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Generalized Derivations via σ -Prime Rings

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ABSTRACT

Let R be a $2-$torsion free $\sigma-$prime ring with involution σ, U a nonzero $\sigma-$Lie ideal, I a nonzero $\sigma-$ideal of R, S an appropriate subset of R and G a generalized derivation associated with a nonzero derivation δ of R commuting with σ. It is shown that the commutativity of a $\sigma-$prime ring R admitting a generalized derivation G satisfying one of the conditions: (P_1) $[\partial(x), G(y)] = \pm [x, y]$ and (P_2) $\partial(x) \circ G(y) = \pm\, x \circ y$, for every $x, y \in S$.

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Keywords: Generalized derivation, σ-Lie ideal, ring with involution, σ- prime ring.

INTRODUCTION

Throughout the paper, R will represent an associative ring with center $Z(R)$. Recall that a ring R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, R^0 is an opposite ring of a prime ring R with $(x, y) = (y, x)$ $\forall\, x, y \in R$, and an additive mapping $x \to \sigma(x)$ of R into itself is called an involution σ if it satisfies (i) $\sigma(\sigma(x)) = x$, (ii) $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in R$. If R has an involution σ, then R is said to be σ-prime if $aRb = (0) = aR\sigma(b)$ implies $a = 0$ or $b = 0$. Every prime ring equipped with an involution σ is σ-prime but the converse need not be true in general. As an example, taking $S = R \times R^0$, then S is not prime if $(0, a)S(a, 0) = (0)$. But if we take $(a,b)S(x, y) = (0)$ and $(a,b)S\sigma((x, y)) = (0)$, then $a\,Rx \times yRb = (0)$ and $a\,Ry \times xRb = (0)$, and thus $a\,Rx = yRb = aRy = xRb = (0)$ that shows R is σ-prime (see [5], for reference). An ideal I of R is a σ-ideal if I is invariant under σ (i.e. $\sigma(I) = I$). An additive subgroup U of R is said to be a Lie-ideal if $[u, r] \in U$ for all $u \in U$, $r \in R$. A Lie ideal U is called a $\sigma-$Lie ideal if $\sigma(U) = U$. Next, suppose U is a Lie (resp. $\sigma-$Lie) ideal of R. Then U is called a square closed Lie (resp. $\sigma-$Lie) ideal if $u^2 \in U, \forall u \in U$ and define a set of symmetric and skew symmetric elements of R as follows $S_{a_\sigma}(R) = \{x \in R | \sigma(x) = \pm x\}$.

Now, we recall the basic commutator identities. If for any $x, y, z \in R$, then:

- $[xy, z] = x[y, z] + [x, z]y$; $[x, yz] = y[x, z] + [x, y]z$.
- $[xy, z] = x[y, z] + [x, z]y$; $[x, yz] = y[x, z] + [x, y]z$
- $(x y) \circ z = x(y \circ z) - [x, z] y = (x \circ z)y + x[y, z]$;
- $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y] z$.

An additive map G: R → R is called a generalized derivation associated with ∂ if there exists a derivation ∂: R → R (an additive map ∂: R → R is called a derivation if $\partial(xy) = \partial(x)y + x\partial(y)$ holds for all x, y ∈ R) such that $G(xy) = G(x)y + x\partial(y)$ for all $x, y \in R$ (see [2] for references). It is noted that the notion of generalized derivation covers the notions of derivation and a left multiplier ($G(xy) = G(x)y$, for all $x, y \in R$). In particular: For a fixed $a \in R$, the map $\partial_a: R \to R$ defined by $\partial_a(x) = [a, x]$ for all $x \in R$ is a derivation which is said to be an inner derivation. An additive map $G_{a,b}: R \to R$ is called a generalized inner derivation if $G_{a,b}(x) = ax + xb$ for some fixed $a, b \in R$. It is easy to see that if $G_{a,b}(x)$ is a generalized inner derivation, then

$G_{a,b}(x)(xy) = G_{a,b}(x)y + x\partial_{-b}(y)$ for all $x, y \in R$, where ∂_{-b} is an inner derivation.

