Marshall-Olkin Power Hazard Rate Distribution with Associated Minification Process and Application

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ABSTRACT

In this paper, a new three-parameter lifetime distribution of Marshall-Olkin family of distributions with the baseline model as power hazard rate distribution called the Marshall-Olkin Power Hazard rate distribution is introduced and its various structural properties are discussed. We examined different statistical properties like hazard rate function, quantile function, raw moment, incomplete moment and related measures, mean deviation, Bonferroni and Lorenz curve, order statistics, Renyi's entropy, stress-strength reliability, stochastic ordering, and compounding and geometric extreme stability. The model parameter estimation is carried out using maximum likelihood estimation method. The model's flexibility is evaluated by two real datasets by comparing it with different related models, and its application in time series is studied by the Markovian first-order autoregressive minification process.

Keywords: Marshall-Olkin family of distributions, power hazard rate distribution, minification process, quantile function.

INTRODUCTION

The demand for the probabilistic approach to data modelling in fields such as agriculture, demography, economics, engineering, epidemiology, finance and medicine, is arguably increasing. Since different data sets may have varying properties, it is usually expedient to examine the characteristics of a set of data before fitting a distribution to the data. Important characteristics of data that have to be put into consideration while searching for an appropriate distribution for data include skewness, and kurtosis whether the data are discrete or continuous. Distributions with different properties abound in the literature. They include exponential distribution which has a right-skewed shape with a constant hazard rate function, Weibull and gamma distributions having asymmetrical shapes with decreasing, increasing and constant hazard rate functions, log-normal distribution which has a right-skewed shape with increasing and decreasing hazard rate function, and so on.

In practice, data with unique features are encountered. One way of ensuring that an adequate distribution is fitted to the data is to consider a new distribution. A new distribution can be introduced by generalizing an existing one. Recently, authors have paid attention to the development of existing distributions or suggestions of new flexible distributions that are suitable for data with any degree of complexity (Mohiuddin et al. 2021; Mustafa 2024). Several techniques for generalizing distributions have been pointed out by a quite number of scholars. Among these methods, the Marshall-Olkin (1997) procedure has attracted the attention of scholars because of its ability to handle complex failure patterns in reliability engineering, and its performance in survival and risk analyses. Primarily, the MO method is aimed at inducing an additional parameter in a distribution, resulting in to Marshall-Olkin extended distribution (Jose 2011).

If $G(x;\omega)$ and $g(x;\omega)$ denotes the cumulative distribution function (cdf) and probability density function (pdf) of the original distribution depending on the vector parameters ω, then the survival function, cdf and pdf of Marshall-Olkin family of distributions, are, respectively, defined by

$$\bar{F}_{MO}(x;\alpha,\omega)=\frac{\alpha\,\bar{G}(x;\omega)}{1-\bar{\alpha}\,\bar{G}(x;\omega)} \qquad \omega;\alpha>0 \quad x>0, \tag{1}$$

$$F_{MO}(x;\alpha,\omega)=\frac{G(x;\omega)}{1-\bar{\alpha}\bar{G}(x;\omega)} \qquad \omega;\alpha>0 \quad x>0, \tag{2}$$

and

$$f_{MO}(x;\alpha,\omega)=\frac{\alpha\,g(x;\omega)}{\left(1-\bar{\alpha}\bar{G}(x;\omega)\right)^2} \qquad \omega;\alpha>0 \quad x>0, \tag{3}$$

where, $\bar{G}(x;\omega)=1-G(x;\omega)$, $\bar{\alpha}=1-\alpha$, and α as the tilt parameter which makes the distribution more flexible and robust than the baseline distribution. When $\alpha=1$, the distribution becomes the baseline distribution.

The Marshall-Olkin (MO) family of distribution is usually possessed with an interesting hazard rate function and a flexible shape parameter allowing for varying degrees of skewness and heavy-tailedness than the baseline distribution. The MO is a simple method and has been applied by many authors to generate flexible distributions. For instance, the technique had been applied to the exponential, Weibull, Pareto, logistic, gamma, Burr type XII, beta, Lomax, and log-normal distributions amongst others, to generate Marshall-Olkin extended exponential distribution (Roa et al. 2009), Marshall-Olkin extended Weibull family (Santos-Neto et al. 2014), Marshal-Olkin Pareto distribution (Alice and Jose, 2003), Marshall-Olkin logistic distribution (Alice and Jose, 2005), Marshall-Olkin extended Burr type XII distribution (Al-Saiari et al. 2014), Marshall-Olkin beta distribution (Jose et al. 2009), Marshall-Olkin extended Lomax distribution (Ghitany et al. 2007), Marshall-Olkin power log-normal (Gui, 2013), and others.

Real lifetime distributions which include the power hazard rate distribution usually have a specific hazard rate function which describes how its failure rate changes over time.

Mugdadi (2005) introduced the power hazard rate function as

$$h(x;\theta,\beta)=\theta x^{\beta} \qquad x>0, \tag{4}$$

and its related pdf, cdf, and sf are defined in (5), (6) and (7) respectively as

$$g(x;\theta,\beta)=\theta x^{\beta}\,e^{-\frac{\theta}{\beta+1}x^{\beta+1}} \qquad x>0, \tag{5}$$

$$G(x;\theta,\beta)=1-e^{-\frac{\theta}{\beta+1}x^{\beta+1}} \qquad x>0, \tag{6}$$

and

$$\bar{G}(x;\theta,\beta)=e^{-\frac{\theta}{\beta+1}x^{\beta+1}} \qquad x>0, \tag{7}$$

where, $\theta>0$ and $\beta>-1$ are scale and shape parameters.

The power hazard rate distribution PHRD has a monotonic hazard rate function depending on the value of

β. Its hazard rate function increases when $\beta > 0$, decreases when $-1 < \beta < 0$ and remains constant when $\beta = 0$ (Mustafa 2024). This makes it a useful alternative to some distributions when modelling monotone hazard rates (Mustafa and Khan 2022). Again, well-known classical distributions like exponential, Rayleigh and one-parameter Weibull distribution are essentially special cases of the PHRD.

Sequel to the introduction of this distribution, studies based on its properties, extensions and applications to complete and censored schemes have been undertaken by researchers. In particular, Kinaci (2014) discussed the stress-strength reliability of PHRD. Mugdadi and Min (2009) discussed the Bayes estimation of PHRD using complete and Type II censored samples, El-Sagheer (2015) explained the estimations of the parameters for PHRD using a progressive Type II censoring scheme while Travirdizade and Nematollahi (2016) discussed the parameter estimation of PHRD using record data. El-Sagheer et al. (2022) studied the Bayesian and non-Bayesian approaches to the lifetime performance index of PHRD using the Type II progressive censored sample. Al-Morshedy et al. (2022) investigated the estimation of shape and scale parameters, sf, hrf of PHRD using A Type II adaptive progressive censored scheme. Khan and Mustafa (2022) introduced a generalization of PHRD called the weighted power hazard rate distribution with applications and investigated many of its properties. Khan and Mustafa (2022) introduced the transmuted PHRD and discussed its properties and applications to lifetime data. Mustafa and Khan (2022) introduced the length-biased PHRD and studied some properties and applications. Mustafa and Khan (2023) introduced a generalization of PHRD using the sine function and discussed its statistical properties and applications.

This study aims to introduce a new generalization of the PHRD using the MO generalization of distribution. With an additional parameter (tilt), the Marshall Olkin Power Hazard rate MOPHR distribution is expected to be more flexible in modelling a wide range of data than the PHRD. The remaining part of the paper is organized as follows: Section 2, is concerned with the definition and statistical properties of the new distribution. The MOPHR first-order Autoregressive process is discussed in Section 3. The Maximum likelihood estimation of the parameters of the distribution is discussed in Section 4. In section 5, a simulation study and numerical application to a real dataset are performed, and the MOPHR distribution is compared with other distributions. The last section contains the concluding remarks.

DEFINITION AND STATISTICAL PROPERTIES OF THE NEW DISTRIBUTION

Here, we provide the pdf, cdf, sf and hazard rate function (hrf) of the proposed distribution. Due consideration is also given to its statistical properties, among which are quantile function, raw moment, incomplete moment and related measures, mean deviation, Bonferroni and Lorenz curve, order statistics, Renyi's entropy, stress-strength reliability, stochastic ordering, and compounding and geometric extreme stability.

