INTERNATIONAL JOURNAL OF RESEARCH AND SCIENTIFIC INNOVATION (IJRSI)
ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XI Issue IX September 2024

Some Results on a Class of Harmonic Univalent Functions Defined by Generalised Derivative Operator

Entisar El-Yagubi*, Samira Ziada

Mathematical Department, Gharyan University, Faculty of Science, Gharyan, Libya

***Corresponding Author**

DOI: https://doi.org/10.51244/IJRSI.2024.1109033

Received: 28 August 2024; Accepted: 18 September 2024; Published: 03 October 2024

ABSTRACT

Harmonic function is one of an important branches of complex analysis. The first study of complex – valued, harmonic mappings defined on a domain $D \subset \mathbb{C}$ was given by Clunie and Sheil-Small [1]. Harmonic functions have been studied by different researchers such as Silverman [6]. In the present paper, a new class of harmonic univalent functions will be introduced. Various properties of functions belong to this class which include coefficient bounds, growth bounds, a closure property, extreme points, neighborhood and a convex combination will be obtained.

Keywords: Univalent functions, harmonic functions, derivative operator, distortion inequalities.

INTRODUCTION

Let $U = \{z \in \mathbb{C}: |z| < 1\}$ be the open unit disc and let S_H denote the class of all complex valued, harmonic, sense-preserving, univalent functions f in $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$ and expressed as $f(z) = h(z) + \overline{g(z)}$ where h and g belong to the linear space $H(\mathbb{U})$ of all analytic functions on $\mathbb{U}$ take the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ and\ g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{1.1}$$

Thus for each $f \in S_H$ takes the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad z \in \mathbb{U}. \tag{1.2}$$

Clunie and Sheil-Small [1] proved that S_H is not compact and the necessary and sufficient condition for f to be locally univalent and sense-preserving in any simply connected domain $\mathbb{U}$ is that $|h'(z)| > |g'(z)|$.

Darus and Ibrahim [2] introduced the generalized derivatives operator, denoted by $\mathcal{D}_{\delta,\beta,\lambda}^{k} f(z)$ for $f \in \mathcal{A}$ as follows:

$$\mathcal{D}_{\delta,\beta,\lambda}^{k} f(z) = z + \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^k a_n z^n, \tag{1.3}$$

where $\delta \geq 0,\ \beta > 0,\ \lambda > 0, ,\ \delta \neq \lambda, k \in \mathbb{N}_0 = \{0,1,2,\dots\}$.

In this paper the operator $\mathcal{D}_{\delta,\beta,\lambda}^{k} f(z)$ will be introduced for $f = h + \overline{g}$ where h and g given by (1.1) and it will be given by

$$\mathcal{D}_{\delta,\beta,\lambda}^{k} f(z) = \mathcal{D}_{\delta,\beta,\lambda}^{k} h(z) + (-1)^K \overline{\mathcal{D}_{\delta,\beta,\lambda}^{k} g(z)}, \ \ z \in \mathbb{U}, \tag{1.4}$$

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where,

$\mathcal{D}^{k}_{\delta,\beta,\lambda}h(z) = z + \sum_{n=2}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^{k} a_n z^n,$

$\mathcal{D}^{k}_{\delta,\beta,\lambda}g(z) = \sum_{n=1}^{\infty} [\beta(n-1)(\lambda-\delta)+1]^{k} b_n z^n,$ (1.5)

for $\delta \geq 0,\ \beta > 0,\ \lambda > 0,\ \delta \neq \lambda$, and $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$.

