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Hyperplane-Open Weak Topologies in R^n

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ABSTRACT

A general procedure of constructing hyperplane-open weak topology on R^n is exposed. The process also leads to the formulation of matrix-open weak topology on R^n. This is a very interesting discovery since we never expected that a matrix of points in the Cartesian plane could come out as an open set.

Keywords: Weak Topology, Hyperplane-Open Weak Topology, Vertical (or Horizontal) Line Open Topology
Mathematics Subjects Classification (MSC) 2020: 54A05, 54A10

INTRODUCTION

This paper resulted from an attempt to better appreciate the concept of weak topology by taking a constructive approach. We sought to construct concrete examples of weak topologies on concrete sets. This approach opens up for us a vast vista which we began to explore. We hope that what has been thrown open is rich enough to cater to the quest of a plethora of inquisitive minds.

MAIN RESULTS—THE HYPERPLANE-OPEN (OR LINE-OPEN) WEAK TOPOLOGIES IN R^N

Throughout this work, R will denote the set of real, N the set of natural numbers, and R^n the product of n copies of R, for each $n \in N$

We know that in general, lines (curved or straight) are not open in the usual topology of the Cartesian plane R^2. It is also known that lines—curved or straight—are open subsets of R^2 when the discrete topology is assumed. Is there another topology on R^2, coarser than the discrete topology, in which all lines (vertical or horizontal, short or long) are open? This question has neither been posed before nor answered in any way till this moment. If such topologies exist, are they weak topologies and what does their landscape look like? We are here set to introduce another topology on R^2 (and indeed on R^n) with respect to which lines (even if only straight lines) are open.

Let us recall that the usual topology of R^2 is generated by the projection maps when the factor spaces of R^2 are themselves endowed with their own usual topologies. Let us now start by bringing forth constructible and easy to visualize examples of weak topologies such as

1. line-open topology on R^n, $n > 2$;
2. plane-open topology on R^n, $n \geq 3$;
3. hyperplane-open topology on R^n generally.

Construction 1 *Consider R^2. Let the horizontal factor space be endowed the discrete topology* (R, D) *and the vertical factor space be endowed with the usual topology of R,* (R, u)*. Then the coarsest topology on R^2 with respect to which the projection maps p_1 and p_2 are continuous, is called the vertical line open topology of R^2 because vertical lines (of all lengths) are among the basic open sets of this topology.*

To appreciate the nature of this topology on R^2, we recall that singletons $\{x\}$ are open in the horizontal factor

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space, which we are to call R_1. Therefore $p_1^{-1}(\{x\})$ is a sub-basic open set in this weak topology on R^2. Such a set is an infinite (in length) vertical line passing through the point (x, 0) in the plane. That is,

$$p_1^{-1}(\{x_0\}) = \{(x_0, y): y \in R\} \dots\dots\dots\dots\dots\dots (1)$$

where x_0 is any fixed real number along the horizontal axis. Hence, the basic open sets of this weak topology on R^2 include vertical lines of all lengths; the lines with finite length do not contain their endpoints as elements. To see this, we recall that a basic open set in the vertical factor space R_2 is an open interval (a, b). Thus, in the formation of this weak topology, this factor space will donate sub-basic sets of the form

$$p_2^{-1}\{(a,b)\} = \{(x,y) \in R^2: a < y < b\} \dots\dots\dots\dots\dots (2)$$

which are infinite horizontal strips. The intersection of sets of type (2) with those of type (1) results in finite-length vertical lines, which are now the basic open sets of the weak topology.

Vertical lines are not the only open sets in this topology on R^2. In the discrete topological factor space—the horizontal axis—every other type of set (apart from singletons) is still open. In particular, sets of the form (a, b), (a, b], [a, b), and [a, b] are all open. Hence the usual open rectangles, open circles, in short all open polygons, vertically half-open, half-closed and vertically closed rectangles in R^2 are all open. It then follows that this weak topology is strictly stronger than the usual euclidean topology of the plane R^2 and yet strictly weaker than the discrete topology of R^2, since for instance singletons are not open in this topology.

