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Study on the Concreate Model for the Development of Supergeometry as a New Category of Supermanifolds

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ABSTRACT

In this present paper sheaves, notion of GH^{∞} supermanifolds, graded manifolds, morphism, *Z*-expansion functions of variables in $B_L^{m,n}$, sheaves of graded commutative B_L-algebra, G-supermanifolds, topologies of rings of G-functions are studied. We establish a theorem demonstrating that the *Z*-expansion on an isometry onto its image and prove a related metric isomorphism.

Keywords: Supermanifolds, Sheaf, Morphism, Graded G-module, Seminorms $p_K^{I,\mu}$

INTRODUCTION

The category of G-supermanifolds [3], [4] provides a consistent and concrete model for the development of supergeometry. In order to supply exact motivation for the development these objects and also for historical argument we strated with a brief representation graded manifolds, these are basically introduced by Berzin and Le$\hat{\imath}$tes [8], [14]; although the most widespread treatment can be found in Kostant [12] and Manin [13]. In this way G-supermanifolds could be expanded more consistant or concreate compared to traditional model. Graded manifolds [2] play a directrole in the theory developed in this paper, in that some results holding in that category can be applied as they are in the context of G-supermanifolds. The discussion of the relationship between G-supermanifolds and the axiomatics for supermanifolds proposed by Rathstein [19]. The classes of G^{∞}, GH^{∞} and H^{∞} supersmooth functions are used which allow us to define supermanifolds in the sense of Rogers [15], [16], [17], [18]; the discussion of their short comings leads us to introduce the notion of G-supermanifolds and *Z*-expansion. In the present work a theorem on an isometry and its image of *Z*-expansion and on a metric isomorphism is established.

PRELIMINARIES

The original idea of geometric approach to supermanifolds [10] is to patch open sets in $B_L^{m,n}$ by means of transition functions which fulfill a suitable 'smoothness' condition. We call generically supersmooth functions such as G^{∞}, GH^{∞} and H^{∞} functions. These functions are introduced in a unified manner, in terms of a morphism called *Z*-expansion, which maps functions of real variables into functions of variables in $B_L^{m,n}$. Unless otherwise stated, whenever referring, explicitly or implicitly, to a topology on $B_L^{m,n}$, we mean its ∇-vectorspace topology. Throughout this paper we assume to choose integers *L*, *m* and *n* with $L > 0$ and $m, n \geq 0$ subject to the condition $L \geq n$. For every integer L' such that $0 \leq L' \leq L$ the exterior algebra $B_{L'}$ is regarded as a subalgebra of B_L so that B_L acquires a structure of a graded $B_{L'}$-module which is not free unless $L' = 0$ or $L' = L$. We recall that the graded vector space associated with $B_L^{m|n}$ according to the procedure is simply $\nabla^m \oplus \nabla^n$. We denote by $\sigma^{m,n} : B_L^{m,n} \to \nabla^m$ the restriction of

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the augmentation map to $B_L^{m,n}$.

For any C^∞ differentiable manifold X, let us denote by $\mathrm{C}_{L'}^{\infty}(W)$ the graded algebra of $B_{L'}$-valued C^∞ functions on the open set $W \subset X$. For each integer $L' \le L$ and any $U \subset \nabla^m$, the Z-expansion is the morphism of graded algebras

$$Z_{L'} : \mathrm{C}_{L'}^{\infty}(U) \to \mathrm{C}_{L}^{\infty}((\sigma^{m,0})^{-1}(U)),$$

defined by the formula (cf. [18])

$$Z_{L'}(h)(x) = h(\sigma^{m,0}(x) + \sum_{j=1}^{L} \frac{1}{j!} D^{(j)} h_{\sigma^{m,0}(x)}(s^{m,0}(x), \ldots, s^{m,0}(x)) \quad (2.1)$$