We further denote by $\overline{S_H}$ the subclass of S_H consist of harmonic functions of the form

$f_k = h + \overline{g_k},$ (1.6)

where

$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$

$$g_k(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n, \quad z \in U, \quad |b_1| < 1.$$

A class of harmonic univalent functions is introduced as the following:

Definition 1.1. *The function $f = h + \overline{g}$ defined by (1.2) is in the class $S^{K}_{H}(\delta,\beta,\lambda,\alpha)$* if

$\Re\left\{\frac{\mathcal{D}^{k+1}_{\delta,\beta,\lambda}f(z)}{\mathcal{D}^{k}_{\delta,\beta,\lambda}f(z)}\right\} \geq \alpha,$ (1.7)

where $0 \leq \alpha < 1, \delta \geq 0,\ \beta > 0,\ \lambda > 0,\ \delta \neq \lambda, k \in \mathbb{N}_0 = \{0,1,2,\dots\}$.

Note that, the class $\overline{S}^{0}_{H}(0,0,0,\alpha) \equiv S_H(\alpha)$ is the class of sense-preserving harmonic univalent functions f which are starlike of order α in $\mathbb{U}$, was studied by Jahangiri [3] and the class $\overline{S}^{k}_{H}(0,1,1,\alpha)$ is the class of Salagean-type harmonic univalent functions introduced by Jahangiri et al. [4].

We further denote by $\overline{S_H}^{k}(\delta,\beta,\lambda,\alpha)$, the subclass of $S^{k}_{H}(\delta,\beta,\lambda,\alpha)$,where $\overline{S_H}^{k}(\delta,\beta,\lambda,\alpha) = S^{k}_{H}(\delta,\beta,\lambda,\alpha) \cap \overline{S_H}$.

MAIN RESULTS

In the next theorem, a sufficient coefficient bound related to the class $S^{k}_{H}(\delta,\beta,\lambda,\alpha)$ shall be obtained

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.2). Furthermore, let*

$\sum_{n=2}^{\infty} \Omega^k[\Omega-\alpha]|a_n| + \sum_{n=1}^{\infty} \Omega^k[\Omega+\alpha]|b_n| \leq 1-\alpha$ (2.1)

Where

$\Omega^k = [\beta(n-1)(\lambda-\delta)+1]^k, \Omega = [\beta(n-1)(\lambda-\delta)+1],$

$a_1 = 1, 0 \leq \alpha < 1, \delta \geq 0,\ \beta > 0,\ \lambda > 0,\ \delta \neq \lambda, k \in \mathbb{N}_0,$ then $f \in S^{k}_{H}(\delta,\beta,\lambda,\alpha)$.

Proof. According to (1.7), we have

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$\Re\left\{\frac{\mathcal{D}^{k+1}_{\delta,\beta,\lambda}f(z)}{\mathcal{D}^{k}_{\delta,\beta,\lambda}f(z)}\right\} \geq \alpha.$

This is equivalent to $\Re\left(\frac{A(z)}{B(z)}\right) > \alpha,$

where $A(z) = \mathcal{D}^{k+1}_{\delta,\beta,\lambda}f(z)$ and $B(z) = \mathcal{D}^{k}_{\delta,\beta,\lambda}f(z)$.

Using the fact that, $\Re(w) > \alpha$ if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$|A(z) + (1-\alpha)B(z)| \geq |A(z) - (1+\alpha)B(z)|.$

Substituting values of $A(z)$ and $B(z)$, and with simple calculations, we get

$|\mathcal{D}^{k+1}_{\delta,\beta,\lambda}f(z) + (1-\alpha)\mathcal{D}^{k}_{\delta,\beta,\lambda}f(z)| \geq |\mathcal{D}^{k+1}_{\delta,\beta,\lambda}f(z) - (1+\alpha)\mathcal{D}^{k}_{\delta,\beta,\lambda}f(z)|.$

$|z + \sum_{n=2}^{\infty} \Omega^{k+1} a_n z^n + (-1)^{k+1} \sum_{n=1}^{\infty} \Omega^{k+1} \overline{b_n} \overline{z^n}$

$+(1-\alpha)[z + \sum_{n=2}^{\infty} \Omega^{k} a_n z^n + (-1)^{k} \sum_{n=1}^{\infty} \Omega^{k} \overline{b_n} \overline{z^n}|$

