value problems. The rate of convergence (ROC) analysis, as mentioned in the article, further supports the BHNTM's effectiveness in providing accurate solutions as the step size decreases. These findings contribute to the method's credibility and applicability in scientific and engineering contexts where such equations arise.

Table 6 showcases the maximum absolute errors obtained using both the Block Nyström Method (BNM) from Jator and Manathunga (2018) and the proposed Block Hybrid Nystrom-Type Method (BHNTM) at different values of N, where N represents the number of iterations. The results clearly demonstrate the superiority of the BHNTM over the BNM in terms of accuracy.

The Rate of Convergence (ROC) analysis is performed, and it indicates that the BHNTM behaves like an order seven (7) method. This observed order of convergence ($p = 7$) aligns with the theoretical order of the BHNTM method, confirming the method's high accuracy.

Table 7 provides a comprehensive comparison of absolute errors for Problem 2 across various x values. These comparisons involve different methods, including Jalilian (2010), Raja *et al.* (2016) using Genetic Algorithm (GA), Active-Set Method (ASM), and RK4 Method (RKM), as well as Jator &Manathunga (2018), and the $BHNTM_{1,5}$. The results highlight that the $BHNTM_{1,5}$ consistently outperforms other methods across different x values, demonstrating its superior accuracy. Notably, the $BHNTM_{1,5}$achieves extremely low absolute errors, often reaching values close to zero, which indicates its high precision in approximating the exact solution.

In summary, the results findings for Problem 2 affirm the exceptional accuracy and efficiency of the BHNTM in solving initial value problems of Bratu-type equations. The method consistently outperforms other numerical methods, demonstrating its potential as a reliable and accurate solver for such nonlinear differential equations. The Rate of Convergence analysis further supports the BHNTM's theoretical order of convergence, reinforcing its credibility as a numerical method for scientific and engineering applications where precision is crucial.

CONCLUSION

The well-known Bratu problems arise in different variety of applications and many researchers have drawn attention to solve them. The difficulties that exist in these problems due to the strong nonlinear terms is overcome here. The main characteristics of the proposed method is solving the problem without reducing them into their initial value equivalent, taking away the computational burdens and time wastage attached to them. It is also observed that increasing the off-grid points results in improved accuracy. Numerical results confirmed that the BHNTM is preferred to reported methods in literature in terms of accuracy and consistency.

Declaration of competing interest

The authors declare that there is no known conflicting interests.

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