for all $h \in \mathrm{C}_{L'}^{\infty}(U)$ and all $x \in (\sigma^{m,0})^{-1}(U)$; here the j-th Fre′cht differential $D^{(j)} h_{\sigma^{m,0}(x)}$ at the point $\sigma^{m,0}(x)$ acts on $B_L^{m,0} \times \cdots \times B_L^{m,0}$ (j-times) simply by extending by $(B_L)_0$-linearity its action on $\nabla^m \times \cdots \times \nabla^m$. The mapping $\mathrm{S}^{m,0}: B_L^{m,0} \to \mathbf{n}_L^{m,0}$ is the projection onto the second component of the direct sum $B_L^{m,0} = \nabla^m \oplus \mathbf{n}_L^{m,0}$.

For each open $U \subset \nabla^m$, $(\sigma^{m,0})^{-1}(U) \subset B_L^{m,0}$ is a subset of $B_L^{m,n}$, so that we can define on the open set $(\sigma^{m,n})^{-1}(U) \subset B_L^{m,n}$ the graded algebra $\mathrm{S}_{L'}((\sigma^{m,n})^{-1}(U))$ formed by the functions having the following expression

$$f(x^1, \ldots, x^m, y^1, \ldots, y^n) = \sum_{\mu \in \Xi_n} f_\mu(x^1, \ldots, x^m) y^\mu \qquad (2.2)$$

where $f_\mu \in Z_{L'}(\mathrm{C}_{L'}^{\infty}(U))$, $(x^1, \ldots, x^m, y^1, \ldots, y^n) \in (\sigma^{m,n})^{-1}(U)$ and $y^\mu = y^{\mu(1)} \ldots y^{\mu(r)}$ if $\mu = \{\mu(1), \ldots, \mu(r)\}$..

We can therefore introduce a sheaf [1] $\mathrm{S}_{L'}$ of graded-commutative $B_{L'}$-algebras over $B_L^{m,n}$ by letting, for each open $V \subset B_L^{m,n}$,

$$\mathrm{S}_{L'}(V) = \mathrm{S}_{L'}((\sigma^{m,n})^{-1} \sigma^{m,n}(V)). \qquad (2.3)$$

The sections of the sheaf $\mathrm{S}_{L'}$ on an open set V are C^∞ functions which show a kind of holomorphic behaviour in the nilpotent directions, in that the coefficients of the various powers of the y's in the equation (2.2) are determined, at every point z of the fibre $(\sigma^{m,n})^{-1}(x)$ of $B_L^{m,n}$ over $x = \sigma^{m,n}(z) \in \nabla^m$, by their germs at x.

We denote by $\hat{\mathrm{S}}_{L'}$ the subsheaf of $\mathrm{S}_{L'}$ whose sections are functions not depending on the odd variables y^a, namely, they have only the first term in the sum (2.2). In other words, the sheaf $\hat{\mathrm{S}}_{L'}$ on $B_L^{m,n}$ is the inverse image under the projection $B_L^{m,n} \to B_L^{m,0}$ of the sheaf $\mathrm{S}_{L'}$ on $B_L^{m,0}$. Then equation (2.2) shows the existence, for any open $U \subset B_L^{m,n}$, of a surjective morphism

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$$\lambda : \hat{\mathrm{S}}_{L'}(U) \otimes_{\nabla} \wedge_{\nabla} \nabla^{n} \to \mathrm{S}_{L'}(U)$$

$$\sum_{\mu \in \Xi_n} f_\mu \otimes y^\mu \mapsto \sum_{\mu \in \Xi_n} f_\mu y^\mu, \qquad (2.4)$$

having identified $\wedge_{\nabla} \nabla^{n}$ with the exterior algebra generated by the y's.

G-SUPERMANIFOLDS

We have seen that the classes of supersmooth functions which is free from inconsistencies and yields a theory aplicable to supersymmetry [5], is nottrivial. In particular it seems rather difficult to combine the following requirements:

(*a*) the sheaf of derivations of the function sheaf under consideration should be locally free;

(*b*) the coefficients of the 'superfield expansion' (2.2), when restricted to real arguments, should take values in a graded-commutative algebra *B;*

(*c*) there should be a good theory of superbundles, and in particular there is a sensible notion of graded tangent space.

