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Matrix-Closed Weak Topologies and the Intersection of the Line-Open Topologies in R^n

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ABSTRACT

In the paper published by the authors in the International Journal of Research and Scientific Innovation (IJRSI), in September 2024, titled Hyperplane-Open Weak Topologies in R^n, a general procedure for constructing hyperplane-open weak topology on R^n was revealed. The process led to, among other things, the formulation of matrix-open weak topology on the Cartesian plane. In the present work, we are to construct a matrix-closed topology on the Cartesian plane. Also, in the first work we constructed horizontal and vertical line-open weak topologies of the plane and showed that the usual topology (of the plane) is actually weaker than the intersection of these two topologies. Then we made a conjecture that the usual topology is actually equal to this intersection; and our reviewers opined that a definite proof (or disproof) of that conjecture was necessarily important, in order to answer an unsettled question regarding the entire work. This task is what we have in addition undertaken in the present work: We have supplied here a definite proof that the usual topology of R^2 and indeed of R^n is actually the intersection of the horizontal line and the vertical line-open weak topologies of R^2 (or R^n).

Keywords: Weak Topology, Hyperplane-Open Weak Topology, Vertical (or Horizontal) Line Open Topology
Mathematics Subjects Classification (MSC) 2020: 54A05, 54A10

INTRODUCTION

This paper is a follow-up on our first one that was published in the international journal or research and scientific innovation (IJRSI).

We recall the following definition of the cofinite topology.

Definition 1.1 Let X be an infinite set and let C = {A ⊂ X: A^c is finite}∪{∅}. Then C is a topology on X, called the cofinite topology on X.

Even though one may loosely say that open sets of a cofiniteType equation here. topology are either finite or have finite complement, it is not quite correct to say that the cofinite topology on an infinite set X is a collection of all subsets of X which are either finite or have finite complement. Reason: Consider the set R of real numbers. If we define cofinite topology on IR using the loose definition above, then a problem will occur as follows. Suppose C is the family of all subsets of R which are either finite or have finite complement. For each natural number n, let $G_n = \{n\}$, the singleton of n. Then each G_n belongs to C, as a finite set. Take the union $\cup_{n=1}^{\infty} G_n$ of all such sets. Then this union does not belong to C since it is infinite and its complement, $\left(\bigcup_{n=1}^{\infty} G_n\right)^c = \bigcap_{n=1}^{\infty} G_n^c$, is infinite because it includes (among others) all the irrational numbers, which are themselves even uncountable.

Some of the well known properties of the cofinite topology C on a set X are as follows:

1. For an infinite set X, the complement of every C-open set (apart from the empty set) is finite—this is the actual complement finite or co-finite property.
2. If X is infinite, then C has infinitely many open sets.
3. If X is infinite, then C is not closed under arbitrary intersections.

4. There is one and only one cofinite topology C on a set X.
5. The cofinite topology C on a set X is always T_1.

Main Results—Matrix-Closed Weak Topologies in R^n

Construction 1 Let us reconsider the projection maps $p_i : R^2 \to R$, where $1 \le i \le 2$, such that $p_1(x, y) = x$ and $p_2(x, y) = y$. We wish to find the weak topology, induced by the projection maps on R^2 when the two factor spaces of R^2 are endowed with the cofinite topology of R.

Let T_{CR} denote the cofinite topology on the set R of real numbers. Let $G \in T_{CR}$ be a nonempty open subset of R in the cofinite topology. Then we know that the complement of G is a finite subset of R, say $G^c = \{g_1, . . . , g_n\}$. The inverse image of G under the first projection map p_1, is $p_1^{-1}(G) =$

$\{(x, y) \in R^2 : p_1((x, y)) \in G\}$

$= \{(x, y) \in R^2 : p_1((x, y)) \in R$ and $p_1((x, y)) \notin G^c\}$

$= \{(x, y) \in R^2 : p_1((x, y)) \notin \{ g_1,g_2, \cdot\cdot\cdot, g_n\}$

$= \{(x, y) \in R^2 : x \notin G^c\}$

$= \{R^2$, without a finite number of infinite vertical lines $\}$

$= \{R^2$, without exactly n vertical infinite lines through the points $g_1, g_2, \cdot\cdot\cdot, g_n$ on the horizontal axis$\}$.