Construction 2 *Consider R^2 but now with the horizontal factor space R_1 endowed with the usual topology and the vertical factor space R_2 endowed with the discrete topology. Then the horizontal line open topology results.*

As with the vertical line open topology of R^2, this open horizontal line topology is generated by the projection maps. It is also easy to see that this topology is finer than the usual topology on R^2 and coarser than the discrete topology of R^2. It can also be observed that these two topologies, on R^2, are not comparable. The intersection of these two topologies is finer than the euclidean topology of R^2.

Proposition 2.1 *Let τ_v, τ_h and τ_u denote respectively the open vertical line topology, the open horizontal line topology, and the usual topology of the Cartesian plane R^2. Then*

1. $\tau_u \leq \tau_v$;
2. $\tau_u \leq \tau_h$; and hence
3. $\tau_u \leq \tau_v \cap \tau_h$.

Proof:

1 and 2 are obvious from the discussions so far. This further implies 3. ⊙

Remark

We have just proved that the usual topology is weaker than the intersection of the vertical line and the horizontal line-open topologies on R^2. Is the usual topology strictly weaker, or is it actually equal to this intersection? Answer: Since we cannot (at least for now) find any open set in this intersection which is not open in τ_u, our conjecture is that this intersection is actually equal to τ_u. There is however here a need for further researches to concretize this conclusion.

We are now on the process of working out the proof or disproof—which is not available now—of our conjecture above; the result however is what we hope to bring out soon or later in a subsequent publication of our researches in these areas. The reader should bear with us now, as we promise to later bring the conclusion (whatever it may be) of this conjecture. Also in doing this, we shall then expand the scope of the implication of what we have done in this research in order to compare and contrast it with what obtains in other weak topologies in terms of topological properties. In particular, we shall investigate the topological invariants that are preserved or altered under the kind of constructions we have done.

Construction 3 *Let $n \geq 3$ and let $X = R^n$ be the product of n copies of R. Let the projection maps $p_i : X \longrightarrow R_i$, for $1 \leq i \leq n$, be defined in the usual way by $p_i(\bar{x}) = x_i$, where $\bar{x} = (x_1,x_2,\cdots,x_n)$, for all $\bar{x} \in X$. Let m factor subspaces $(1 \leq m < n)$ be endowed with discrete topology and let the remaining $n - m$ factor subspaces retain the usual topology of R. Then the* hyperplane-open topology *of X (= R^n) is the coarsest topology on R^n relative to which the projection maps are continuous.*

In R^2, $p_i^{-1}(\{x_{i_0}\})$ is a straight, infinite line perpendicular to the *i*th axis, $1 \leq i \leq 2$, a 1-dimensional hyperplane perpendicular to the *i*th axis; for any fixed point x_{i0} in the *i*th factor space. In R^3, $p_i^{-1}(\{x_{i_0}\})$ is a straight, infinite plane (a 2-dimensional hyperplane) perpendicular to the *i*th axis, $1 \leq i \leq 3$; for any fixed point x_{i0} in the *i*th factor space. In R^n $(n \geq 4)$, $p_i^{-1}(\{x_{i_0}\})$ is a hyperplane (of dimension $n - 1$), $1 \leq i \leq n$; for any fixed point x_{i_0}in the *i*th factor space. However, if $n-1$ factor spaces are endowed with the discrete topology, and the *n*th factor space with the usual topology, then the basic open set

$$p_1^{-1}(\{x_0^1\}) \bigcap p_2^{-1}(\{x_0^2\}) \bigcap \cdots \bigcap p_{n-1}^{-1}(\{x_0^{(n-1)}\}) = \bigcap_{i=1}^{n-1} p_i^{-1}(\{x_0^i\})$$

results in a one-dimensional hyperplane; a straight line (parallel to the *n*th factor space which has the usual topology). So, in the product $X = R^n$, lines are open in the hyperplane open topology if $n - 1$ factor spaces are endowed with the discrete topology (and the *n*th factor space has the usual or possibly any other topology on R).