$\geq |z + \sum_{n=2}^{\infty} \Omega^{k+1} a_n z^n + (-1)^{k+1} \sum_{n=1}^{\infty} \Omega^{k+1} \overline{b_n} \overline{z^n}$

$-(1+\alpha)[z + \sum_{n=2}^{\infty} \Omega^{k} a_n z^n + (-1)^{k} \sum_{n=1}^{\infty} \Omega^{k} \overline{b_n} \overline{z^n}|.$

$|(2-\alpha)z + \sum_{n=2}^{\infty} \Omega^{k}[\Omega + (1-\alpha)] a_n z^n$

$-(-1)^{k} \sum_{n=1}^{\infty} \Omega^{k}[\Omega - (1-\alpha)] \overline{b_n} \overline{z^n}|$

$-| -\alpha z + \sum_{n=2}^{\infty} \Omega^{k}[\Omega - (1+\alpha)] a_n z^n$

$+(-1)^{k} \sum_{n=1}^{\infty} \Omega^{k}[-\Omega - (1+\alpha)] \overline{b_n} \overline{z^n}|$

$\geq 2(1-\alpha)|z| - \sum_{n=2}^{\infty} \Omega^{k}[2\Omega - 2\alpha]|a_n||z|^n$

$- \sum_{n=1}^{\infty} \Omega^{k}[2\Omega + 2\alpha]|\overline{b_n}||\overline{z}|^n$

$\geq 2(1-\alpha)|z|\{1 - \sum_{n=2}^{\infty} \Omega^{k}\left[\frac{\Omega-\alpha}{1-\alpha}\right]|a_n||z|^{n-1}$

$- \sum_{n=1}^{\infty} \Omega^{k}\left[\frac{\Omega+\alpha}{1-\alpha}\right]|\overline{b_n}||\overline{z}|^{n-1}\}$

≥ 0

by assumption. Hence the proof is complete.

Theorem 2.2. *Let $f_k = h + \overline{g_k}$ be given by (1.6). Then*

$f_k \in \overline{S_H}^{k}(\delta, \beta, \lambda, \alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} \Omega^{k}[\Omega - \alpha]|a_n| + \sum_{n=1}^{\infty} \Omega^{k}[\Omega + \alpha]|b_n| \leq 1 - \alpha, \quad (2.2)$$

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where

$\Omega^k = [\beta(n-1)(\lambda-\delta)+1]^k, \Omega = [\beta(n-1)(\lambda-\delta)+1]$

$a_1 = 1, 0 \leq \alpha < 1, \delta \geq 0,\ \beta > 0,\ \lambda > 0,\ \delta \neq \lambda, k \in \mathbb{N}_0$.

Proof. Since $\overline{S}_H^k(\delta, \beta, \lambda, \alpha) \subset S_H^k(\delta, \beta, \lambda, \alpha)$, we only need to prove the "only if" part of the theorem. Note that a necessary and sufficient condition for $f_k = h + \overline{g_k}$ given by (1.6) to be in $\overline{S_H}^k(\delta, \beta, \lambda, \alpha)$ is that

$$\Re\left\{\frac{\mathcal{D}_{\delta,\beta,\lambda}^{k+1} f(z)}{\mathcal{D}_{\delta,\beta,\lambda}^{k} f(z)}\right\} \geq \alpha,$$

which is equivalent to

$$\Re\left\{\frac{\mathcal{D}_{\delta,\beta,\lambda}^{k+1} f_k(z) - \alpha \mathcal{D}_{\delta,\beta,\lambda}^{k} f_k(z)}{\mathcal{D}_{\delta,\beta,\lambda}^{k} f_k(z)}\right\} \geq 0$$

$$\Re\left\{\frac{(1-\alpha)z - \sum_{n=2}^{\infty} \Omega^k[\Omega-\alpha] a_n z^n - (-1)^k \sum_{n=1}^{\infty} \Omega^k[\Omega+\alpha] b_n \overline{z^n}}{z - \sum_{n=2}^{\infty} \Omega^k a_n z^n + (-1)^k \sum_{n=1}^{\infty} \Omega^k b_n \overline{z^n}}\right\} \geq 0.$$