The sf, cdf, pdf and hrf of MOPHR distribution

Survival function (sf)

In defining the sf of the distribution, we make use of (1) and (7) to obtain

$$\overline{F}(x) = \frac{\alpha e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{1-\overline{\alpha}\, e^{-\frac{\theta}{\beta+1}x^{\beta+1}}} \qquad x > 0 \quad \alpha, \beta, \theta > 0 . \qquad (8)$$

Cumulative density function (cdf)

The related cdf is obtained using (2), (6), and (7) as

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$$F(x)=\frac{1-e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{1-\bar{\alpha}\,e^{-\frac{\theta}{\beta+1}x^{\beta+1}}} \qquad x>0 \quad \alpha,\beta,\theta>0\,. \tag{9}$$

Probability distribution function (pdf)

Differentiating (9) with respect to x yields the pdf of the new distribution as

$$f(x)=\frac{\alpha\theta x^{\beta}e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{\left(1-\bar{\alpha}\,e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right)^{2}} \qquad x>0 \quad \alpha,\beta,\theta>0\,, \tag{10}$$

where, α is the additional parameter. Henceforth, if a non-negative continuous random variable X has a Marshall-Olkin power hazard rate MOPHR distribution with parameters α, β and θ, we write X ~ MOPHR(α, β, θ).

Remark: There are four special cases of this distribution.

i. When $\alpha=1$ and $\beta=0$ the distribution reduces to the exponential distribution.
ii. When $\beta=0$ the distribution reduces to the Marshall-Olkin Exponential (MOE) distribution.
iii. When $\alpha=1$, $\theta=1$, and $\beta=1$ the distribution reduces to the Rayleigh distribution.
iv. When $\theta=1$ and $\beta=1$ the distribution reduces to the Marshall-Olkin extended Rayleigh (MOER) distribution,
v. When $\alpha=1$ and $\beta=\theta-1$ the distribution reduces to One parameter Weibull (OPW) distribution, and
vi. When $\beta=\theta-1$ the distribution reduces to the Marshall-Olkin One parameter Weibull (MOOPW) distribution.

Figure 1 contains the pdf of the MOPHR distribution and shows that the distribution can be unimodal and asymmetry.

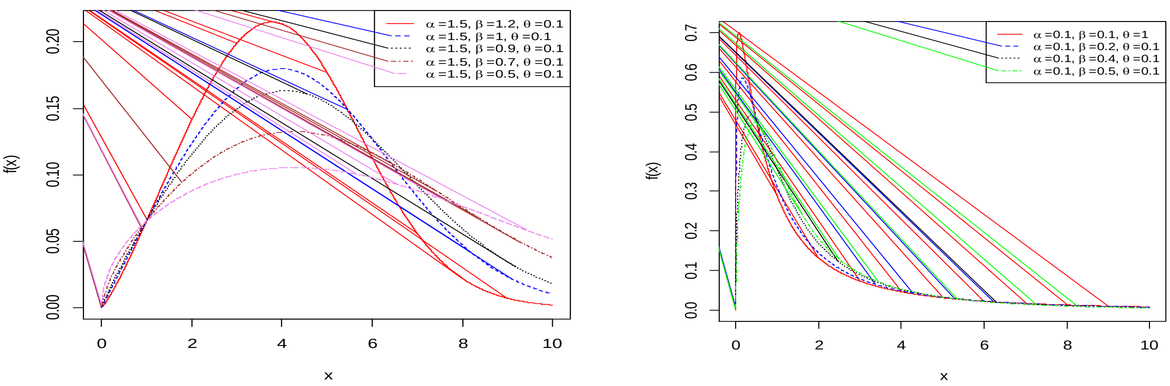

Figure 1: The graph of the probability density function of MOPHR(α,β,θ) distribution

Hazard rate function

The hazard rate function (hrf) for the Marshall-Olkin power hazard rate distribution is

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{\theta x^{\beta}}{1-\bar{\alpha}\,e^{-\frac{\theta}{\beta+1}x^{\beta+1}}} \tag{11}$$

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The plots of hrf of MOPHR distribution are presented in Figure 2. From this Figure 2, the distribution can have a decreasing hrf or a non-monotonic hrf.

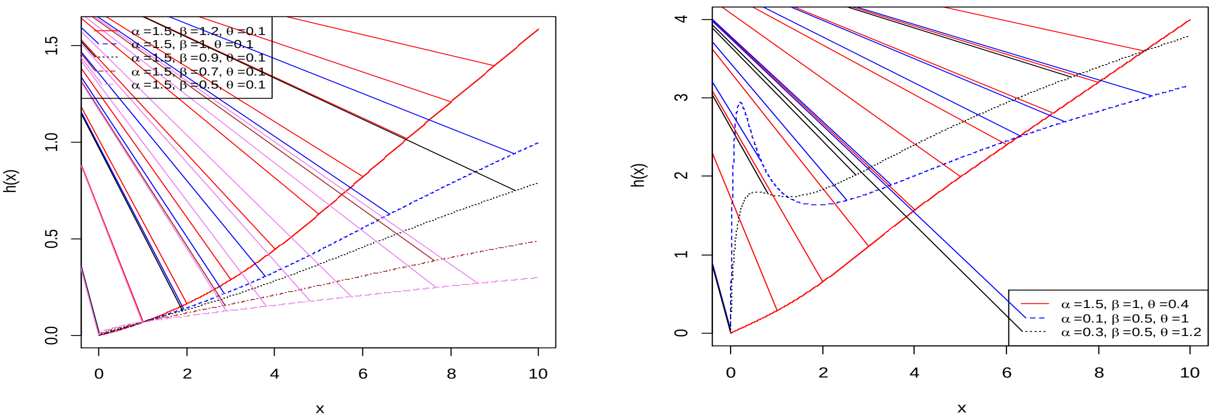

Figure 2: The graph of the hazard rate function of MOPHR(α, β, θ).

Quantile function (qf) and random number generator

Suppose $q \in (0,1)$. The qth quantile of the MOPHR distribution, denoted by x_q is determined by solving for x_q in the following equation:

$$F(x_q) = q.$$

Using Eq. (9) for $F(x_q)$, we have

$$x_q = \left(-\tfrac{\beta+1}{\theta} \ln\left[\frac{1-q}{1-\bar{\alpha} q} \right] \right)^{\frac{1}{\beta+1}} . \qquad (12)$$

The random number generator is obtained using Eq. (12)

Let $U \sim U(0,1)$. Then the random variable X is generated by

$$X = \left(-\tfrac{\beta+1}{\theta} \ln\left[\frac{1-U}{1-\bar{\alpha} U} \right] \right)^{\frac{1}{\beta+1}} . \qquad (13)$$

Usefulness expression for the pdf of MOPHR distribution

To determine moments and other properties of the distribution, it is worthwhile to consider an alternative representation of the pdf of the MOPHR distribution which facilitates the derivation of the properties.

If /z/<1, then for any real number λ, we have the negative binomial series

$$(1-z)^{-\lambda} = \sum_{i=0}^{\infty} \frac{\Gamma(\lambda+i) z^i}{\Gamma(\lambda)\Gamma(i+1)} = \sum_{i=0}^{\infty} \frac{\Gamma(\lambda+i) z^i}{\Gamma(\lambda)\, i!}, \qquad (14)$$

where, $\Gamma(.)$ is the gamma function. Whenever $\alpha < 1$,

$$f(x) = \alpha\theta x^{\beta} e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\left(1-\bar{\alpha}e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right)^{-2} = \alpha\theta x^{\beta} e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\sum_{i=0}^{\infty}(i+1)\bar{\alpha}^{i} e^{-\frac{\theta i}{\beta+1}x^{\beta+1}}$$

$$= \alpha\theta\sum_{i=0}^{\infty}(i+1)\bar{\alpha}^{i} x^{\beta} e^{-\frac{(i+1)\theta}{\beta+1}x^{\beta+1}} \qquad (15)$$

If $\alpha > 1$, then

$$f(x) = \alpha\theta x^{\beta} e^{-\frac{\theta}{\beta+1}x^{\beta+1}} \alpha^{-2}\sum_{l=0}^{\infty}(l+1)\left(\frac{\alpha-1}{\alpha}\right)^{l}\left(1-e^{-\frac{\theta i}{\beta+1}x^{\beta+1}}\right)^{l}$$

$$= \alpha^{-1}\theta x^{\beta} e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\sum_{i=0}^{\infty}(-1)^{i} e^{-\frac{\theta i}{\beta+1}x^{\beta+1}}\sum_{l=i}^{\infty}\binom{l}{i}(l+1)\left(\frac{\alpha-1}{\alpha}\right)^{l}$$

$$= \alpha^{-1}\sum_{i=0}^{\infty}\sum_{l=i}^{\infty}(-1)^{i}\binom{l}{i}\frac{(l+1)}{(i+1)}\left(\frac{\alpha-1}{\alpha}\right)^{l}(i+1)\theta x^{\beta} e^{-\frac{(i+1)\theta}{\beta+1}x^{\beta+1}} \qquad (16)$$

Combining Equation Eq. (15) and Eq. (16) leads to

$$f(x) = \sum_{l=i}^{\infty} v_i g_{ph}(x;\beta,(i+1)\theta) \quad , \qquad (17)$$

where,

$$v_i = \begin{cases} \alpha\bar{\alpha}^{i}, & \alpha < 1 \\ \alpha^{-1}\sum_{l=i}^{\infty}\frac{(-1)^{i}(l+1)}{(i+1)}\binom{l}{i}\left(\frac{\alpha-1}{\alpha}\right)^{l}, & \alpha > 1 \end{cases},$$

and $g_{ph}(x;\beta,(i+1)\theta)$ is the density function of a continuous random variable having the power hazard distribution parameters β and $(i+1)\theta$.