These difficulties can be overcome by introducing a new category of supermanifolds [6], called G-supermanifolds, characterized in terms of a sheaf G on $B_L^{m,n}$, which is in a sense a 'completion' of $\mathrm{GH}_{L'}$ (condition $L - L' \geq n$ is assumed to hold). More precisely, we define the sheaf of graded-commutative B_L-algebras on $B_L^{m,n}$

$$\mathrm{G}_{L'} \equiv \mathrm{GH}_{L'} \otimes_{B_{L'}} B_L \qquad (3.1)$$

It is convenient to introduce an evaluation morphism $\delta : \mathrm{G}_{L'} \to \mathrm{C}_L$ (we denote by C_L the sheaf of B_L-valued continuous functions on $B_L^{m,n}$), by extending by additivity the mapping

$$\delta(f \otimes a) = fa \qquad (3.2)$$

Proposition 3.1 *The image of* δ *is isomorphic to the sheaf* G^∞ of G^∞ *functions on* $B_L^{m,n}$. *The morphism* δ *is injective when restricted to the subsheaf* $\hat{\mathrm{G}}_{L'} = \hat{\mathrm{G}}\hat{\mathrm{H}}_{L'} \otimes_{B_{L'}} B_L$.

Proof. The first claim is evident in view of the definition of the sheaf of G^∞ functions. In order to prove that $\delta : \hat{\mathrm{G}}_{L'} \to \hat{\mathrm{G}}^\infty$ is an isomorphism, we exhibit the inverse morphism $\lambda : \hat{\mathrm{G}}^\infty \to \hat{\mathrm{G}}_{L'}$. Given an open set $U \subset B_L^{m,n}$, every $f \in \hat{\mathrm{G}}^\infty(U)$, can written in accordance with equation (2.1), in the form

$$f = \sum_{\mu \in \Xi_n} Z_0\left(\hat{f}^{\mu}\right)_{|U} \beta_\mu, \qquad (3.3)$$

where the $\hat{f}^{\mu}$'s are suitable sections of $C^{\infty}{}_{\nabla}{}^{m}\left(\sigma^{m,n}(U)\right)$. After letting $\lambda(f)=Z_0\left(\hat{f}^{\mu}\right)_{|U}\otimes\beta_{\mu}$, we verify that $\lambda\circ\delta=id=\delta\circ\lambda$.

Corollary 3.2 *Given two integers* L', L'' *satisfying the condition* $L-L'\geq n$ *there is a canonical isomorphism of sheaves of graded commutative* B_L*-algebras* $G_{L'} > G_{L''}$.

Proof. Proposition 3.1. entails the isomorphism $\hat{G}_{L'}\simeq\hat{G}_{L''}$. On the other hand, for any open $U\subset B_L^{m,n}$,the surjective isomorphism gives

$$\hat{G}_{L'}\simeq\hat{G}_{L''}\otimes_{\nabla}\wedge_{\nabla}\nabla^{n}, \qquad (3.4)$$

so that our claim is proved.

Therefore, it is possible to introduce on $B_L^{m,n}$ a canonical sheaf of graded commutative B_L-algebras G, formally defined as the isomorphism class of the sheaves $G_{L'}$ while L' varies among the non-negative integers such that $L-L'\geq n$. Alternatively, one can assume $L\geq 2n$ and take once for all $L'=[L/2]$, the biggest integer less than $L/2$ (cf. [17]). A subsheaf $\hat{G}$, of germs of sections of G‘ not depending on the odd variables’ is defined in the same fashion and one obtains the isomorphism

$$G\simeq\hat{G}\otimes_{\nabla}\wedge_{\nabla}\nabla^{n} \qquad (3.5)$$

Proposition 3.3 *There is an isomorphism of sheaves of graded* B_L-modules $\mathrm{Der}\,G\simeq\mathrm{Der}\,GH\otimes_{B_{L'}}B_L$.