The above condition must holds for all values of z, $|z| = r < 1$. Choosing z on the positive real axis where $0 \leq z = r < 1$. We have

$$\left\{\frac{(1-\alpha) - \sum_{n=2}^{\infty} \Omega^k[\Omega-\alpha] a_n r^{n-1} - (-1)^k \sum_{n=1}^{\infty} \Omega^k[\Omega+\alpha] b_n \overline{r^{n-1}}}{1 - \sum_{n=2}^{\infty} \Omega^k a_n r^{n-1} + (-1)^k \sum_{n=1}^{\infty} \Omega^k b_n \overline{r^{n-1}}}\right\} \geq 0. \quad (2.3)$$

If the condition (2.2) does not hold, then the numerator in (2.3) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in (0,1) for which the quotient in (2.3) is negative. This contradicts the required condition for $f_k \in \overline{S_H}^k(\delta, \beta, \lambda, \alpha)$ and so the proof is complete.

In this section, growth bounds for $f_k \in \overline{S_H}^k(\delta, \beta, \lambda, \alpha)$ are obtained.

Theorem 2.3. *Let* $f_k \in \overline{S_H}^k(\delta, \beta, \lambda, \alpha)$, *then for* $|z| = r < 1$, *we have*

$$|f_k(z)| \leq (1 + |b_1|)r + \frac{1}{\Omega^k}\left(\frac{1-\alpha}{\Omega-\alpha} - \frac{\Omega^k(\Omega+\alpha)}{\Omega-\alpha}|b_1|\right)r^2,$$

$$|f_k(z)| \geq (1 - |b_1|)r - \frac{1}{\Omega^k}\left(\frac{1-\alpha}{\Omega-\alpha} - \frac{\Omega^k(\Omega+\alpha)}{\Omega-\alpha}|b_1|\right)r^2,$$

where $\Omega^k = [\beta(\lambda-\delta)+1]^k, \Omega = [\beta(\lambda-\delta)+1]$.

$a_1 = 1, 0 \leq \alpha < 1, \delta \geq 0,\ \beta > 0,\ \lambda > 0,\ \delta \neq \lambda, k \in \mathbb{N}_0$

Proof. The first inequality will be proved. The argument for the second inequality is similar and will be omitted. Let $f_k \in \overline{S_H}^k(\delta, \beta, \lambda, \alpha)$. Taking absolute value of f_k, we obtain

$$|f_k(z)| \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n$$

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$\leq (1+|b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^2$

$\leq (1+|b_1|)r + \frac{1-\alpha}{\Omega^k(\Omega-\alpha)} \sum_{n=2}^{\infty} \left(\frac{\Omega^k(\Omega-\alpha)}{1-\alpha} (|a_n| + |b_n|) \right) r^2$

$\leq (1+|b_1|)r + \frac{1-\alpha}{\Omega^k(\Omega-\alpha)} \left(1 - \frac{\Omega^k(\Omega+\alpha)}{1-\alpha} |b_1| \right) r^2$

$\leq (1+|b_1|)r + \frac{1}{\Omega^k} \left(\frac{1-\alpha}{\Omega-\alpha} - \frac{\Omega^k(\Omega+\alpha)}{\Omega-\alpha} |b_1| \right) r^2.$

Next, we prove closure property related to the class $\overline{S_H}^k(\delta, \beta, \lambda, \alpha)$.

Theorem 2.4. *Let the functions $f_{k_i}(z)$ defined by*

$$f_{k_i}(z) = z - \sum_{n=2}^{\infty} |a_{n,i}| z^n + (-1)^k \sum_{n=1}^{\infty} |b_{n,i}| \overline{z^n},$$

be in the class $\overline{S_H}^k(\delta, \beta, \lambda, \alpha)$, for every $i = 1, 2, \ldots, m$.