A number of authors [1, 2, 3, 15, 16] have established a vast theory concerning derivations and generalized derivations of prime and semi prime rings. Oukhtite et al.[6] gave rise to an extension of prime rings in the form of σ -prime rings and proved numerous results which hold true for prime rings (see for references [5 - 13]). In their papers [15, 16], Huang contributed to the newly emerged theory by extending the results of ([2, 3]). In this context, Khan et al. ([3, 4]) extended results concerning derivations and generalized derivations of σ -prime rings to some more general settings. In [1] , Ashraf et al. studied the commutativity of a prime ring R admitting a generalized derivation F with associated derivation d satisfying any one of the conditions: (i) $d(x) \circ F(y) = 0$, (ii) $[d(x), F(y)] = 0$, (iii) $d(x) \circ F(y) \pm x \circ y = 0$, (iv) $(d(x) \circ F(y)) \pm [x, y] = 0$, (v) $[d(x), F(y)] \pm [x, y] = 0$, and (vi) $[d(x), F(y)] \pm x \circ y = 0$ for all $x, y \in I$, where I is non zero ideal of prime ring R. Huang [15] obtained similar results by considering Lie ideals instead. Motivated by above observations it is natural to ask a question: Under what additional conditions the above properties hold for σ -prime rings on σ-Lie ideals. To answer this question in affirmative and the aim of this paper is to extend some results for σ -prime rings.

MAIN RESULTS

Theorem 2.1: Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose R admits a generalized derivation F associated with a derivation ∂, commuting with σ, such that $\partial(x) \circ G(y) = \pm x \circ y$, for all $x, y \in U$. If $G = 0$ or $\partial \neq 0$ then $U \subseteq Z(R)$.

Theorem 2.2: Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose R admits a generalized derivation F associated with a derivation d, commuting with σ, such that $[\partial(x), G(y)] \pm [x, y] = 0$, for all $x, y \in U$, then $U \subseteq Z(R)$.

In order to prove our main results, we need the following known results.

Result 2.1([12, Lemma 4]) If $U \subseteq Z(R)$ is a σ-Lie ideal of a 2-torsion free σ-prime ring R and $a, b \in R$ such that $aUb = \sigma(a)Ub = 0$ or $aUb = aU\sigma(b)$, then $a = 0$ or $b = 0$.

Result 2.2 ([9, Theorem 1.1])Let R be a 2-torsion free σ-prime ring, U a nonzero Lie ideal of R and ∂ a nonzero derivation of R which commutes with σ. If $\partial^2(U) = 0$,then $U \subset Z(R)$.

Result 2.3 ([9, Lemma 2.3]) Let $0 \neq U$ be a σ-Lie ideal of a 2-torsion free σ-prime ring R.
If $[U, U] = 0$, then $U \subset Z(R)$.

Result 2.4 ([7, Lemma 2.2]) Let R be a 2-torsion free σ-prime ring and U a nonzero σ-Lie ideal of R. If ∂ is a derivation of R which commutes with σ and satisfies $\partial(U) = 0$, then either $\partial = 0$ or $U \subseteq Z(R)$.
Next, we prove the following.

Result 2.5 Let R be a 2-torsion free σ-prime ring and U a nonzero σ-Lie ideal of R. If ∂ is a derivation of R which commutes with σ such that $\partial(x) \circ y = 0$ for all $x, y \in U$, then either $\partial = 0$ or $U \subseteq Z(R)$.

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Proof: Let $U \not\subset Z(R)$. We have

$$\partial(x) \circ y = 0 \text{ for all } x, y \in U. \quad (1)$$

Setting y by y z in (1), we obtain

$$y\,[\partial(x), z] = 0 \text{ for all } x, y, z \in U. \quad (2)$$

Replacing y by $\partial^2(x)\,y$ in (2), we obtain

$$\partial^2(x)\,y\,[\partial(x), z] = 0 \text{ for all } x, y, z \in U.$$

or $$\partial^2(x)\,U\,[\partial(x), z] = 0 \text{ for all } x, z \in U. \quad (3)$$

Let $x \in U \cap S_a\sigma(R)$. Since ∂ commutes with σ, application of Result 2.1 in the relation (3),

yields $[\partial(x), z] = 0$ or $\partial^2(x) = 0$ for all $z \in U$.