Raw moments

Using Eq. (17), the rth crude moment of the MOPHR distribution given by

$$\mu_r' = E(X^r) = \sum_{i=0}^{\infty} v_i \int_0^{\infty} x^r g_{ph}(x;\beta,(i+1)\theta)dx = \sum_{i=0}^{\infty} v_i \int_0^{\infty}(i+1)\theta\, x^{\beta+r}\, e^{-\frac{(i+1)\theta}{\beta+1}x^{\beta+1}} dx$$

$$= \sum_{i=0}^{\infty}\frac{(i+1)v_i\theta}{\beta+1}\left(\frac{\beta+1}{\theta(i+1)}\right)^{\frac{\beta+r+1}{\beta+1}}\Gamma\left(\tfrac{\beta+r+1}{\beta+1}\right). \qquad (18)$$

The mean of the MOPHRD is obtained by putting r = 1 into (18).

$$\mu_1' = \sum_{i=0}^{\infty} v_i \frac{(i+1)\theta}{\beta+1}\left(\frac{\beta+1}{\theta(i+1)}\right)^{\frac{\beta+2}{\beta+1}}\Gamma\left(\tfrac{\beta+2}{\beta+1}\right) \qquad (19)$$

The variance of the MOPHRD is obtained as follows

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$$\sigma^2 = \mu_2' - (\mu_1')^2$$

where μ_2' is obtained by putting r = 2 into (18). Then

$$\therefore \quad \sigma^2 = \mu_1'\left[\left(\frac{\beta+2}{\beta+1}\right)\left(\frac{\beta+1}{\theta(i+1)}\right)^{\frac{1}{\beta+1}} - \mu_1'\right] \tag{20}$$

Table 1: Moments, skewness (Sk) and kurtosis (Kur) for MOPHR (α, β, θ) distribution

α	B	θ	E(X)	E(X^2)	E(X3)	E(X4)	Var(X)	Sk	Kur
1	0	0.5	2.000	8.000	48.000	348.000	4.000	2.000	9.000
	0	1	1.000	2.000	6.000	24.000	1.000	2.000	9.000
	0.5	0.5	1.878	5.152	18.000	75.111	1.626	1.072	4.390
	0.5	1	1.183	2.044	4.500	11.829	0.645	1.072	4.390
	1	0.5	1.773	4.000	10.635	32.000	0.858	0.631	3.245
	1	1	1.253	2.000	3.760	8.000	0.429	0.631	3.245
1.5	0	0.5	1.751	5.551	26.919	177.098	2.885	1.591	7.651
	0	1	0.876	1.388	3.365	11.069	0.721	1.591	7.651
	0.5	0.5	1.786	4.164	12.489	45.128	1.393	0.501	3.31
	0.5	1	1.125	1.653	3.122	7.107	0.553	0.501	3.31
	1	0.5	1.758	3.503	8.267	22.202	0.816	-0.065	2.415
	1	1	1.243	1.751	2.923	5.551	0.408	-0.065	2.415
2	0	0.5	1.641	4.575	19.973	120.525	2.531	1.298	6.763
	0	1	0.821	1.144	2.497	7.533	0.633	1.298	6.763
	0.5	0.5	1.758	3.728	10.293	34.598	1.384	0.114	2.613
	0.5	1	1.107	1.48	2.573	5.449	0.549	0.114	2.613
	1	0.5	1.776	3.283	7.242	18.298	0.889	-0.472	1.932
	1	1	1.256	1.641	2.56	4.575	0.445	-0.472	1.932

Table 1 displays the basic descriptive statistics of MOPHR distribution which include the first four raw moments, variance, coefficient of skewness (Sk) and coefficient of kurtosis (Kur) for MOPHR distribution for different parameters values.

We may infer from the table that the MOPHR distribution takes different shapes depending on the values of the parameters. Thus, MOPHR distribution can be right skewed, left skewed, platykurtic and leptokurtic.

Incomplete moment and related measures

The incomplete moment is useful and plays a key role in determining some of the statistical properties of the distribution. The rth incomplete moment for the MOPHR distribution using Eq. (17) is

$$J_r(z) = \int_0^z x^r f(x)\,dx \quad = \sum_{i=0}^{\infty} \frac{(i+1)v_i\theta}{\beta+1}\left(\frac{\beta+1}{\theta(i+1)}\right)^{\frac{\beta+r+1}{\beta+1}} \gamma\left(\tfrac{(i+1)\theta x^{\beta+1}}{\beta+1}, \tfrac{\beta+r+1}{\beta+1}\right), \tag{21}$$

where, $\gamma(y,\delta) = \int_0^y t^{\delta-1}e^{-t}dt$ represents the incomplete gamma function.

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Mean deviation

For the MOPHR distribution, the mean deviation about the mean μ_1' is

$$\delta_1 = \int_0^\infty / x - \mu_1' / f(x)dx == 2\mu_1' F(\mu_1') - 2J_1(\mu_1'). \quad (22)$$

And, the mean deviation about the median m is

$$\delta_2 = \int_0^\infty / x - m / f(x)dx == \mu_1' - 2J_1(m) . \quad (23)$$

Bonferroni and Lorenz curve

Let p be a given probability value. The Bonferroni curve and Lorenz curve for the MOPHR distribution respectively are defined using Eq. (22) and Eq. (19) as

$$B(p) = \frac{J_1(q)}{p\mu_1'}, \quad (24)$$

and

$$L(p) = \frac{J_1(q)}{\mu_1'} . \quad (25)$$

Order statistics

Here, we look at another medium of understanding and characterizing the behaviour of the MOPHR distribution by determining the order statistics which is crucial and also helps in identifying the distributional properties of the distribution.

Given a random sample $X_1, X_2, \ldots, X_n$ from the MOPHR distribution, Let $X_{i:n}$ be the ith order statistics. The pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)\bar{F}^{n-i}(x)\left(1 - \bar{F}(x)\right)^{i-1}$$

$$= \frac{n!}{(i-1)!\,(n-i)!} f(x)\sum_{j=1}^{i-1}(-1)^j \binom{i-1}{j} \bar{F}^{n+j-i}(x) \quad (26)$$

Substituting Eq. (8) and Eq. (9) in Eq. (26), we have

$$= \frac{n!\alpha\,\theta x^{\beta}}{(i-1)!\,(n-i)!}\sum_{j=0}^{i-1}(-1)^j \binom{i-1}{j} \frac{\alpha^{n-i+j}\, e^{-\frac{\theta(n-i+j)}{\beta+1}x^{\beta+1}}}{\left(1 - \bar{\alpha}\, e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right)^{n-i+j+2}} . \quad (27)$$

If $\alpha \in (0,1)$

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$$f_{i:n}(x)=\frac{n!\alpha\,\theta x^{\beta}e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{(i-1)!\,(n-i)!}\sum_{j=0}^{i-1}(-1)^{j}\binom{i-1}{j}\alpha^{n-i+j}e^{-\frac{(n-i+j)\theta}{\beta+1}x^{\beta+1}}\sum_{k=0}^{\infty}\overline{\alpha}^{k}\binom{(n-i+j+2)+k-1}{k}e^{-\frac{k\theta}{\beta+1}x^{\beta+1}}$$

$$=\frac{n!}{(i-1)!\,(n-i)!}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\overline{\alpha}^{k}(-1)^{j}\binom{i-1}{j}\alpha^{n-i+j+1}\binom{n-i+j+k+1}{k}\theta x^{\beta}e^{-\frac{(n-i+j+k+1)\theta}{\beta+1}x^{\beta+1}}$$

$$f_{i:n}(x)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}v_{i,j,k}\,g_{ph}(\beta,(n-i+j+k+1)\theta)\,, \qquad (28)$$

where, $v_{i,j,k}=\dfrac{n!\;\theta\alpha^{n-i+j+1}\overline{\alpha}^{k}(-1)^{j}\binom{i-1}{j}\binom{n-i+j+k}{k-1}}{(i-1)!\,(n-i)!}$, and $g_{ph}(\beta,(n-i+j+k+1)\theta)=(n-i+j+k+1)\theta x^{\beta}e^{-\frac{(n-i+j+k+1)\theta}{\beta+1}x^{\beta+1}}$

