On Commutatativity of Primitive Rings with Some Identities

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***Abstract*: - In this paper, we prove that some results on commutativity of primitive rings with some identities**

***Key Words:* Commutative ring, Non associative primitive ring, Central**

I. INTRODUCTION

In this paper, we first study some commutativity theorems of non-associative primitive rings with some identities in the center. We show that some preliminary results that we need in the subsequent discussion and prove some commutativity theorems of non-associative rings and also non-associative primitive ring with $(ab)^2 – ab \in Z(R)$ or $(ab)^2 –ba \in Z(R)$ $\forall$ a , b in R is commutative. We also prove that if R is a non-associative primitive ring with identity $(ab)^2 – b(a^2b) \in Z(R)$ for all a, b in R is commutative. Also we prove that if R is an alternative prime ring with identity $b(ab^2)a – (ba^2)b \in Z(R)$ for all a, b in R, then R is commutative. Some commutativity theorems for certain non-associative rings, which are generalization for the results of Johnsen and others and R.N. Gupta, are proved in this paper. Johensen, Outcalt and Yaqub proved that if a non-associative ring R satisfy the identity $(ab)^2 = a^2 b^2$ for all a, b in R, then R is commutative. The generalization of this result proved by R.D. Giri and others states that if R is a non-associative primitive ring satisfies the identity $(ab)^2 – a^2 b^2 \in Z(R)$, where Z(R) denoted the center, then R is commutative.

A modification of Johnsen`s identity viz., $(ab)^2 = (ba)^2$ for all a, b in R for a non -associative ring R which has no element of additive order 2, is commutative was proved by R.N. Gupta [1]. R.D. Giri and others [2] generalized Gupta`s result by taking $(ab)^2 – (ba)^2 \in Z(R)$.

II. MAIN RESULTS

Theorem 2.1 : If R is a 2-torsion free non- associative ring with unity satisfying $(ab)^2 = (ba)^2$, then R is commutative.

Proof : Let a, b $\in$ R.

Then $[a(1 + b)^2] = [(1 +b) a]^2$

i.e., $(a + ab)^2 = (a + ba)^2$

i.e., $a^2 + a(ab) + (ab) a + (ab)^2 = a^2 + a(ba) + (ba) a + (ba)^2$

i.e., $a(ab) + (ab) a = a(ba) + (ba) a.$ 2.1

substituting a by (1 + a) in 2.1., we get

$(1 + a) (b + ab) + (b + ab) (1 + a) = (1 + a) (b + ba) + (b + ba) + (b + ba) (1 + a)$

By simplifying, ,

$b + ab + ab + a (ab) + b + ba + ab + (ab) a = b + ba + ab + a(ba) + b +ba +ba + ba +(ba) a.$

Using 2.1, we get

$2(ab – ba) = 0$, i.e., ab = ba.

Hence R is commutative.

Theorem 2.2 : If R is a 2 – torsion free non-associative primitive ring with unity

such that $(ab)^2 – (ba)^2 \in Z(R)$, for all a, b in R , then R is commutative.

Proof : Given $(ab)^2 - (ba)^2 \in Z(R)$ ….2.2

Replacing b by (b+1) in 2.2, and using 2.2, we obtain

$a(ab) + (ab) a – a(ba) – (ba)a \in Z(R).$ ...2.3

Now replacing a by a + 1 in 2.3, and using 2.3.,

we achieve, $2ab – 2ba \in Z(R)$.

i.e., $2(ab - ba) \in Z(R)$.

Since R is a 2-torsion free ring, $ab – ba \in Z(R)$.

We conclude that R is commutative.

Now we present, some examples to see that the unity and 2-torsion free are essential in theorems 2.2 and 2.3

Example 2.1 : The restriction on R, being 2 - torsion free in theorem 2.1 is essential one. For if we consider the ring R of quaternion’s over the field of order 4 namely splitting field of $a^2 + a + 1$ over Z_2, then it is not of 2-torsion free but satisfies the identity of theorem 2.1. Yet it is non-commutative.

Example 2.2: Theorem 2.2 is false for rings without unity. In fact any nilpotent ring of index ≤ 4 and any nil ring of index 2 will trivially satisfy $(ab)^2 = (ba)^2$, but such rings may not be commutative. As an example let F be any field define an algebra A over F with basis { a, b, c} , where ab = c, all other products zero. A is nilpotent of index 3, A is not commutative.