Then the convex combination of f_{k_i} denoted by $\sum_{i=1}^{m} t_i f_{k_i}(z)$ are also in the class $\overline{S_H}^k(\delta, \beta, \lambda, \alpha)$, where $\sum_{i=1}^{m} t_i = 1$,

$0 \leq t_i \leq 1$.

Proof. According to the definition of convex combination of f_{k_i}, we can write

$\sum_{i=1}^{m} t_i f_{k_i}(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{m} t_i |a_{n,i}| \right) z^n$

$+(-1)^k \sum_{n=1}^{\infty} \left(\sum_{i=1}^{m} t_i |b_{n,i}| \right) \overline{z^n}.$

Further, since $f_{k_i}(z)$ are in $\overline{S_H}^k(\delta, \beta, \lambda, \alpha)$, for every $(i = 1, 2, \ldots, m)$, then by (2.2), we have

$$\sum_{n=2}^{\infty} \frac{\Omega^k[\Omega-\alpha]}{1-\alpha} \left(\sum_{i=1}^{m} t_i |a_{n,i}| \right) + \sum_{n=1}^{\infty} \frac{\Omega^k[\Omega+\alpha]}{1-\alpha} \left(\sum_{i=1}^{m} t_i |b_{n,i}| \right)$$

$$= \sum_{i=1}^{m} t_i \left[\sum_{n=2}^{\infty} \frac{\Omega^k[\Omega-\alpha]}{1-\alpha} |a_{n,i}| + \sum_{n=1}^{\infty} \frac{\Omega^k[\Omega+\alpha]}{1-\alpha} |b_{n,i}| \right]$$

$\leq 1.$

Therefore $\sum_{i=1}^{m} t_i f_{k_i} \in \overline{S_H}^k(\delta, \beta, \lambda, \alpha)$.

Next, we present the extreme points related to the class $\overline{S}_H^k(\delta, \beta, \lambda, \alpha)$.

Theorem 2.5. *A function $f \in \overline{S}_H^k(\delta, \beta, \lambda, \alpha)$ if and only if f can be expressed as*

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)), \tag{2.4}$$

where

$$h_1(z) = z, \qquad h_n(z) = z - \frac{1-\alpha}{\Omega^k(\Omega-\alpha)} z^n, \qquad n \geq 2,$$

$$g_n(z) = z + (-1)^k \frac{1-\alpha}{\Omega^k(\Omega-\alpha)} \overline{z^n}, \qquad n \geq 1, \qquad \sum_{n=1}^{\infty} (X_n + Y_n) = 1, \qquad X_n \geq 0\,, Y_n \geq 0,$$

$\Omega^k = [\beta(n-1)(\lambda-\delta)+1]^k$, $\Omega = [\beta(n-1)(\lambda-\delta)+1]$, $0 \leq \alpha < 1$, $\delta \geq 0, \beta > 0$, $\lambda > 0$, $\delta \neq \lambda, k \in \mathbb{N}_0$.

In particular, the extreme points of $\overline{S}_H^k(\delta, \beta, \lambda, \alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Note that for f of the form (2.4), we may write

$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$

$= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} + (-1)^k \sum_{n=1}^{\infty} \frac{1-\alpha}{\Omega^k(\Omega-\alpha)} Y_n \overline{z^n}$.

Then

$\sum_{n=2}^{\infty} \left(\frac{\Omega^k(\Omega-\alpha)}{1-\alpha}\right) \frac{1-\alpha}{\Omega^k(\Omega-\alpha)} X_n + \sum_{n=1}^{\infty} \left(\frac{\Omega^k(\Omega-\alpha)}{1-\alpha}\right) \frac{1-\alpha}{\Omega^k(\Omega-\alpha)} Y_n$

$= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n$

$= \sum_{n=1}^{\infty} X_n - X_1 + \sum_{n=1}^{\infty} Y_n$

$= 1 - X_1$

$\leq 1.$

So, $f \in \overline{S}_H^k(\delta, \beta, \lambda, \alpha)$. Conversely, suppose that $f \in \overline{S}_H^k(\delta, \beta, \lambda, \alpha)$.