In a similar way, the inverse image of G under the second projection map is $p_2^{-1}(G) = \{(x, y) \in R^2 : p_2((x, y)) \in G\}$

$= \{(x, y) \in R^2 : p_2((x, y)) \in R$ and $p_2((x, y)) \notin G^c\}$

$= \{(x, y) \in R^2 : p_2((x, y)) \notin \{ g_1, g_2, \cdot\cdot\cdot, g_n\}$

$= \{(x, y) \in R^2 : y \notin G^c,\}$

$= \{R^2$, without a finite number of infinite horizontal lines $\}$

$= \{R^2$, without exactly n horizontal infinite lines through the points $g_1, g_2, \cdot\cdot\cdot, g_n$ on the vertical axis$\}$.

The sets $p_1^{-1}(G)$ and $p_2^{-1}(G)$ are among the sub-basic sets of this weak topology on R^2 generated by the projection maps. It follows that the base (or basis) for this weak topology includes, among other sets, the set $U = p_1^{-1}(G) \bigcap p_2^{-1}(G) =$

$\{(x, y) \in R^2 : x, y \notin \{g_1, \cdot\cdot\cdot, g_n\}\} =$

$\{(x, y) \in R^2 : x \notin \{g_1, \cdot\cdot\cdot, g_n\}$ and $y \notin \{g_1, \cdot\cdot\cdot, g_n\}\} = \{(x, y) \in R^2 : x \in G^c$ and $y \in G^c\} = R^2 - (G^c \times G^c)$.

$= R^2 - (\{g_1, \cdot\cdot\cdot, g_n\} \times \{g_1, . . ., g_n\})$

$= R^2 - \{(g_1,g_1),(g_1,g_2),\cdots,(g_1,g_n),(g_2,g_1),(g_2,g_2),\cdots,(g_2,g_n),\cdots,(g_n,g_n)\}$. = the entire plane without a square $n \times n$ matrix of coordinate points So the basic open set U equals the entire plane minus a finite square $n \times n$ matrix of coordinate points. Since a set is open if and only if its complement is closed (in any topological space), the closed complement of the open set U is the set $U^c = M$, an $n \times n$ matrix of coordinate points in the plane. The closed matrix M is shown below.

$$M = \begin{bmatrix} (g_1, g_n) & (g_2, g_n) \cdots (g_n, g_n) \\ \vdots & \\ (g_1, g_2) & (g_2, g_2) \cdots (g_n, g_2) \\ (g_1, g_1) & (g_2, g_1) \cdots (g_n, g_1) \end{bmatrix}$$

If G, H $\in T_{CR}$ are two open sets in the cofinite topology of R such that $G \neq H$, $H^c = \{h_1, \cdots, h_m\}$, $G^c = \{g_1, \cdots, g_n\}$ and $m \neq n$, then following the steps we have taken before we shall have a basis element as

$V = p^{-1}_1(G) \cap p^{-1}_2(H) = \{(x, y) \in R^2 : x \in G^c \text{ and } y \in H^c\} = R^2 - (G^c \times H^c)$.

$= R^2 - (\{g_1, \cdots, g_n\} \times \{h_1, \ldots, h_m\})$

$= R^2 - \{(g_1,h_1),(g_1,h_2),\cdots,(g_1,h_m),(g_2,h_1),(g_2,h_2),\cdots,(g_2,h_m),\cdots,(g_n,h_m)\}$. Then the closed set which is the complement of V is the n×m matrix M shown below.

$$M = \begin{pmatrix} (g_1, h_m) & \cdots & (g_n, h_m) \\ \vdots & \ddots & \vdots \\ (g_1, h_1) & \cdots & (g_n, h_1) \end{pmatrix}$$

In fact all kinds of matrices (column, row vectors, and others) of coordinate points are closed sets of this topological space.