Let $x \in U$. Since $x - \sigma(x) \in U \cap S_a\sigma(R)$, from (3) in combination with Result 2.1, it follows that

$$[\partial(x - \sigma(x)), z] = 0 \text{ or } \partial^2(x - \sigma(x)) = 0 \text{ for all } w \in U.$$

Now we break the proof in two steps.

Step1: Take $[\partial(x - \sigma(x)), z] = 0$. This implies that

$$[\partial(x), z] = [\partial(\sigma(x)), z] = \sigma([d(x), w]).$$

In view of (3), we have

$$[\partial(x), z] = 0 \text{ or } \partial^2(x) = 0 \text{ for all } x \in U.$$

Step 2: Let $\partial^2(x - \sigma(x)) = 0$. Since ∂ commutes with σ, we have

$$\partial(\partial(x) - \partial(\sigma(x))) = 0$$

or $$\partial^2(x) - \partial(\sigma(\partial(x))) = 0$$

or $$\partial^2(x) = \sigma(\partial^2(x)).$$

Thus, $\partial^2(x) \in S_a\sigma(R)$ and again by virtue of Result 2.1 and (3), we get

$$[\partial(x), z] = 0 \text{ or } \partial^2(x) = 0 \text{ for all } x \in U. \quad (4)$$

One can conclude that, for all $x \in U$ we have either $[\partial(x), z] = 0$ or $\partial^2(x) = 0$.

Clearly, U is a union of two additive subgroups Y and Z, where

$$Y = \{x \in U \mid [\partial(x), z] = 0, \text{ for all } z \in U\} \text{ and } Z = \{x \in U \mid \partial^2(x) = 0\}.$$

But a group cannot be a union of two of its subgroups and thus U = Y or U = Z.

If U = Y, then by virtue of above obtained result (The relation (3) is same as obtained in proof of the [14, Theorem 3.2], yields $\partial = 0$), we get either $\partial = 0$ or $U \subseteq Z(R)$.

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If U = Z, then by application of Result 2.2 we get either $\partial = 0$ or $U \subseteq Z(R)$.

Hence, both the cases yield either $\partial = 0$ or $U \subseteq Z(R)$. But, if ∂ is non-zero, then U must be contained in Z(R).

Remark 2.1: The technique used in proof of Theorem 3.2 from equation (4) onwards till end would be needed further in the paper.

Now, we are in a position to prove our main results.

Proof of Theorem 2.1. First, we divide the condition in two parts.

i. $\partial(x) o\, G(y) - x\, o\, y = 0$, *for all* $x, y \in U$

ii. $\partial(x) o\, G(y) + x\, o\, y = 0$, *for all* $x, y \in U$

Taking G = 0 in (i), then $x\, o\, y = 0$ for all $x, y \in U$. (2.1)

For any $z \in U$, replacing y by yz in (2.1), we get, $(x\, o\, y)z - y[x, z] = 0$. This implies

$y\,[x, z] = 0$. For all $y \in U$, we have $U[x, z] = 0$ or $\sigma(1)\, U\,[x, z] = 0$.

Since U is σ-Prime Lie ideal of R, we have

$$[x, z] = 0 \text{ for all } x, z \in U. \quad (2.2)$$

In view of Result 2.3, (2.2) implies that $U \subseteq Z(R)$.

Now assume that $\partial \neq 0$. We have

$$\partial(x)\ o\ G(y) = \pm\, x\ o\ y \text{ for all } x, y \in U. \quad (2.3)$$

For any $z \in U$, replacing y by y z in (2.3), we get

$$(\partial(x)\ o\ y)\ \partial(z) - y\,[\partial(x), \partial(z)] - G(y)[\partial(x), z] + y[x, z] = 0 \quad (2.4)$$

Replacing z by $z\partial(x)$ in (2.4) and applying (2.4), we obtain

$$(\partial(x)\ o\ y)\ z\ \partial^2(x) - y[\partial(x), z\partial^2(x)] + yz\,[x, \partial(x)] = 0 \quad (2.5)$$

Substitute y by z y in (2.5) and using (2.5), we have

$$[\partial(x), z]\ U\ \partial^2(x) = 0 \text{ for all } x, z \in U. \quad (2.6)$$

Consequently, by application of Result 2.1 in (2.6), we get $U \subseteq Z(R)$.