Set

$$X_n = \frac{\Omega^k(\Omega-\alpha)}{1-\alpha} |a_n|, 0 \leq \alpha < 1, 0 \leq X_n \leq 1, \qquad n \geq 2,$$

$$Y_n = \frac{\Omega^k(\Omega-\alpha)}{1-\alpha} |b_n|, 0 \leq \alpha < 1, 0 \leq Y_n \leq 1, \qquad n \geq 1,$$

we define

$X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$.

Therefore, f can be written as

$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \overline{\sum_{n=1}^{\infty} |b_n| z^n}$

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XI Issue IX September 2024

$$= z - \sum_{n=2}^{\infty} \frac{(1-\alpha)X_n}{\Omega^k(\Omega-\alpha)} z^n + (-1)^k \overline{\sum_{n=1}^{\infty} \frac{(1-\alpha)Y_n}{\Omega^k(\Omega-\alpha)} z^n}$$

$$= z + \sum_{n=2}^{\infty} (h_n(z) - z)X_n + \sum_{n=1}^{\infty} (g_n(z) - z)Y_n$$

$$= \sum_{n=2}^{\infty} h_n(z)X_n + \sum_{n=1}^{\infty} g_n(z)Y_n + z(1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n)$$

$$= \sum_{n=1}^{\infty} (h_n(z)X_n + g_n(z)Y_n)$$

as required.

Following Avci and Zlotkiewicz [5], we refer to the γ −neighborhood of a function $f \in \mathcal{S}_H^*(\alpha)$ will be defined by

$$\mathcal{N}_\gamma(f) = \{F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n z^n}:$$

$$\sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \leq \gamma\}.$$

In our case, let us define the generalized γ −neighborhood of f to be the set

$$\mathcal{N}_\gamma^D(f) = \{F(z): \sum_{n=2}^{\infty} (\beta(n-1)(\lambda-\delta)+1)^k$$

$$\left[(\Omega-\alpha)\left(|a_n - A_n| + (\Omega+\alpha)\right)|b_n - B_n|\right]$$

$$+(1+\alpha)|b_1 - B_1| \leq (1-\alpha)\gamma\}. \qquad (2.5)$$

Now, we see the following theorem:

Theorem 2.6. *Let $f \in \mathcal{S}_H^k(\delta, \beta, \lambda, \alpha)$ be given by (1.2). If f satisfies the conditions*

$$\sum_{n=2}^{\infty} n(\beta(n-1)(\lambda-\delta)+1)^k((\Omega-\alpha)|a_n| + (\Omega+\alpha)|b_n|)$$

$$\leq (1-\alpha) - (1+\alpha)|b_1|, \qquad (2.6)$$

where

$\Omega^k = [\beta(n-1)(\lambda-\delta)+1]^k, \Omega = [\beta(n-1)(\lambda-\delta)+1], 0 \leq \alpha < 1, \delta \geq 0, \quad \beta > 0,\ \lambda > 0,\ \delta \neq \lambda, k \in \mathbb{N}_0,$

$$\gamma \leq \frac{1}{2}\left(1 - \frac{1+\alpha}{1-\alpha}|b_1|\right), \qquad (2.7)$$

then $N_\gamma^D(f) \subset \mathcal{S}_H^k(\delta, \beta, \lambda, \alpha)$.