Proof. By virtue of the surjective isomorphism for any open $U\subset B_L^{m,n}$, it is enough to show that $\mathrm{Der}\,\hat{G}\simeq \mathrm{Der}\,GH\otimes_{B_{L'}}B_L$. By identifying $\hat{G}$ with $\hat{G}^{\infty}$, we define a morphism $\eta:\mathrm{Der}\,\hat{G}^{\infty}\to\mathrm{Der}\hat{G}\hat{H}\otimes_{B_{L'}}B_L$ given by

$$\eta(D)(f)=\sum_{\mu\in\Xi_n}D\left(Z_0\left(\hat{f}^{\mu}\right)\right)\otimes\beta_{\mu},$$

where f has been factorized according to equation (3.3). It easily verified that η is an isomorphism.

Proposition 3.4 $\mathrm{Der}\,G$ *is a locally free graded* G*-module on* $B_L^{m,n}$, *of rank* (*m*, *n*). *On every open set* $U\subset B_L^{m,n}$, $\mathrm{Der}\,G(U)$ *is generated over* $G(U)$ *by the derivations*

$$\left\{\frac{\partial}{\partial x^{i}},\frac{\partial}{\partial y^{\alpha}}\mid i=1\ldots m,\ \alpha=1\ldots n\right\}$$

defined as follows:

$$\frac{\partial}{\partial x^{i}}(f\otimes a)=\frac{\partial f}{\partial x^{i}}\otimes a,\quad i=1\ldots m;\quad \frac{\partial}{\partial y^{\alpha}}(f\otimes a)=\frac{\partial f}{\partial y^{\alpha}}\otimes a, \quad \text{where} \quad \alpha=1\ldots n. \qquad (3.6)$$

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Definition 3.5 *An* (*m*,*n*) *dimensional G-supermanifold is a graded locally ringed* B_L-space (M ,A) *satisfying the following conditions:*

(*a*) *M is a Hausdorff, paracompact topological space;*

(*b*) (M ,A) *is locally isomorphic with* ($B_L^{m,n}$,G);

(*c*) *denoting by* C_L^M *the sheaf of continuous* B_L*-valued functions on M, there exists a morphism of sheaves of* B_L*-algebras* $\delta^M : A \to \mathrm{C}_L^M$ *which is locally compatible with the evaluation morphism* (3.2) *and with the isomorphisms ensuing from condition* (*b*).

Thus, by assumptions, any point $z \in M$ has a neighbourhood U such that:

(i) there is an isomorphism of graded locally ringed spaces

$$\left(\bar{\phi},\phi\right):\left(U, \mathrm{A}_{|U}\right) \tilde{\to} \left(\bar{\phi}(U), \mathrm{G}_{|\bar{\varphi}(U)}\right), \qquad (3.7)$$

(ii) the following diagram commutes:

$$\begin{array}{ccc} \mathrm{G}|_{\bar{\varphi}(U)} & \xrightarrow{\ \phi\ } & \mathrm{A}_{|U} \\ \delta \downarrow & & \downarrow \delta^M \\ \mathrm{C}_{L|\bar{\phi}(U)} & \xrightarrow{\ \bar{\phi}^*\ } & \mathrm{C}_{L|U}^M \end{array}, \qquad (3.8)$$

where $\bar{\phi}^*$ is the ordinary pull-back associated with the mapping $\bar{\phi}$.

If there is no confusion, the evaluation morphism δ^M will be denoted simply by δ. The image of the sheaf A through δ is a sheaf on *M* of graded-commutative B_L-algebras, denoted by A^∞.

Proposition 3.6

(*a*) *The atlas* $\mathrm{U}^\infty = \{ \left(U_i, \bar{\phi}_i\right), i \in \subseteq \}$ *endows M with a structure of* G^∞ *supermanifold of the same dimension as* (*M*,A).

(*b*) *The* G^∞ *structure sheaf of M coincides with* A^∞.