It is well known that a Boolean ring satisfies $a^2 = a$, for all $a \epsilon R$ and this implies commutativity. Similarly we can see the properties of rings in which $(ab)^2 = ab$ for each pair of elements $a, b \epsilon R$. In [3] Quadri and others proved that an associative semi prime ring in which $(ab)^2 - ab \epsilon Z(R)$ is commutative. In this direction we prove that a 2 – torsion free non – associative ring with unity satisfying $(ab)^2 = ab \epsilon Z(R)$ is commutative. We give an example to show that the unity is essential in the hypothesis. Also, We prove that a non – associative primitive ring (not necessarily having unity) satisfying $(ab)^2 – ab$ (or) $(ab)^2 – ba$ is central for all $a, b \epsilon R$ is commutative.

First we prove the following theorem:

Theorem 2.3 : Let R be a 2 – torsion free non – associative ring with unity satisfying $(ab)^2 - ab \in Z(R)$ for all a, b in R. Then R is commutative.

Proof : By hypothesis $(ab)^2 - ab \in Z(R)$. ....2.4.

Replacing a by a + 1 in 2.4. and using 2.4., we get

$(ab)b + b(ab) + b^2 - b \in Z(R)$. ....2.5.

Again replacing a by a + 1 in 2.5. and using it, we obtaing $2b^2 \in Z(R)$

Since R is a 2 – torsion free, $b^2 \in Z(R)$ ....2.6.

Replacing b by ab in. 2.6.

we get $(ab)^2 \in Z(R)$ . ...2.7.

But by hypothesis $(ab)^2 - ab \in Z(R)$,

hence we get $ab \in Z(R)$. ...2.8.

Now again replacing a by a + 1 in 2.8.,

we get $ab + ba \in Z(R)$ ...2.9.

From the equations 2.8. and 2.9. we obtain $b \in Z(R)$ for all $b \in R$.

Hence R is commutative.

Theorem 2.4. : Let R be a 2 – torsion free non- associative ring with unity satisfying $(ab)^2 - ba \in Z(R)$ for all a, b in R . Then R us commutative.

Proof : Given $(ab)^2 - ba \in Z(R)$ 2.10

Replacing a by z + 1 in 2.3.10. and using 2.10., we get

$(ab)b + b(ab) + b^2 – b \in Z(R)$ ...2.11

Again replacing a by a + 1 in 2.11. and using 2.11.,

we obtain $2b^2 \in Z(R)$

Since R is a 2 torsion free, then $b^2 \in Z(R)$. ....2.12.

Now replacing b by ab in 2.12.. we get

$(ab)^2 \in Z(R)$. 2.13

But by hypothesis $(ab)^2 - ba \in Z(R)$.

Hence we have $ba \in Z(R)$ 2.14

Now again replacing a by a + 1 in 2.14 , we get

$ba + b \in Z(R)$. 2.15

Using 2.14 and 2.15 , we obtain $b \in Z(R)$ for all $b \in R$,

then R is commutative.

Theorem 2.5 : If R is a 2 – torsion free primitive ring which satisfy

$(ab)^2 - ab \in Z(R)$ for all a, b in R, then R is commutative.

Proof : By hypothesis , $(ab)^2 - ab \epsilon Z(R)$. 2.16

Replacing a by a + b in 2.16 and using 2.16,

we obtain$(ab) b^2 + b^2 (ab) + b^4 – b^2 \epsilon Z(R)$. 2.17

Now replacing a by b in $(ab)^2 - ab \epsilon Z(R)$, we get

$b^4 - b^2 \epsilon Z(R)$. 2.18

Using 2.3.17 and 2.3.18, we obtain

$(ab) b^2 + b^2 (ab) \epsilon Z(R)$. 2.19

We replacing a by a + b in 2.19, then $(ab) b^2 + b^4 + b^2 (ab) + b^4 \epsilon Z(R)$.

By . 2.12 $b^4 + b^4 \epsilon Z(R)$., i.e., $2b^4 \epsilon Z(R)$.

Since R is a 2 – torsion free ring, $b^4 \epsilon Z(R)$. 2.20

Using 2.18 and 2.20, we obtain

$b^2 \epsilon Z(R)$. 2.21

Taking b by ab in 2.21, we get $(ab)^2 \epsilon Z(R)$.

But by hypothesis $(ab)^2 - ab \epsilon Z(R)$.

Hence , $ab \epsilon Z(R)$. 2.22

Replacing b by a + b in 2.3.21, we get $a^2 + b^2 + ab + ba \epsilon Z(R)$.

Since $a^2 , b^2 \epsilon Z(R)$., we get

$ab + ba \epsilon Z(R)$. 2.23

From 2.22 and 2.23, $ba \epsilon Z(R)$. Hence $ab - ba \epsilon Z(R)$.