For example in R^3, exactly 2 factor spaces (only) have to be endowed with the discrete topology for lines to emerge really as open sets. If we give all three factor spaces of R^3 the discrete topology, then the resulting open line topology would coincide with the discrete topology of R^3. If only 1 factor space of R^3 is given the discrete topology, then the resulting weak topology will have no lines as open sets. The weak topology will have 2-dimensional hyperplanes (planes) as basic open sets. This is because sets of the form

$$p_1^{-1}(\{x_0\}) = \{(x_0, y, z) : y, z \in R\} \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

are two-dimensional planes.

Remark

1. We observe that *m* has to be strictly less than *n* in the last construction since otherwise we would get the discrete topology of R^n.
2. What actually happens is that if we endow 2 factor spaces of R^3 with the discrete topology and the remaining 1 factor space with the usual topology of R, then the line-open (weak) topology results. If we endow 1 factor space of R^2 with discrete topology, then these open lines will *all* be parallel to one axis of R^2; parallel to the vertical axis if the horizontal factor space is endowed with the discrete topology, and vice versa. In R^3, with 2 factor spaces given the discrete topology, all the open lines will be parallel to the only 1 factor space retaining the usual topology, and perpendicular to the plane of the two other factor spaces.

Construction 4 *Let $X = \{x_1,x_2,x_3,\cdots,x_n\}$ be any finite set of real numbers, and let 2^X be the power set of X. Then $\{R, 2^X\}$ is a topology on R, called (and introduced in this work as) the X-topology on R. The point-open weak topology on R^n is the weak topology, on R^n, generated by the projection maps when the factor spaces are each endowed with the X-topology.*

REMARK/EXAMPLES

We observe that actually some points of R^n (as singletons) are open in this weak topology while the others are not. This is why we call this *the* point-open *weak* topology of R^n. For example, let $X = \{x_1, x_2\}$; then $2^X = \{\emptyset, X, \{x_1\},\{x_2\}\}$ and the X-topology on R is $\{\emptyset, X,\{x_1\},\{x_2\}, R\}$. Let the factor subspaces R_1 and R_2 (horizontal and

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vertical respectively) of R^2 be, each, endowed with this X-topology. Then the only singletons open in the weak topology of R^2, generated by the projection maps this time, are

$$p_1^{-1}(\{x_1\})\bigcap p_2^{-1}(\{x_1\}) = \{(x_1, x_1)\},$$

$$p_1^{-1}(\{x_1\})\bigcap p_2^{-1}(\{x_2\}) = \{(x_1, x_2)\},$$

$$p_1^{-1}(\{x_2\}) \cap p_2^{-1}(\{x_1\}) = \{(x_2, x_1)\},$$

$$p_1^{-1}(\{x_2\}) \cap p_2^{-1}(\{x_2\}) = \{(x_2, x_2)\}.$$

If, say, all three factor spaces of R^3 are given this particular X-topology, then the only open singletons of R^3 in the resulting weak topology would be

$\{(x_1, x_1, x_1)\}$, $\{(x_1,x_1,x_2)\}$, $\{(x_1,x_2,x_1)\}$, $\{(x_2,x_2,x_2)\}$,$\{(x_2,x_2,x_1)\}$,$\{(x_2,x_1,x_2)\}$,$\{(x_2,x_1,x_1)\}$, $\{(x_1, x_2, x_2)\}$, a total of only 8 singletons in R^3.

We also note that some matrices of coordinate points (grid points) in the Cartesian plane R^2 are open sets in this X-topology induced weak topology.