Given that $\alpha>1$, then

$$f_{i:n}(x)=\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{j=0}^{i-1}v_{i,j,k,m}\,g_{ph}(\beta,(n-i+j+m+k+1)\theta)\,, \qquad (29)$$

where, $v_{i,j,k,m}=\dfrac{n!\theta\overline{\alpha}^{-1}(1-\alpha^{-1})^{k}(-1)^{j+m}\binom{i-1}{j}\binom{n-i+j+m}{m-1}\binom{m}{k}}{(i-1)!\,(n-i)!}$

and $g_{ph}(\beta,(n-i+j+m+1)\theta)=(n-i+j+m+1)\theta x^{\beta}e^{-\frac{(n-i+j+m+1)\theta}{\beta+1}x^{\beta+1}}$

Therefore, the rth raw moment of the ith order statistic for the MOPHR distribution is

$$E\left(X^{r}{}_{i:n}\right)=\begin{cases}\displaystyle\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}v_{i,j,k}\left(\frac{\beta+1}{n-i+j+k+1}\right)^{\frac{r}{\beta+1}}\Gamma(\tfrac{\beta+r+1}{\beta+1}) & \alpha\in(0,1)\\ \displaystyle\sum_{k=0}^{\infty}\sum_{m=0}^{k}\sum_{j=0}^{i-1}v_{i,j,k,m}\left(\frac{\beta+1}{n-i+j+m+1}\right)^{\frac{r}{\beta+1}}\Gamma(\tfrac{\beta+r+1}{\beta+1}) & \alpha>1\end{cases} \qquad (30)$$

Renyi’s Entropy

Here, we look at the property that measures the uncertainty or randomness of the MOPHR distribution.

$$I_{R}(\lambda)=(1-\lambda)^{-1}\log\left[\int_{0}^{\infty}f^{\lambda}(x)dx\right]\;\lambda>0,\;\lambda\neq1 \qquad (31)$$

where, $f^{\lambda}(x)=\dfrac{(\alpha\theta)^{\lambda}x^{\beta\lambda}e^{-\frac{\theta\lambda}{\beta+1}x^{\beta+1}}}{\left(1-\overline{\alpha}e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right)^{2\lambda}}$.

If $\alpha\in(0,1)$ and Eq. (14) is applied to express $f^{\lambda}(x)$, then we have

$$f^{\lambda}(x) = (\alpha\theta)^{\lambda}\sum_{i=0}^{\infty}\frac{x^{\beta\lambda}e^{-\frac{(i+\lambda)\theta}{\beta+1}x^{\beta+1}}\Gamma(2\lambda+i)\bar{\alpha}^{i}}{\Gamma(2\lambda)i!} \quad (32)$$

On the other hand, if $\alpha > 1$, then

$$f^{\lambda}(x) = \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}(\alpha^{-1}\theta)^{\lambda}(-1)^{i}x^{\beta\lambda}e^{-\frac{(i+\lambda)\theta}{\beta+1}x^{\beta+1}}\binom{k}{i}\frac{\Gamma(2\lambda+k)}{\Gamma(2\lambda)k!} \quad (33)$$

Therefore, for $\lambda > 0$ and $\lambda \neq 1$, using (32) and (33) in (31), we have

$$I_R(\lambda) = \begin{cases} (1-\lambda)^{-1}\log\left[\sum_{i=0}^{\infty} w_i\left(\frac{1}{\beta+1}\right)\left(\frac{\beta+1}{(i+\lambda)\theta}\right)^{\frac{\beta\lambda+1}{\beta+1}}\Gamma\left(\frac{\beta\lambda+1}{\beta+1}\right)\right], & if\ \alpha \in (0,1) \\ (1-\lambda)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} w_{i,k}\left(\frac{1}{\beta+1}\right)\left(\frac{\beta+1}{(i+\lambda)\theta}\right)^{\frac{\beta\lambda+1}{\beta+1}}\Gamma\left(\frac{\beta\lambda+1}{\beta+1}\right)\right], & if\ \alpha > 1 \end{cases}, \quad (34)$$

where, $w_i = \frac{(\alpha\theta)^{\lambda}\bar{\alpha}^{i}\Gamma(2\lambda+i)}{\Gamma(2\lambda)\, i\,!}$ and $w_{i,k} = \frac{(\alpha^{-1}\theta)^{\lambda}(-1)^{i}\Gamma(2\lambda+k)}{\Gamma(2\lambda)\, k\,!}\binom{k}{i}$

Stress-Strength Reliability

In reliability theory, the stress-strength model is crucial in describing the life of a component having a random strength X_1, subject to a random stress X_2. Here, the stress-strength reliability is given by $R = \Pr(X_2 < X_1)$.

Given that $X_1 \sim MOPHR(\alpha_1, \beta, \theta_1)$ and $X_2 \sim MOPHR(\alpha_2, \beta, \theta_2)$ are independent MOPHR random variables with the same shape parameter β.

$$R = \int_0^{\infty} f_1(x)F_2(x)\,d(x) \quad (35)$$

In Eq. (35), $f_1(x)$ is the pdf of X_1 while $F_2(x)$ is the cdf of X_2.

Applying Eq. (9), we have that for $\alpha \in (0,1)$

$$F_2(x) = \frac{1-e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}}}{1-\bar{\alpha}_2 e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}}} = \sum_{j=0}^{\infty}\bar{\alpha}_2^{\,j}\left(1-e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}}\right)e^{-\frac{\theta_2 j}{\beta+1}x^{\beta+1}}. \quad (36)$$

For $\alpha > 1$,

$$F_2(x) = \frac{1-e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}}}{1-\bar{\alpha}\left(1-(1-e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}})\right)} = \sum_{j=0}^{\infty}\alpha^{-1}\sum_{m=j}^{\infty}(-1)^{j}(1-\tfrac{1}{\alpha})^{m}\binom{m}{j}\left(1-e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}}\right)e^{-\frac{\theta_2 j}{\beta+1}x^{\beta+1}} \quad (37)$$

Hence,

$$R=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}v_i\,z_j\int_0^{\infty}\left(1-e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}}\right)e^{-\frac{\theta_2}{\beta+1}x^{\beta+1}}g_{ph}(x;\beta,(i+1)\theta_1)\,dx$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}v_i\,z_j\,(i+1)\theta_1\left[\frac{1}{(i+1)\theta_1+\theta_2 j}-\frac{1}{(i+1)\theta_1+(j+1)\theta_2}\right], \tag{38}$$

where,

$$z_j=\begin{cases}\bar{\alpha}_2^{\,j}, & \alpha\in(0,1)\\ \sum_{m=j}^{\infty}\alpha^{-1}(-1)^j\binom{m}{j}(1-\alpha^{-1})^m, & \alpha>1\end{cases}$$

Stochastic ordering

Stochastic ordering is a useful tool that compares the relative magnitude of random variables or distributions.

Theorem 1: Let $X\sim MOPHR(\alpha_1,\beta_1,\theta_1)$ and $Y\sim MOPHR(\alpha_2,\beta_2,\theta_2)$. If $\theta_1=\theta_2$ and $\beta_1=\beta_2$, then $X\prec_{lr}Y$ and hence $X\prec_{hr}Y$ and $X\prec_{st}Y$ if $\alpha_1<\alpha_2$.

Proof.

$$\frac{f_X(x)}{f_Y(x)}=\frac{\left[\alpha_1\theta_1 x^{\beta_1}e^{-\frac{\theta_1}{\beta_1+1}x^{\beta_1+1}}\right]\left[1-\bar{\alpha}_2 e^{-\frac{\theta_2}{\beta_2+1}x^{\beta_2+1}}\right]^2}{\left[\alpha_2\theta_2 x^{\beta_2}e^{-\frac{\theta_2}{\beta_2+1}x^{\beta_2+1}}\right]\left[1-\bar{\alpha}_1 e^{-\frac{\theta_2}{\beta_1+1}x^{\beta_1+1}}\right]^2}$$

For $\beta_1=\beta_2=\beta$ and $\theta_1=\theta_2=\theta$, then

$$\frac{f_X(x)}{f_Y(x)}=\frac{\alpha_1\left[1-\bar{\alpha}_2 e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right]^2}{\alpha_2\left[1-\bar{\alpha}_1 e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right]^2} \tag{39}$$

$$\ln\left(\frac{f_X(x)}{f_Y(x)}\right)=\frac{\ln\alpha_1+2\ln\left[1-\bar{\alpha}_2 e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right]}{\ln\alpha_2+2\ln\left[1-\bar{\alpha}_1 e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right]} \tag{40}$$

$$\frac{\partial\ln\left(\frac{f_X(x)}{F_Y(x)}\right)}{\partial x}=\frac{2\theta(\alpha_1-\alpha_2)x^{\beta}e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{\left(1-\bar{\alpha}_1 e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right)\left(1-\bar{\alpha}_2 e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right)} \tag{41}$$

It can be deduced that $\frac{\partial\ln\left(\frac{f_X(x)}{F_Y(x)}\right)}{\partial x}<0$, if $\alpha_1<\alpha_2$. Under this condition, $\frac{f_X(x)}{f_Y(x)}$ decreases in x and $X\prec_{lr}Y$, $X\prec_{hr}Y$ and $X\prec_{st}Y$.