If we endow the three factor spaces of R^3 with the cofinite topology, the product topology (i.e. the weak topology generated by the projection maps) that would result on R^3 will include cuboids of finite number of coordinate points: One can see this as the result of taking a matrix of coordinate points already formed in the plane through a distance along a line segment parallel to the third coordinate axis. We will still regard this as a matrix-closed topology because R^2 in the cofinite topology-induced weak topology is a subspace of R^3 in the cofinite topology-induced weak topology. In the same way R^4, R^5, etc., R^n in the cofinite topology-induced weak topology are matrix-closed topological spaces.

Main Results—The Intersection of the Line-Open Topologies of R^n

First, we recast how we constructed the vertical line-open and the horizontal line-open topologies of the Cartesian plane.

Construction 2 Consider R^2. Let the horizontal factor space be endowed the discrete topology (R, D) and the vertical factor space be endowed with the usual topology (R, u) of R. Then the coarsest topology on R^2 with respect to which the projection maps p_1 and p_2 are continuous is called the vertical line open topology of R^2 because vertical lines (of all lengths) are among the basic open sets of this topology.

Construction 3 Consider R^2 but now with the horizontal factor space R_1 endowed with the usual topology and the vertical factor space R_2 endowed with the discrete topology. Then the horizontal line open topology results.

Proposition 3.1 Let τ_u, τ_v and τ_h denote respectively the usual topology, the vertical line-open topology, and the horizontal line-open topology of the Cartesian plane R^2. Then $\tau_u = \tau_v \cap \tau_h$

Proof:

Let $\tau = \tau_v \cap \tau_h$ and let B^* be a basis for τ. Let D represent the discrete topology on R and E the usual topology of R. Let $B \in B^*$ be a basis element for τ. Then B is τ-open. This implies that $B \in \tau_v$ and $B \in \tau_h$. Now, since $B \in \tau_v$, it is of the form $B = U \times V$ where U is an open set in the horizontal factor space of R^2 (which has the discrete topology of R under τ_v), and V is an open set in the vertical factor space that is endowed with the usual topology E of R. Also, since $B \in \tau_h$, it is of the form $B = U' \times V'$ where U' is an open set in the horizontal factor space (which now has the usual topology of R under τ_u), and V' is an open set in the vertical factor space which is endowed with the discrete topology of R. Then we have

$U \times V = B = U' \times V'$.

$\Longrightarrow U \times V = U' \times V'$.

It is known that two vectors are equal if and only if their corresponding components are equal. Likewise two Cartesian products of sets are equal if and only if their corresponding factors are equal. It follows from the last equality, therefore, that $U = U'$ and $V = V'$. Since (remember) U is an open set in the discrete topology of R and U' is an open set in the usual topology of R, it follows that the subset $U = U'$ of R is open in both the discrete and the usual topologies of R. The only subsets of R that are open in these two topologies on R are the usual topology-open sets, because E is strictly weaker than D. This means that $U = U'$ is an open set in the usual topology E on R. In a similar way, $V = V'$ is an E-open subset of R. Therefore $B = U \times V$ is a basis element in the usual topology τ_u of R^2. Since $B \in B^*$ is an arbitrary basis element for τ, it follows that every τ-open set is τ_u-open. Because we have proved in our first paper on this topic that every τ_u-open set is τ-open, we now have $\tau_u \leq \tau \leq \tau_u$. That is, $\tau_u \leq \tau_v \cap \tau_h \leq \tau_u$. Hence $\tau_u = \tau_v \cap \tau_h$.

REMARK

1. If we appropriately vary the topologies endowed the factor spaces of R^3, R^4, or R^n between the usual and the discrete topologies of R—as we did in constructions 2 and 3—we would get the line-open topologies of R^3, R^4, or R^n respectively.
2. If we copiously follow the steps in the proof above and write an arbitrary basic element of the intersection of the various line-open topologies of R^n in two ways as done above, we would find that the usual topology of R^n equals this intersection of the various line-open topologies of R^n. In other words, the usual euclidean topology of R^n is the intersection of the various line-open topologies of R^n if the factor spaces are appropriately varied between the discrete and the usual topologies of R.
3. Further research may now be carried out on finding a well-known topology (on any set) that emerges as the intersection of several other topologies on the set.

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