For example, we observe that

$$p_1^{-1}(X) = p_1^{-1}(\{x_1, x_2\}) = \{(x, y) \in R^2 : x = x_1\}\bigcup\{(x, y) \in R^2 : x = x_2\}$$

$= \{(x_1, y) \in R^2\}\cup\{(x_2, y) \in R^2\}$ = two vertical infinite lines and

$$p_2^{-1}(X) = p_2^{-1}(\{x_1, x_2\}) = \{(x, y) \in R^2 : y = x_1\}\bigcup\{(x, y) \in R^2 : y = x_2\}$$

$= \{(x, x_1) \in R^2\}\cup\{(x, x_2) \in R^2\}$ = two horizontal infinite lines.

Therefore

$$p_1^{-1}(X) \cap p_2^{-1} = p_1^{-1}(\{x_1, x_2\}) \cap p_2^{-1}(\{x_1, x_2\}) = \{x_1, x_2\} \times \{x_1, x_2\}$$

$= \{x_1,x_2\} \times \{x_1,x_2\} = \{(x_1,x_1), (x_1,x_2), (x_2,x_1), (x_2,x_2)\}$; a 2×2 matrix of four coordinate points. The matrix M is shown below.

$$M = \begin{bmatrix} (x_1, x_2) & (x_2, x_2) \\ (x_1, x_1) & (x_2, x_1) \end{bmatrix}$$

Construction 5 *Let R^n be the Cartesian product of n copies of R. Let $X \subset R$ be any proper subset of R. Let $\tau_X = 2^X \cup \{R\}$ be the X-topology on R. Let m ($m < n$) factor spaces of R^n be endowed with this X-topology on R, and the remaining $n - m$ factor spaces be endowed with the usual topology of R. Then from the factor spaces having the X-topology, $p_i^{-1}(\{x_{i_0}\})$ is a hyperplane of dimension $n - m$, for each $1 \le i \le m < n$. Their intersections*

$$\bigcap_{i=1}^{m} p_i^{-1}(\{x_{i_0}\}). \dots\dots\dots\dots\dots\dots (3)$$

are hyperplanes of dimension $n - m$. If $n = m + 1$, so that $n - 1 = m$, then the intersection (3) *would be a 1-dimensional hyperplane (i.e. a straight line) in R^n. If $n = m+2$, then* (3) *would be a 2-dimensional hyperplane. And so on.*

Note

Since X is a proper subset of R, 2^X is not equal to the discrete topology of R. Hence the X-topology τ_X on R is

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strictly weaker than the discrete topology of R. Therefore, even if all the factor spaces of R^n are given this X-topology on R, the resulting weak topology on R^n (generated by the projection maps) would still be strictly weaker than the discrete topology of R^n. And then some singletons (probably infinitely many) of R^n would be open in the hyperplane-open weak topology on R^n. These singletons are sets of the form

$$\bigcap_{i=1}^{n} p_i^{-1}(\{x_{i_0}\}). \dots\dots\dots (4)$$

where $x_{i_0} \in X \subset R$. It is important to compare and make the contrast between (3) and (4).

REFERENCES

1. Angus E. Taylor and David C. Lay; An Introduction to Functional Analysis; Second Edition, John Wiley and Sons, New York (1980).
2. Chidume C.E.; Applicable Functional Analysis: Fundamental Theorems with Applications; International Center for Theoretical Physics, Trieste, Italy (1996).
3. Edwards R.E.; Functional Analysis: Theory and Applications; Dover Publications Inc., New York (1995).
4. H.L. Royden; Real Analysis; Third Edition, Prentice-Hall of India Private Limited, New Delhi (2005).
5. Jawad Y. Abuhlail; On the Linear Weak Topology and Dual Pairings Over Rings; Internet (2000).
6. Rudin W.; Functional Analysis; McGraw-Hill, New York (1973).
7. Sheldon W. Davis; Topology; McGraw-Hill Higher Education, Boston (2005).
8. Titchmarsh E.C.; Theory of Functions; Second Edition, Oxford University Press, Oxford (1939).
9. Wada J.; Weakly Compact Linear Operators on Function Spaces; Osaka Math.J.13(1961), 169-183.
10. Willard Stephen; General Topology; Courier Dover Publications (2004).