Compounding and geometric extreme stability

Suppose that $\overline{G}(x\,|\,\delta)$ is the conditional survival function of a continuous random variable X given a continuous random variable δ, where $-\infty < x < \infty$, and $-\infty < \delta < \infty$. Let δ be a random variable having the density $m(\delta)$. Consequently, a distribution whose survival function is

$$\overline{F}(x) = \int_{-\infty}^{\infty} \overline{G}(x\,|\,\delta)\, m(\delta)\, d\delta\,,\ -\infty < x < \infty \qquad (42)$$

is known as a compounding distribution with mixing density $m(\delta)$. In Theorem 2, we establish that the $MOPHR(\alpha,\beta,\theta)$ distribution can be derived through the mixing density which is an exponential distribution.

Theorem 2: Let the conditional survival function of a continuous random variable X given $\Delta = \delta$ be

$$\overline{G}(x\,|\,\delta) = \exp\left(\delta(1 - e^{-\frac{\theta}{\beta+1}x^{\beta+1}})\right)\ x, \delta > 0,\ \theta > 0 \text{ and } \beta > -1 \qquad (43)$$

Let Δ have an exponential distribution with density

$$m(\delta) = \alpha e^{-\alpha\delta}, \quad \alpha > 0, \delta > 0. \qquad (44)$$

Then, the compound distribution is the $MOPHR(\alpha,\beta,\theta)$ distribution.

Proof.

From Eq. (42), we have

$$\overline{F}(x) = \int_{0}^{\infty} \overline{G}(x\,|\,\delta)\, m(\delta)\, d\delta$$

Putting Eq. (43) and Eq. (44) into Eq. (42), we have

$$\overline{F}(x) = \alpha \int_{-\infty}^{\infty} e^{\delta(1 - e^{-\frac{\theta}{\beta+1}x^{\beta+1}})} e^{-\alpha\delta} d\delta = \alpha \int_{0}^{\infty} e^{-\delta(e^{\frac{\theta}{\beta+1}x^{\beta+1}} - \overline{\alpha})} d\delta$$

$$= \frac{\alpha}{(e^{\frac{\theta}{\beta+1}x^{\beta+1}} - \overline{\alpha})} = \frac{\alpha e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{1 - \overline{\alpha} e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}$$

Therefore, X follows the $MOPHR(\alpha,\beta,\theta)$ distribution.

For $\delta > 0$, the distribution with survival function (43) is a generalization of some existing distributions such as the Chen distribution (when $\beta = \theta - 1$) and the Gompertz distribution (when $\beta = 0$).

Other procedures for deriving the MOPHR distribution are summarized in theorem 3.

Theorem 3: Let $\{X_i, i > 1\}$ be a sequence of *iid* $PHR(\beta,\theta)$ random variables. Suppose N follows a geometric distribution with parameters α, where $0 < \alpha < 1$ and $P(N = n) = \alpha(1-\alpha)^{n-1}$, $n = 1, 2, \ldots$ Then:

(i) min($X_1, X_2, \ldots, X_N$) follows an $MOPHR(\alpha,\beta,\theta)$ distribution, i.e. geometric minimum stable.

(ii) $\max(X_1, X_2, \ldots, X_N)$ has an $MOPHR(\frac{1}{\alpha}, \beta, \theta)$ distribution, i.e. geometric maximum stable.

Proof.

The survival function of $\min(X_1, X_2, \ldots, X_N)$ is given by

$$P(\min(X_1, X_2, \ldots, X_N) > x) = \sum_{n=1}^{\infty} P(X_1 > x, \ldots, X_N > x) P(N = n)$$

$$P(\min(X_1, X_2, \ldots, X_N) > x) = \sum_{n=1}^{\infty} \left(\overline{G}(x)\right)^n \alpha (1-\alpha)^{n-1} = \frac{\alpha \overline{G}(x)}{1-(1-\alpha)\overline{G}(x)}$$

which is the survival function of the Marshall-Olkin Power Hazard rate $MOPHR(\alpha, \beta, \theta)$ distribution.

The survival function of $\max(X_1, X_2, \ldots, X_N)$ has the form

$$P(\max(X_1, X_2, \ldots, X_N) > x) = 1 - P(\max(X_1, X_2, \ldots, X_N) \le x))$$

$$= 1 - \sum_{n=1}^{\infty} P(X_1 \le x, \ldots, X_N \le x) P(N = n)$$

$$= 1 - \sum_{n=1}^{\infty} \left(G(x)\right)^n \alpha (1-\alpha)^{n-1} = 1 - \frac{\alpha G(x)}{1-(1-\alpha)G(x)}$$

$$= \frac{\overline{G}(x)}{\alpha(1-(1-\frac{1}{\alpha})\overline{G}(x))} = \frac{\frac{1}{\alpha}\overline{G}(x)}{1-(1-\frac{1}{\alpha})\overline{G}(x)}$$

which is the survival function of the Marshall-Olkin Power Hazard rate $MOPHR(\frac{1}{\alpha}, \beta, \theta)$ distribution.

Limiting distribution of sample extremes

To derive limiting distributions of $X_{1:n}$ and $X_{n:n}$ corresponding to the $MOPHR(\alpha, \beta, \theta)$, the following asymptotic results that are available in Arnold et al., (1992) and Kotz and Nadarajah (2001) are notable.

(i) For minimum order statistic, we have $\lim_{n\to\infty} P(X_{1:n} \le c_n^* + d_n^* t) = 1 - \exp(-t^a)$, $t > 0$, $a > 0$, $c_n^* = F^{-1}(0)$, and $d_n^* = F^{-1}(\frac{1}{n}) - F^{-1}(0)$. If and only if $F^{-1}(0)$ is finite and for all $t > 0$ and $a > 0$.

$$\lim_{\varepsilon\to 0^+} \frac{F(F^{-1}(0)+\varepsilon t)}{F(F^{-1}(0)+\varepsilon t)} = t^a$$

(ii) For the largest order statistic, $\lim_{n\to\infty} P(X_{n:n} \le c_n + d_n x) = \exp(-e^{-x})$ where, $-\infty < x < \infty$, $c_n = F^{-1}(1-\frac{1}{n})$, and $d_n = (nf(c_n))^{-1}$. If and only if

$$\lim_{x\to F^{-1}(1)} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = 0$$

Consider a random sample $X_1, X_2, \ldots, X_n$ of size n from a continuous distribution whose pdf and cdf are

$f(x)$ and $F(x)$ respectively. Then $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ and $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ are sample minima and maxima in that order.

The theorem below provides the limiting distributions of the $X_{1:n}$ and $X_{n:n}$ from the MOPHR.

Theorem 4: Consider an $MOPHR(\alpha, \beta, \theta)$ distribution whose corresponding smallest and largest order statistic are $X_{1:n}$ and $X_{n:n}$ respectively: Then

$$\lim_{n\to\infty} P(X_{1:n} \le c_n^* + d_n^* t) = 1 - \exp(-t^a),$$

Where, $t > 0$, $a > 0$, $c_n^* = 0$, and $d_n^* = \left(-\frac{\beta+1}{\theta}\ln\left(\frac{n-1}{n-2}\right)\right)^{\frac{1}{\beta+1}}$

$$\lim_{n\to\infty} P(X_{n:n} \le c_n + d_n x) = \exp(-e^{-x})$$

Where, $-\infty < x < \infty, c_n = F^{-1}(1-\frac{1}{n})$, and $d_n = (nf(c_n))^{-1}$

Proof:

(i) For the MOPHR distribution $F^{-1}(0) = 0$ is finite

Now, $\lim_{\varepsilon\to 0^+} \dfrac{F(F^{-1}(0)+\varepsilon t)}{F(F^{-1}(0)+\varepsilon)} = \lim_{\varepsilon\to 0^+} \dfrac{F(\varepsilon t)}{F(\varepsilon)}$