If R is a primitive ring, ten R has a maximal right ideal which contains no non – zero ideal of R .Consequently, we obtain $(ab - ba) R = 0$,

which further yields $ab - ba = 0$

Due to primitivity of R. Hence R is commutative.

Theorem 2.6 : Let R be a 2 – torsion free primitive ring which satisfy the identity $(ab)^2 – ba \epsilon Z(R)$. for all a, b in R . Then R is commutative.

Proof : Given (ab)2 - ba $\epsilon Z(R)$. 2.24

Replacing a by a + b in 2.24, and using 2.3.24, we obtain

$(ab)b^2 + b2\,(ab) + b4 - b2 \in Z(R)$. 2.25

Replacing a by y in 2.24, we get

$b^4 - b^2 \in Z(R)$. 2.26

Using 2.25 and 2.26, we get

$(ab)\,b^2 + b^2(ab) \in Z(R)$. 2.27

Now we replacing a by (a + b) in 2.3.27 , then $(ab)b^2 + b^4 + b^2\,(ab) + b^4 \in Z(R)$.

But by 2.27, $b^4 + b^4 \in Z(R)$., i.e., $2b^4 \in Z(R)$.

Since R is a 2 – torsion free ring, then $b^4 \in Z(R)$. 2.28

Using 2.26 and 2.28, we get $b^2 \in Z(R)$. 2.29

Now replacing b by ab in 2.29, $(ab)^2 \in Z(R)$.

By assumption, $(ab)^2 - ba \in Z(R)$. Hence, $ba \in Z(R)$. ..2.30

Replacing b by (a + b) in 2.30 ,

we get, $ab \in Z(R)$. 2.31

Hence, $ab - ba \in Z(R)$.

Now using the same argument as in the proof of theorem 2.5 we conclude that R is commutative.Now we give examples showing that unity in the statement of the theorems is essential.

Example : Let R =

$$\left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \Big/ a, b, c \in Z \right\}$$ a, b, c $\in$ Z

Clearly, R is not commutative though it satisfy the relations $(ab)^2 - ab \in Z(R)$ or $(ab)^2 - ba \in Z(R)$, for all a, b in R.

Ram Awatar [4] generalized Gupta`s [5] result and proved that if R is an associative semi prime ring in which $ab^2 a - ba^2 b$ is central, then R is commutative. In this section we show that if R is an alternative prime ring in which $(ab^2)\,a - (ba^2)\,b$ is central, then R is commutative.

Now we prove the following theorem.

Theorem 2.7 : Let R be a non – associative primitive ring with unity satisfying $(ab)^2 - b(a^2b) \in Z(R)$

for all a, b in R . Then R is a commutative.

Proof: By hypothesis $(ab)^2 - b(a^2b) \in Z(R)$ 2.32

For all a, b in R.

Replacing a by a+1 in 2.32, we get $((a+1)b)^2 - b((a+1)^2b) \in Z(R)$.

i.e., $(ab + b)^2 - b(a^2b + 2ab + b) \in Z(R)$.

Using 2.32 , we obtain $(ab)b - b(ab) \in Z(R)$ 2.33

Now replacing b by (b+1) in 2.3.33 and using 2.33 ,we get $ab - ba \in Z(R)$.

If R is a primitive ring then R has a maximal right ideal which contains no non-zero ideal of R. Consequently, we obtain $(ab - ba)\,R = 0$. This further yields $ab - ba = 0$ due to primitivity of R. Hence R is commutative.

Theorem 2.3.8: Let R be an alternative prime ring with $(ab^2)\,a - (ba^2)b \in Z(R)$ for all a, b in R. Then R is commutative.

Proof : First we shall prove that $Z(R) \neq (0)$

Let us suppose that $Z(R) = (0)$

Hence by hypothesis, $(ab^2)\,a = (ba^2)b$, ...2.34

for all a, b in R.

Replacing b by $b+b^2$ in 2.3.34. we obtain $(a(b^2 + b^4 + 2b^3))\,a = (ba^2 + b^2a^2)\,(b+b^2)$

i.e., $(ab^2)a + (ab^4)\,a + 2(ab^3)a = (ba^2)\,b + (ba^2)b^2 + (b^2a^2)b + (b^2a^2)b^2$

i.e., $2(ab^3)\,a = (b^2a^2)b + (ba^2)\,b^2$ 2.35

Since $(b^2a^2)b = (b(ba^2))b = b(ba^2)b) = b((ab^2)a = ((ba)b^2)a = (ba)(b^2a)$

and $(ba^2)b^2 = ((ba)a)b^2 = (ba)\,(ab^2)$

Hence 2.35 reduced to, $2\,(ab^3)a = (ba)(b^2\,a + ab^2)$ 2.36

If R is not 2 –torsion free, 2.36, becomes $(ba)\,(b^2a + ab^2) = 0$

With a = (a+b), this gives $(ba + b^2)\,(b^2a + b^3 + ab^2 + b^3) = 0$

i.e., $b^2(b^2a + ab^2) = 0$ 2.37

put a = ra in 2.37, then we get

$b^2(b^2(ra) + (ra)b^2) = 0$ 2.38

since $b^2(b^2r) = b^2\,(rb^2)$

From 2.37 and 2.38, we have $b^2(r(b^2a + ab^2)) = 0$.