It is clear that G-supermanifolds generalize the notion of GH^∞ supermanifolds; indeed, if $\left(M, \mathrm{GH}^M\right)$ is a GH^∞ supermanifold [7], the pair *(M*,A), with A= $\mathrm{GH}^M \otimes_{B_{L'}} B_L$, is a G-supermanifold. The resulting G-

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supermanifold will be called the trivial extension of the original GH^{∞} supermanifold [19].

Graded tangent space. As a consequence of Proposition 3.3., the sheaf $\mathrm{D}er\,\mathrm{A}$ of graded derivations on a G-supermanifold *(M,*A) is locally free, with local bases given by the derivations

$$\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^\alpha} \mid i=1\ldots m,\ \alpha=1\ldots n\right\}$$

associated with a local coordinate system $\left(x^1,\ldots,x^m,y^1,\ldots,y^n\right)$.

Definition 3.7 *The graded tangent space* $T_z(M,A)$ *at a point* $z\in M$ *is the graded* B_L*-module whose elements are the graded derivations* $X:\mathrm{A}_z\to B_L$.

The graded tangent space $T_z(M,\mathrm{A})$ is quite evidently free of rank (*m, n*)and the elements $\left(\frac{\partial}{\partial x^i}\right)_z, \left(\frac{\partial}{\partial y^\alpha}\right)_z$ defined by

$$\left(\frac{\partial}{\partial x^i}\right)_z (f) = \frac{\partial \tilde{f}}{\partial x^i}(z), \qquad \left(\frac{\partial}{\partial y^\alpha}\right)_z (f) = \frac{\partial \tilde{f}}{\partial y^\alpha}(z) \qquad \text{for all } f\in A_z,$$

yield a graded basis for it. Furthermore, there is a canonical isomorphism of graded B_L-modules

$$T_z(M,A) \tilde{\to} (\mathrm{D}er\,\mathrm{A})_z / (\mathrm{L}_z.(\mathrm{D}er\,\mathrm{A})_z),$$

where L_z is the ideal of germs in which vanish when evaluated, i.e.

$$\mathrm{L}_z = \left\{ f\in A_z \,\middle|\, \hat{f}(z)=0 \right\}.$$

TOPOLOGIES OF RINGS OF G-FUNCTIONS.

In order to introduce the notions of morphisms and products of G-supermanifolds, and to discuss Rothstein's axiomatics, we need to topologize in a suitable way the rings of sections of the structure sheaves of G-supermanifolds [9]. This will parallel the analogous study performed in the case of graded manifolds [2].

Let (*M,*A) be a G-supermanifold and let $\|\ \|$ denote the l^1 norm in B_L; for every open subset $U\subset M$ the rings $\mathrm{A}(U)$ of A can be topologized by means of the seminorms $p_{L,K}:\mathrm{A}(U)\to\nabla$ defined by

$$p_{L,K}(f) = \max_{z\in K} \| \delta(L(f))(z) \|$$

where L runs over the differential operators of A on U and $K\subset U$ is compact. The above topology is also given by the family of seminorms

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$$p_K^I(f)=\max_{\substack{z\in K\\ |J|\le I,\ \mu\in\Xi_n}}\left\|\delta\left(\left(\frac{\partial}{\partial x}\right)^J\left(\frac{\partial}{\partial y}\right)_\mu f\right)(z)\right\|, \quad (4.1)$$

where K runs over the compact subsets of a coordinate neighbourhood W with coordinates $\left(x^1,\ldots,x^m,y^1,\ldots,y^n\right)$. Under this

form it is clear that this topology makes $\mathrm{A}(U)$ into a locally convex metrizable graded algebra. The next results will allow to prove that $\mathrm{A}(U)$ is complete, so that it is in fact a graded Fr e′ chet algebra. Without loss of generality, we may assume that $(M,\mathrm{A})=\left(B_L^{m,n}\ \mathrm{G}\ \right)$. With reference to the isomorphism (3.5), we topologize the rings $\hat{\mathrm{G}}(U)$ by means of the seminorms

$$\hat{p}_K^I(f)=\max_{\substack{z\in K\\ |J|\le I}}\left\|\delta\left(\left(\frac{\partial}{\partial x}\right)^J f\right)(z)\right\|. \quad (4.2)$$