Finding the limit by direct substitution of $\varepsilon = 0$ results in an indeterminate case. Hence, L'Hopital's rule needs to be given due consideration. By L'Hopital's rule

$$\lim_{\varepsilon\to 0^+} \frac{F(\varepsilon t)}{F(\varepsilon)} = t \lim_{\varepsilon\to 0^+} \frac{F(\varepsilon t)}{F(\varepsilon)} = t \lim_{\varepsilon\to 0^+} \left(\frac{\alpha\theta(t\varepsilon)^{\beta} e^{-\frac{\theta}{\beta+1}(t\varepsilon)^{\beta+1}} \left(1-\bar{\alpha} e^{-\frac{\theta}{\beta+1}\varepsilon^{\beta+1}}\right)^2}{\alpha\theta\varepsilon^{\beta} e^{-\frac{\theta}{\beta+1}\varepsilon^{\beta+1}} \left(1-\bar{\alpha} e^{-\frac{\theta}{\beta+1}t\varepsilon^{\beta+1}}\right)^2} \right)$$

$$= t^{\beta+1} \lim_{\varepsilon\to 0^+} \left(\frac{e^{-\frac{\theta}{\beta+1}(t\varepsilon)^{\beta+1}} \left(1-\bar{\alpha} e^{-\frac{\theta}{\beta+1}\varepsilon^{\beta+1}}\right)^2}{e^{-\frac{\theta}{\beta+1}\varepsilon^{\beta+1}} \left(1-\bar{\alpha} e^{-\frac{\theta}{\beta+1}(t\varepsilon)^{\beta+1}}\right)^2} \right) = t^{\beta+1} \lim_{\varepsilon\to 0^+} \left(\frac{(1-\bar{\alpha})^2}{(1-\bar{\alpha})^2} \right) = t^{\beta+1}$$

Thus, the survival function $\bar{F}_1(t)$ of the sample minima from the $MOPHR(\alpha, \beta, \theta)$ is of the Weibull type. i.e.

$$\bar{F}_1(t) = 1 - e^{-t^a},$$

where $a = \beta + 1$. Also, $c_n^* = F^{-1}(0) = 0$, and $d_n^* = F^{-1}(\frac{1}{n}) = \left(-\left(\frac{\beta+1}{\theta}\right)\ln\left(\frac{n-1}{n-\alpha}\right)\right)^{\frac{1}{\beta+1}}$

This brings the proof to a logical conclusion.

(ii) In the case of the MOPHR distribution

$$\lim_{x\to\infty}\frac{d}{dx}\left(\frac{1}{h(x)}\right)=\lim_{x\to\infty}\frac{d}{dx}\left(\theta^{-1}x^{-\beta}(1-\bar{\alpha}e^{-\frac{\theta}{\beta+1}x^{\beta+1}})\right)$$

$$=\lim_{x\to\infty}\left(\theta^{-1}(-\beta\, x^{-\beta-1}(1-\bar{\alpha}e^{-\frac{\theta}{\beta+1}x^{\beta+1}})+\bar{\alpha}\theta\, x^{\beta-\beta}e^{-\frac{\theta}{\beta+1}x^{\beta+1}}\right)=0$$

This implies that $P(X_{n:n}\le c_n+d_n x)=\exp(-e^{-x})$, $-\infty<x<\infty$.

That is the asymptotic distribution of the sample maxima from the MOPHR distribution of the Gumbel type.

MOPHR AR(1) MINIFICATION PROCESS

Here, we elucidate a minification process using the proposed MOPHR distribution.

Theorem 5: Let a first-order autoregressive minification process be defined as

$$X_n=\begin{cases} v_n & \text{with probability } \alpha \\ \min(X_{n-1},v_n) & \text{with probability } 1-\alpha \end{cases} \qquad (45)$$

where, $0<\alpha<1$ and $\{v_n,n\ge 1\}$ is a sequence of *iid* random variables independent of $\{X_n\}$. Then, $\{X_n,n\ge 0\}$ is a stationary Markovian first-order autoregressive process with the $MOPHR(\alpha,\beta,\theta)$ marginal distribution if and only if $\{v_n\}$ is distributed as $PHR(\beta,\theta)$.

Proof.

From Eq. (45), we obtain

$$\bar{F}_X(x)=\alpha\bar{F}_{v_n}(x)+(1-\alpha)\bar{F}_{X_{n-1}}(x)\bar{F}_{v_n}(x) \qquad (46)$$

Based on the stationarity assumption, Eq. (46) is reduced to

$$\bar{F}_X(x)=\alpha\bar{F}_{v_n}(x)+(1-\alpha)\bar{F}_X(x)\bar{F}_{v_n}(x) \qquad (47)$$

Hence,

$$\bar{F}_X(x)=\frac{\alpha\bar{F}_{v_n}(x)}{(1-\alpha)\bar{F}_{v_n}(x)} \qquad (48)$$

Replacing, $\bar{F}_{v_n}(x)$ in Eq. (48) by this sf of the PHR distribution in Eq. (7) yields

$$\bar{F}_X(x)=\frac{\alpha\, e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{1-(1-\alpha)e^{-\frac{\theta}{\beta+1}x^{\beta+1}}} \qquad (49)$$

This indicates that $\bar{F}_X(x)$ is the sf of the MOPHR(α,β,θ) distribution.

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In contrast, let

$$\bar{F}_{X_n}(x) = \frac{\alpha e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}{1-(1-\alpha)e^{-\frac{\theta}{\beta+1}x^{\beta+1}}}$$

Then under the stationarity assumption and solving for $\bar{F}_{v_n}(x)$ in Eq. (48), we obtain

$$\bar{F}_{v_n}(x) = \frac{\bar{F}_X(x)}{\alpha+(1-\alpha)\bar{F}_X(x)} = e^{-\frac{\theta}{\beta+1}x^{\beta+1}} \quad (50)$$

Therefore, v_n has the PHR(β, θ) distribution. Remarkably, the process can be easily shown to be stationary.

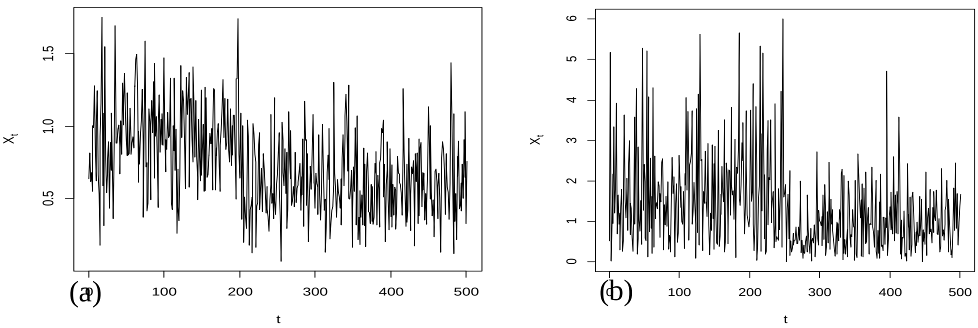

Figure 3: Sample path behaviour of the MOPHR minification process with parameter values (a) α = 0.4, β = 2, θ = 3, length = 500 and (b) α = 0.5, β = 5, θ = 5, length = 500

MAXIMUM LIKELIHOOD ESTIMATION

The maximum likelihood estimation method is used to estimate the unknown parameters for MOPHR distribution. Given a random sample $x_1, x_2, \ldots, x_n$ of size n from the MOPHR distribution with pdf in (10), the log-likelihood function $\ell = \ell(\wedge)$ of the vector of parameters $\wedge = (\alpha, \beta, \theta)^T$ is

$$\ell = \ln\left(\prod_{i=1}^{n} \frac{\alpha\theta x_i^{\beta} e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}}{\left(1-(1-\alpha)e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}\right)^2}\right) \quad (51)$$

$$\ell = n\ln\alpha + n\ln\theta + \beta\sum_{i=1}^{n}\ln x_i - \frac{\theta}{\beta+1}\sum_{i=1}^{n}x_i^{\beta+1} - 2\sum_{i=1}^{n}\ln\left(1-(1-\alpha)e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}\right)$$

Consider the score vector,

$$U(\wedge) = \begin{bmatrix} u_\alpha \\ u_\beta \\ u_\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial l}{\partial \alpha} \\ \frac{\partial l}{\partial \beta} \\ \frac{\partial l}{\partial \theta} \end{bmatrix}$$

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Then

$$u_\alpha = \frac{n}{\alpha} + 2\sum_{i=1}^{n} \frac{e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}}{1-(1-\alpha)e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}} \quad (52)$$

$$u_\beta = \sum_{i=1}^{n} x_i + \theta(1+\beta)^{-2}\sum_{i=1}^{n} x_i^{\beta+1} - \theta(\beta+1)^{-1}\sum_{i=1}^{n} x_i^{\beta+1}\ln x_i + 2(1-\alpha)(\beta+1)^{-1}\theta\sum_{i=1}^{n}\frac{(\ln x_i - (\beta+1)^{-1})x_i^{\beta+1}e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}}{1-(1-\alpha)e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}} \quad (53)$$

$$u_\theta = \frac{n}{\theta} - \theta(\beta+1)^{-1}\sum_{i=1}^{n} x_i^{\beta+1} - 2(1-\alpha)(\beta+1)^{-1}\sum_{i=1}^{n}\frac{x_i^{\beta+1}e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}}{1-(1-\alpha)e^{-\frac{\theta}{\beta+1}x_i^{\beta+1}}} \quad (54)$$

Maximum likelihood estimates $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ of α, β and θrespectively are obtained by solving the equations $u_\alpha = 0$, $u_\beta = 0$ and $u_\theta = 0$ simultaneously. It is not possible to solve the equations analytically. As a consequence, a numerical approach such as the Newton-Raphson can be employed to find the solution of the system of the equations.