We write this as $b^2\,r\,(b^2a + ab^2) = 0$

Since R is prime, either $b^2 = 0$ or $b^2 a + ab^2 = 0$. i.e.., $b^2 \in Z(R) = 0$.

Thus in either case $b^2 = 0$ for every b in R.

If R is 2- torsion free, we replace b by $b + b^3$ in 2.33, and get

$2(ab^4)a = (b^3\,a^2)b + (ba^2)b^3$ or

$2(b^2a^2)b^2 = b^2((ba^2)\,b) + ((ba^2)b)b^2 = b^2((ab^2)a) + ((ab^2)\,a)b^2$

We write this as $(b^2a^2)b^2 - b^2((ab^2)a) = ((ab^2)a)b^2 - (b^2a^2)b^2$ or

$(b^2a)\,(ab^2 - b^2a) = (ab^2 - b^2a)b^2$

We replacing a by a + b : Then we get

$b^3 (ab^2 – b^2a) = (ab^2 –b^2a)b^3$ 2.39

For all a, b in R

Let Ib^2 be the inner derivation by b^2 i.e., a → ab^2 - b^2a and Ib^3 be the inner derivation by b^3. Then 2.39 becomes $1b^3$ $1b^2$ (a) = 0

Thus the product of these derivation is again a derivation. we can conclude

that either b^2 or b^3 in Z(R), i.e., b^2 or b^3 is zero.

If $b^3 = 0$, then 2.35, becomes $(b^2a^2)b + (ba^2)b^2 = 0$

Substituting a +b for a, we get

$(b^2 a^2 +b^3 +2b^2 (ab))b + (ba^2 +b^3 +2b(ab))b^2 = 0$

i.e., $2(b^2a)b^2 +2(b(ab^3)b^2) = 0$

Then we get $2(b^2a)b^2 = 0$ or $(b^2a)b^2 = 0$, Then $b^2 = 0$

Thus if Z (R) = (0), then $b^2 = 0$ for every b in R.

Then 0 = (a +b) = (ab)a or a R a = 0

Then a = 0 or R = 0, a contradiction. Therefore Z (R) ≠ (0)

Taking λ ≠ 0 in Z(R) and let a = a+ λ in $(ab^2)a – (ba^2)$ b in Z(R), we get

λ $(ab^2-2(ba)b + b^2 (a))$ in Z(R).

Since R is prime, we must have

$ab^2 – 2(ba)b +b^2$ a in Z(R) 2.40

if λ a is in Z(R), then λ ab - b λ a = 0 = λ (ab –ba)

Then, R λ (ab– ba) = 0 = λ R(ab – ba) and sing λ ≠ 0, we have

ab – ba = 0, i.e., is in Z(R).

In 2.3.40., let a = ab and get

ab^2 - 2(ba)b + $(b^2a)b$ in Z(R), then b is in Z(R),

unless $ab^2 – 2(ba)b + b^2a = 0$. So if b is not in Z(R),

$ab^2 – 2(ba)b + b^2a = 0$, for everb a in R, and

b is in Z(R) , then $ab^2 – 2(ba)b + b^2a$ is still zero.

Therefore, $ab^2 +b^2a = 2(ba)b$, ...2.41

for every a, b in R

If R is 2 –torsion free, then R is commutative.

If R is not 2 –torsion free, then 2.3.41 becomes $ab^2 +b^2z = 0$ or b^2 is in Z(R) for every

y in R. Then $(a+b)^2 = a^2 + b^2+ab+ba$ is in Z(R)

i.e. , ab + ba is in Z(R)

Let a = ab and get (ab + ba)b is in Z(R)

Then b is in Z(R) , unless ab + ba = 0, which also means b is in Z(R).

Thus Z(R) = R and R is commutative

We give an eaample sowing that the unity in the statement of the theorem 2.7. is essential.

Example : Let R =

$$\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \Big/ a, b \in Z \right\}$$

a, b ∈ Z, We can easily verify the identity of theorem 2.7. i.e., $(ab)^2 – b(a^2b) \in Z(R)$. But R is not commutative.

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