The tensor product $\hat{\mathrm{G}}(U)\otimes_\nabla\wedge_\nabla\nabla^n$ is in turn given its natural topology, which is induced by the seminorms

$$p_K^{I,\mu}(f)=\hat{p}_K^I\left(f^\mu\right)$$

having set $f=\sum_{\mu\in\Xi_n} f_\mu\otimes y^\mu$.

Lemma 4.1 *The isomorphism* (3.5),

$\mathrm{G}(U)\tilde{\rightarrow}\hat{\mathrm{G}}(U)\otimes_\nabla\wedge_\nabla\nabla^n$, *is a metric isomorphism.*

Proof. A direct majoration argument shows that

$$p_K^I\le\sum_{\mu\in\Xi_n}c_\mu\hat{p}_K^{I,\mu} \text{ where } c_\mu=\max_{\substack{z\in K\\ \nu\in\Xi_n}}\left\|\delta\left(\left(\frac{\partial}{\partial y}\right)_\nu y^\mu\right)(z)\right\|.$$

This shows the continuity of the inverse morphism. We now display the opposite majoration. The seminorm p_K^I is explicitly written as

$$p_K^I(f)=\max_{\substack{z\in K\\ |j|\le I,\nu\in\Xi_n}}\left\|\sum_{\mu\in\Xi_n}\varepsilon_{\mu\nu}\frac{\partial f^\mu}{\partial x^J}(z)\delta\left(\left(\frac{\partial}{\partial y}\right)_\nu y^\mu\right)(z)\right\|, \quad (4.3)$$

with $\varepsilon_{\mu\nu}$ a suitable sign. The seminorms $p_K^{I,\mu}$ are majorated by descending recurrence, starting from the last one, i.e. from $p_K^{I,\omega}$, where ω is the sequence $\{1,2,\ldots,n\}$. Indeed, from (4.3) we obtain $p_K^{I,\mu}\le p_K^I$, since $p_K^{I,\mu}$ is one of the terms over which the maximum (4.3) is taken. For the same reason, if we consider the seminorms p_K^{I,w_i}, I=1,...,n, with $\omega_i = \{1, 2, \ldots, \hat{i}, \ldots, n\}$, we obtain

$$\begin{aligned}
p_K^{I,\omega_i}(f) &= \max_{\substack{z\in K\\ |J|\le I}} \left\| \frac{\partial f^{\omega_i}}{\partial x^J}(z) + \frac{\partial f^{\omega}}{\partial x^J}(z)\delta(y^i)(z) - \frac{\partial f^{\omega}}{\partial x^J}(z)\delta(y^i)(z) \right\| \\
&\le p_K^I(f) + \max_{\substack{z\in K\\ |J|\le I}} \left\| \frac{\partial f^{\omega}}{\partial x^J}(z)\delta(y^i)(z) \right\| \\
&\le (1 + c_{iK})\, p_K^I(f),
\end{aligned}$$

where $c_{iK} = \max_{z\in K} \| \delta(y^i)(z) \|$. The remaining majorations are performed in the same way.

For any open $W \subset \nabla^m$, the space $C^\infty(W) \otimes_\nabla B_{L'}$ is equipped with the usual topology of uniform convergence of derivatives of any order, which is induced by the family of seminorms

$$q_K^I(h) = \max_{\substack{z\in K\\ |J|\le I}} \left\| \left(\frac{\partial}{\partial x}\right)^J h(z) \right\|$$

where *K* is a compact in *W* and the norm is taken in $B_{L'}$. Moreover, since δ is injective when restricted to $\hat{G}$, we may identify the sheaves $\hat{G}$ and $\hat{G}^\infty$.