Hypothesis testing about each of the three parameters can be performed based on the asymptotic distribution of $\hat{\wedge} = (\hat{\alpha}, \hat{\beta}, \hat{\theta})$. If certain regularity conditions are not met, then for large values of n,

$$\sqrt{n}\,(\hat{\wedge} - \wedge) \approx N_3(0, n^{-1}J_n) \quad (55)$$

Where

$$J_n = \begin{pmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\alpha\partial\theta} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\partial\theta} \\ \frac{\partial^2 l}{\partial\theta\partial\alpha} & \frac{\partial^2 l}{\partial\theta\partial\beta} & \frac{\partial^2 l}{\partial\theta^2} \end{pmatrix}$$

is the observed information matrix.

To estimate J_n, we replace each parameter in J_n by corresponding estimate, and The estimate of J_n is denoted by $\hat{J}_n$. Interestingly, the R package is useful in obtaining the solution.

SIMULATION STUDY AND NUMERICAL APPLICATION TO REAL DATA SET

Simulation study

The performance of the MOPHR distribution is analyzed through a simulation study using MLE. The simulation was run with N = 1000 replications for a sample size of n = 25, 50, 75, 100, 200, 500, and 1000, and with some arbitrary choice of values: (α = 2, β = 1, and θ = 3) and (α = 0.5, β = 0.5, and θ = 0.5). Random numbers were generated using Eq. (13) and absolute average bias (AAB) and root mean square error (RMSE) are determined as seen in Oseghale *et al.*, (2023) as

$$\text{Absolute Average Bias} = \left|\frac{1}{n}\sum_{i=1}^{n}(\hat{\xi} - \xi)\right| \quad (56)$$

Root Mean Square Error $= \sqrt{\frac{1}{N}\sum_{i=1}^{n}(\hat{\xi}-\xi)^2}$ (57)

where $\xi = (\alpha, \beta, \theta)$. Simulation results were obtained for different values of α, β, and θ. The values of ABias and RMSEs are shown in Table 2.

Table 2: Simulation results

		α=2, β=1, and θ=3			α=0.5, β=0.5, and θ=0.5		
	n	AE	AAB	RMSE	AE	AAB	RMSE
α	25	1.7114	0.2886	0.5713	0.4635	0.0365	0.1568
	50	1.7437	0.2563	0.4993	0.4754	0.0246	0.1410
	75	1.8091	0.1909	0.4119	0.4762	0.0238	0.1354
	100	1.8075	0.1925	0.3930	0.4841	0.0159	0.1286
	200	1.8310	0.1690	0.3425	0.4861	0.0139	0.1165
	500	1.8682	0.1318	0.2693	0.4962	0.0038	0.1021
	1000	1.8827	0.1173	0.2546	0.4911	0.0089	0.0913
β	25	1.0158	0.0158	0.1587	0.5223	0.0223	0.1275
	50	1.0279	0.0279	0.1275	0.5271	0.0271	0.1087
	75	1.0240	0.0240	0.1208	0.5247	0.0247	0.1017
	100	1.0302	0.0302	0.1127	0.5210	0.0210	0.0969
	200	1.0325	0.0325	0.0961	0.5160	0.0160	0.0830
	500	1.0361	0.0361	0.0832	0.5079	0.0079	0.0695
	1000	1.0319	0.0319	0.0731	0.5117	0.0117	0.0595
θ	25	2.7246	0.2754	0.5147	0.4728	0.0272	0.1275
	50	2.7854	0.2146	0.4173	0.4846	0.0154	0.1076
	75	2.8349	0.1651	0.3385	0.4799	0.0201	0.1038
	100	2.8566	0.1434	0.3085	0.4862	0.0138	0.9715
	200	2.8882	0.1118	0.2483	0.4906	0.0094	0.0846
	500	2.9377	0.0623	0.1655	0.4939	0.0061	0.0707
	1000	2.9450	0.0550	0.1492	0.4914	0.0086	0.0608

It is clearly shown that these values of RMSE decay to zero as the sample size n increases. Therefore, it can be concluded that MLE performs quite well in estimating the parameters of MOPHR distribution.

Numerical application to real data set

In this section, we demonstrate the applicability of MOPHR (α, β, θ) model by fitting it to two real data sets: the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003), and the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli used in Bjerkedal (1960). We compared its fit with the fits of the PHR distribution, exponential (EXP) distribution, MOE distribution, Rayleigh (R) distribution, MOR distribution, OPW distribution and MOOPW distribution.

The MLE method is used to estimate the unknown parameters of models. The performance of the model is evaluated using information criteria (Akaike information criterion (AIC) and Bayesian information criterion (BIC)) and goodness-of-fit statistics (Kolmogorov Smirnov (KS) test statistic, Cramer-Von Mises (CVM) test and Anderson-Darling (AD) test statistic). The smaller values of the negative log-likelihood, AIC and BIC indicate the adequacy of the model while the smaller values of the goodness-of-fit measures indicate better fit of the model to a given data set.

Symbolically, we have the following:

$$KS = \sup_x [G_n(x) - G_{data}(x)] \tag{58}$$

$$AD = -n - \tfrac{1}{n}\sum_{i=1}^{n}\left((2i-1)\ln[G(x_i,\hat{\xi})] + (2n+1-2i)\ln[1-G(x_i,\hat{\xi})]\right) \tag{59}$$

$$CVM = \sum_{i=1}^{n}\left[G(x_i,\hat{\xi}) - \tfrac{2i-1}{2n}\right]^2 + \frac{1}{12n} \tag{60}$$

$$AIC = -2\ell + 2m \tag{61}$$

$$BIC = -2\ell + m\ln(n) \tag{62}$$

where, $G_n(x)$ is the cumulative density function of the hypothetical distribution, $G_{data}(x)$ the empirical distribution function of the observed data, $G(x_i,\hat{\xi})$ is cdf of the specified distribution, i is the ith sample calculated when the data is sorted in ascending order, m is the number of parameters, ℓ is the maximum value of the log-likelihood, and n is the sample size.

Data set 1:

Here, the data set is the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli as reported in Bjerkedal (1960).

Table 3 presents the MLEs values of parameters, the value of negative log-likelihood ℓ, the information criteria and some goodness-of-fit measures for the different distributions. We can observe that the MOPHR distribution has the minimum negative likelihood and AIC values. The associated KS, CMV and AD statistics is also the minimum compared to other distributions. Hence, we conclude that the MOPHR distribution performs well among the eight distributions considered.

The fitted pdf, cdf, quantile-qunantile (Q-Q) and probability-probability (P-P) plots of the MOPHR distribution for the survival times data are displayed in Figure 5. The points in the Q-Q plot and the P-P plot are reasonably in a straight line. Thus, we infer that the distribution yields the best fit for the survival times data.

Table 3: MLEs and associated results for distributions fitted to the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli

Distribution	Parameter	Estimate	ℓ	AIC	BIC	K-S	CVM	AD
PHR	β	0.8253	95.7898	195.5796	200.1329	0.1048	0.16795	1.0071
	θ	0.51697						
MOPHR	α	0.07702	93.0637	192.1275	198.9575	0.0768	0.05814	0.3745
	β	1.7868						
	θ	0.06212						
EXP	θ	0.5655	113.037	228.0741	230.3508	0.2945	1.4036	7.2632
MOE	α	11.6530	96.5582	197.1163	201.6696	0.1067	0.1480	1.0321
	θ	1.5716						
R	θ	0.4783	96.3724	194.7448	197.0215	0.1090	0.2397	1.2686
MOR	α	0.5057	94.9335	193.867	198.4204	0.1166	0.1459	0.9079
	θ	0.3285						

OPW	β	0.0488	127.1014	256.2028	258.4795	0.4889	6.1641	29.8498
MOOPW	α	4.9845	96.9437	197.8875	202.4408	0.1166	0.1694	1.1391
	β	0.2087						