Proof. Let f satisfies (2.6) and $F(z)$ be given by

$$F(z) = z + \overline{B_1 z} + \sum_{n=2}^{\infty} (A_n z^k + \overline{B_n z^n}),$$

which belong to $N_\gamma^D(f)$. In other words, it suffices to show that F satisfies the condition

$$\mathcal{N}_\gamma^D(f) = \sum_{n=2}^{\infty} [\tfrac{\Omega-\alpha}{1-\alpha}|A_n| + \tfrac{\Omega+\alpha}{1-\alpha}|B_n|](\beta(n-1)(\lambda-\delta)+1)^k + \tfrac{1+\alpha}{1-\alpha}|B_1| \leq 1.$$

We observe that

$$\mathcal{N}_\gamma^D(f) = \sum_{n=2}^{\infty} [\frac{\Omega-\alpha}{1-\alpha}|A_n| + \frac{\Omega+\alpha}{1-\alpha}|B_n|]$$

$$(\beta(n-1)(\lambda-\delta)+1)^k + \frac{1+\alpha}{1-\alpha}|B_1|$$

$$= \sum_{n=2}^{\infty} \left[\frac{\Omega-\alpha}{1-\alpha}|A_n - a_n + a_n| + \frac{\Omega+\alpha}{1-\alpha}|B_n - b_n + b_n|\right]$$

$$(\beta(n-1)(\lambda-\delta)+1)^k + \tfrac{1+\alpha}{1-\alpha}|B_1 - b_1 + b_1|$$

$$= \sum_{n=2}^{\infty} [\tfrac{\Omega-\alpha}{1-\alpha}|A_n - a_n| + \tfrac{\Omega+\alpha}{1-\alpha}|B_n - b_n|](\beta(n-1)(\lambda-\delta)+1)^k$$

$$+ \sum_{n=2}^{\infty} \left[\tfrac{\Omega-\alpha}{1-\alpha}|a_n| + \tfrac{\Omega+\alpha}{1-\alpha}|b_n|\right](\beta(n-1)(\lambda-\delta)+1)^k + \tfrac{1+\alpha}{1-\alpha}|B_1 - b_1| + \tfrac{1+\alpha}{1-\alpha}|b_1|$$

$$= \sum_{n=2}^{\infty} [\tfrac{\Omega-\alpha}{1-\alpha}|A_n - a_n| + \tfrac{\Omega+\alpha}{1-\alpha}|B_n - b_n|](\beta(n-1)(\lambda-\delta)+1)^k$$

$$+ \tfrac{1+\alpha}{1-\alpha}|B_1 - b_1| + \sum_{n=2}^{\infty} [\tfrac{\Omega-\alpha}{1-\alpha}|a_n| + \tfrac{\Omega+\alpha}{1-\alpha}|b_n|$$

$$(\beta(n-1)(\lambda-\delta)+1)^k + \frac{1+\alpha}{1-\alpha}|b_1|$$

$$= \gamma + \tfrac{1+\alpha}{1-\alpha}|b_1| + \tfrac{1}{2}\sum_{n=2}^{\infty} n[\tfrac{\Psi-\alpha}{1-\alpha}|a_n| + \tfrac{\Psi+\alpha}{1-\alpha}|b_n|]$$

$$(\beta(n-1)(\lambda-\delta)+1)^k$$

$$\leq \gamma + \tfrac{1+\alpha}{1-\alpha}|b_1| + \tfrac{1}{2}(1 - \tfrac{1+\alpha}{1-\alpha}|b_1|).$$

Now this last expression is never greater than one provided that

$$\gamma \leq 1 - \tfrac{1+\alpha}{1-\alpha}|b_1| - \tfrac{1}{2}(1 - \tfrac{1+\alpha}{1-\alpha}|b_1|) = \tfrac{1}{2}(1 - \tfrac{1+\alpha}{1-\alpha}|b_1|).$$

Remark 2.1. *Other works related to harmonic univalent functions can be found in* [[7]-[10]].

CONCLUSION

In this paper, we obtained some results concerning the coefficient bounds, growth bounds, a closure

ISSN No. 2321-2705 | DOI: 10.51244/IJRSI |Volume XI Issue IX September 2024

property, extreme points, neighborhood and a convex combination of harmonic univalent Function in the open unit disc, which are related to the differential operator. We suggest to introduce a new subclass of *p*-valent starlike functions with negative coefficients in the open unit disc which is defined by a generalised derivative operator.

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