Theorem 4.2 *For any open* $U \subset B_L^{m,n}$ *and all* L' *such that* $0 \le L' \le L$, *the Z-expansion*

$$Z_{L'} : C^\infty(\sigma^{m,n}(U)) \otimes B_{L'} \to \hat{G}(U) \qquad (4.4)$$

is an isometry onto its image. In particular, when $L' = L$, *we obtain a metric isomorphism* $C^\infty(\sigma^{m,n}(U)) \otimes B_L \simeq \hat{G}(U)$, *while , for* $L' = 0$, *we obtain a metric isomorphism* $C^\infty(\sigma^{m,n}(U)) \simeq$

$\hat{H}^\infty(U)$.

Proof. One easily shows that the seminorms which defines the topology in the right-hand side are majorated in terms of the relevant seminorms on the left-hand side. To show the converse, let *K* be a compact subset of an open *W* in ∇^m and I be a nonnegative integer; for any $h \in C^\infty{}_{\nabla^n}(W)$, we have

$$q_K^I(h) \le \max_{\substack{z\in K\\ |J|\le I}} \left\| \left(\frac{\partial}{\partial x}\right)^J Z_{L'}(h)(z) \right\| = \hat{p}_{\tilde{K}}^I(Z_{L'}(h),$$

where $\tilde{K}$ is a compact in $(\sigma^{m,n})^{-1}(W)$ containing *K*. It is clear that the previous minoration implies the thesis.

Proposition 4.3

(*a*) *The functions* $p_k^r : \mathcal{A}(W) \to \nabla$ *are submultiplicative seminorms, in that*

$$P_K^r(f,g) \le 2^{nr} P_K^r(f) P_K^r(g),$$

(*b*) $\mathcal{A}(W)$, *equipped with the topology induced by the seminorms* $\{P_K^r\}$, *where* $r \ge 0$ *and* K *is an arbitrary compact coordinate subset of* W *, is a Frechet algebra.*

Corollary 4.4 *The spaces* $\mathrm{G}(U)$, $\mathrm{H}^{\infty}(U) \otimes_{\nabla} B_L$ *and* $\mathrm{C}^{\infty}(\sigma^{m,n}(U)) \otimes B_L \otimes \wedge \nabla^n$ *are isometrically isomorphic for any open* $U \subset B_L^{m,n}$.

Proposition 4.5 *Let* (M,A) *be a G-supermanifold. For every open* $U \subset M$, *the space* $\mathrm{A}(U)$, *endowed with the topology induced by the semi norms* (4.1) *is a graded* $Fr\mathrm{e}'$ *chet algebra.*

Reasoning as in Proposition 4.3, one proves that the topological algebra $\hat{\mathrm{G}}(U)$ is complete, where using Lemma 4.1 and reasoning as in Proposition 4.3 again, the algebra $\mathrm{G}(U)$ is complete as well. We eventually obtain the results which are Corollary 4.4 and Proposition 4.5.

Example 4.6 The previous Lemma 4.1 and Theorem 4.2 also imply a further result, that will be significant when dealing with morphisoms of G-supermanifolds. For any open $W \subset \nabla^m$, we topologize the space

$$\mathrm{X}^{\infty}(W) \otimes B_L \otimes \wedge \nabla^n \tilde{\to} \mathrm{X}^{\infty}(W) \otimes \wedge \nabla^{L+n}$$

as in Proposition 4.3.

CONCLUSIONS

The *Z*-expansion is the morphism of graded algebras $Z_{L'}$ which is defined by (2.1). A theorem on an isometry onto its image of *Z*-expansion and on a metric isomorphism is derived. This theorem make possible definition of coordinate neighbourhood and odd and even coordinate system and to be able to know about odd symplectic supermanifolds [11] and also it will be helpful to study integration on supermanifolds such as integration on $\nabla_{\mathrm{S}}^{m,n}$ and Rothstein's theory of integration on non-compact supermanifolds. Thus, this theorem is implied a further research, which will be useful when some one author have to deal with morphisms of G-supermanifolds.

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