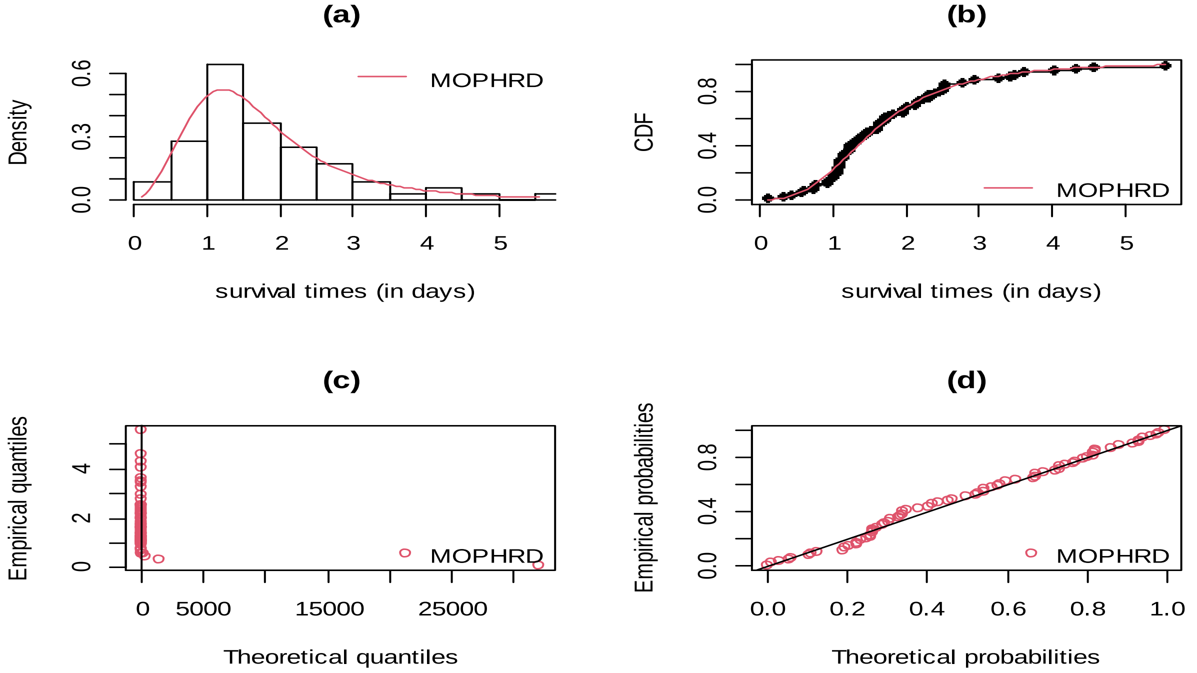

Figure 4: The fitted (a) pdf, (b) cdf, (c) Q-Q plot, and P-P plot of the MOPHR for the survival times data

Data set 2

The remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang (2003).

Table 3 presents the MLEs values of parameters, the value of negative log-likelihood ℓ, information criteria and some goodness-of-fit measures for for MOPHR, PHR, EXP, MOE, R, MOR, OPW, and MOOPW distributions. We observe that the MOPHR distribution has the lowest negative likelihood, AIC and BIC values. Also, the associated KS, CMV and AD statistics is the smallest compared to other distributions. Hence, we conclude that the MOPHR distribution competes favourably among the eight distributions considered using the data.

Table 4: MLEs and associated results for distributions fitted to the remission times (in month) of a random sample of 128 bladder cancer patients

Distribution	Parameter	Estimate	ℓ	AIC	BIC	K-S	CMV	AD
PHR	β	0.0478	414.0869	832.1738	837.8778	0.0670	0.1536	0.9578
	θ	0.0984						
MOPHR	α	0.0639	410.0921	826.1842	834.7403	0.0323	0.0204	0.1712
	β	0.6040						
	θ	0.0054						
EXP	θ	0.1068	414.3419	830.6838	833.5358	0.0846	0.1788	1.1736
MOE	α	1.0555	414.3262	832.6523	838.3564	0.0812	0.1699	1.1129
	θ	0.1099						
R	θ	9.9317	491.2656	984.5312	987.3833	0.3521	6.6134	42.5517

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MOR	α	0.0100	413.2504	830.5007	836.2048	0.0506	0.0813	
	θ	43.4554						
OPW	β	-0.6261	531.9165	1065.833	1068.685	0.6183	19.3877	97.5372
MOOPW	α	15.4551	414.1712	832.3425	838.0466	0.0675	0.1036	0.8248
	β	-0.4583						

The fitted pdf, cdf, quantile-qunantile (Q-Q) and probability-probability (P-P) plots of the MOPHR distribution for the remission times data are given in Figure 4. The points in the P-P plot are reasonably in a straight line. Thus, we infer that the distribution provides the best fit for the remission times data.

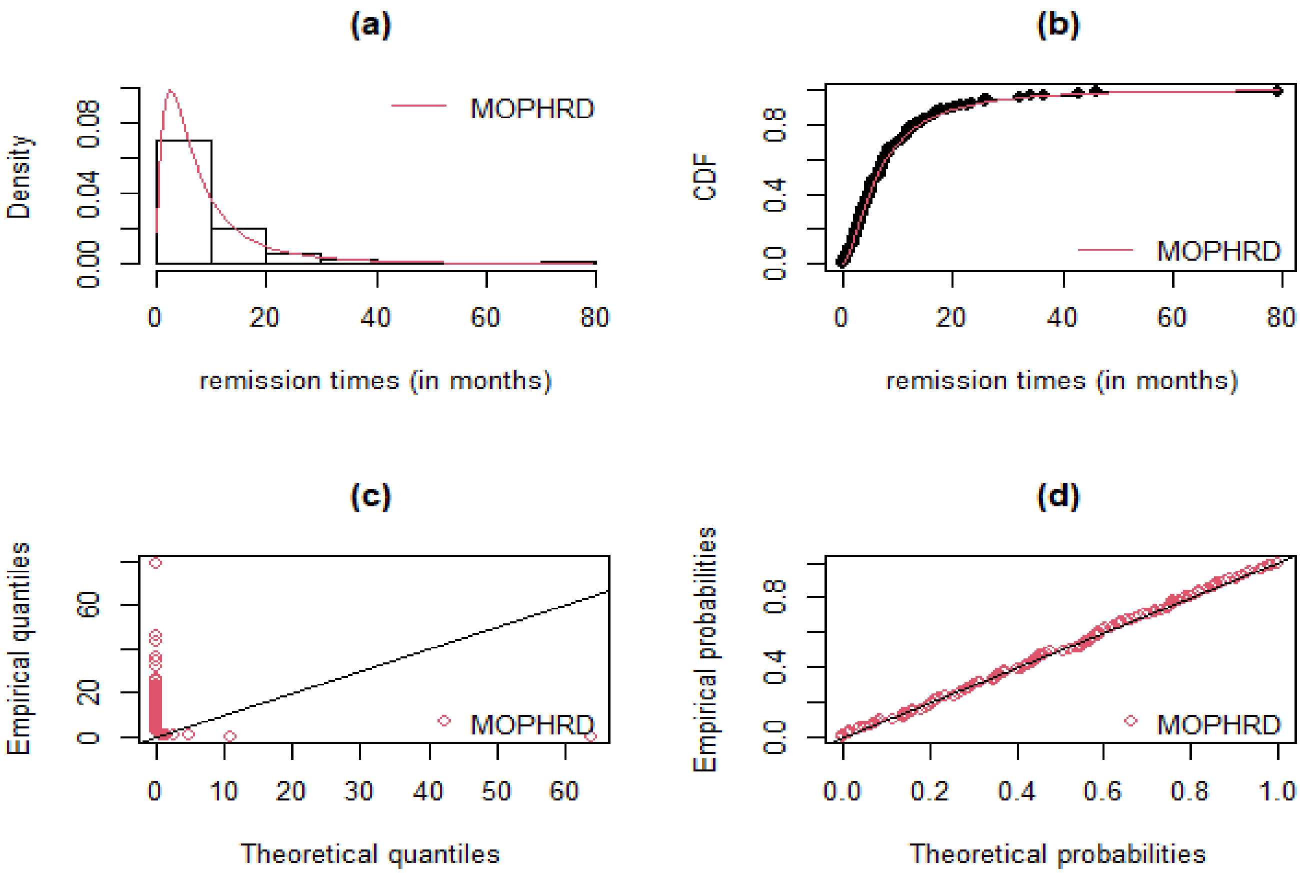

Figure 5: The fitted (a) pdf, (b) cdf, (c) Q-Q plot, and P-P plot of the MOPHR for the remission times data

CONCLUSION

A three-parameter lifetime distribution named, "Marshall-Olkin Power Hazard rate distribution" has been proposed and studied. Its statistical properties including hazard rate function, quantile function, raw moments, incomplete moments and related measures, order statistics, Renyi Entropy, stochastic ordering, stress-strength reliability, compounding and geometric extreme stability, and limiting distribution of sample extremes have been discussed. The application in time series modeling was evaluated by the autoregressive minification process with the MOPHR distribution as marginal. However, the statistical properties and modeling of the autoregressive minification process are considered for further study. The method of maximum likelihood estimation has also been discussed for estimating its parameters. Finally, numerical example using real-life data sets have been considered. The information criteria and goodness-of-fit of MOPHR distribution was determined and as such was compared with that of the PHR, exponential, MOE, Rayleigh, MOR, OPW, and MOOPW distributions. The result obtained has shown that the MOPHR distribution is flexible and performs better than other sub distributions in modeling some real life data.

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