Now using similar techniques, one can also prove the (ii).

Next, suppose $U \not\subset Z(R)$ in (i). We have

$$\partial(x)\ o\ G(y) - [x, y] = 0, \text{ for all } x, y \in U \quad (2.7)$$

Replacing y by y z in (2.7) and using (2.7), we get

$$- G(y)\,[\partial(x), z] + (\partial(x)\ o\ y)\ \partial(z) - y[\partial(x), \partial(z)] = y[x, z] \quad (2.8)$$

Substituting z by $z\partial(x)$ in (2.8) and applying (2.8), we obtain

$$(\partial(x) \text{ o } y)\, z\, \partial^2(x) - y\, z\, [\partial(x), \partial^2(x)] - y\, [\partial(x), z]\, \partial^2(x) = y\, z\, [x, \partial(x)] \qquad (2.9)$$

For any $w \in U$, replacing y by w y in (2.9) and using (2.9), we have

$$[\partial(x), w]\, y\, z\, \partial^2(x) = 0 \text{ for all } x, y, z \in U$$

or $[\partial(x), w]\, U\, \partial^2(x) = 0$ for all $x, w \in U$. (2.10)

Further, we use the techniques as used in the proof of Result 2.1, one concludes the result.

Likewise technique as used in (ii) with necessary variations, the result follows immediately.

Proof of Theorem 2.2.: Suppose $U \not\subset Z(R)$. We have

$[\partial(x), G(y)] = x \text{ o } y$, for all $x, y \in U$ (2.11)

Replacing y by y z in (2.11) and using (2.11), we get

$G(y)\, [\partial(x), z] + [\partial(x), y]\, \partial(z) + y[\partial(x), \partial(z)] = - y[x, z]$ (2.12)

Substituting z by $z\partial(x)$ in (2.12) and applying (2.12), we obtain

$$y\, z\, [\partial(x), \partial^2(x)] + y\, [\partial(x), z]\, \partial^2(x) + [\partial(x), y]\, z\, \partial^2(x) = - y\, z\, [x, \partial(x)] \qquad (2.13)$$

For any $w \in U$, replacing y by w y in (2.13) and using (2.13), we have

$$[\partial(x), w]\, y\, z\, \partial^2(x) = 0 \text{ for all } x, y, z \in U$$

or $[\partial(x), w]\, U\, \partial^2(x) = 0$ for all $x, w \in U$. (2.14)

Applying Result 2.5, in the relation (2.14), we get the desired result.

(ii) Using the similar technique as used in (i) with necessary variations, the result follows.

COROLLARIES

In theorems 2.1 and 2.2, if we put x o y = 0 and [x, y] = 0, then we obtain

Corollary 3.1: *Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose there exists a generalized derivation F associated with a nonzero derivation d, commuting with σ, such that* $\partial(x) \text{ o } G(y) = 0$*, for all* $x, y \in U$*, then* $U \subseteq Z(R)$*.*

Corollary 3.2: *Let R be a 2-torsion free σ-prime ring and U a nonzero square closed σ- Lie ideal of R. Suppose there exists a generalized derivation F associated with a nonzero derivation d, commuting with σ, such that* $[d(x), F(y)] = 0$*, for all* $x, y \in U$*, then* $U \subseteq Z(R)$*.*

Remark 3.1. Recently several authors [1, 3, 4, 5, 6, 10, 11, 13] have proved commutativity theorems for prime and σ-prime rings admitting derivations which are centralizing or commuting on some appropriate subsets of R. Huang [15] obtained similar results by considering Lie ideals. The relationship between the Lie ideals and generalized derivations on prime rings has been established by many authors.

In this sequel, it would be interested to ask a natural question as given below.

Question Let R be a 2-torsion free σ-semi prime ring and L a nonzero σ-Lie ideal of R. If R admits a generalized derivation G associated with a nonzero derivation ∂, commuting with σ, such that one of the following holds: (i) $\partial(x) o\ G(y) = \pm\ x\ o\ y$; and (ii) $[\partial(x), G(y)] = \pm [x, y]\ \forall x, y \in L$. Then R is commutative.

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