An Efficient Two-Step Symmetric Hybrid Block Method for Solving Second-Order Initial Value Problems of Ordinary Differential Equations

F.O. Obarhua

Department of Mathematical Sciences, Federal University of Technology, P. M. B. 704, Akure, Nigeria

Abstract:- **A Linear Multistep Method of order six with two off-grid points is presented for direct numerical integration of second order initial value problems of ordinary differential equations. Several methods are developed using interpolation and collocation approach with special cognizance of two hybrid points which are selected to enhance the accuracy of the block methods. The properties and convergence of the proposed method are discussed. Superiority of the method over existing methods is established by implementing the method on different test problems.**

Keywords: **Symmetric, Hybrid method, Initial value Problems, Block method**

I. BACKGROUND

In this work, initial value second-order problems of the form:

$$y'' = f\left(t,\ y\ \ y'\right),\quad y(a) = \omega_0,\quad y'(a) = \omega_1, \tag{1}$$

is numerically integrated where a, b, ω_0, ω_1 are real numbers.

The mathematical models in engineering and many spheres of human endeavors often lead to initial value problem of ordinary differential equations (1)

Several numerical methods have been designed and proposed in literature for solving second order ordinary differential equations. For example, [1] developed a self-starting linear multistep method and applied it to solve second order IVPs of ODEs directly. Two intra step grid points were considered by means of collocation and interpolation approach. [2] proposed a single-step hybrid block method of order five to solve second order ODEs. In the work, three off-step points were approximated by collocation approach. In the work by [3], continuous hybrid multistep method with Legendre polynomial as the approximate solutions was investigated to obtain the approximation of stiff second order ODEs. Also, two intra step grid points were considered by means of collocation and interpolation approach. Moreso, [4] developed numerical solution of stiff and oscillatory first order differential equations, using the combination of power series and exponential function as basis function. [5] used the same basis function to produce a new numerical integration for the solution of stiff first order ODEs. Most of the methods proposed for the solution of stiff problems are numerically unstable unless the step size is taken to be extremely small and the adoption of implicit A-stable schemes is better for the solution of stiff or stiff oscillatory problems. Above all, most proposed numerical methods implemented in block modes were problem dependent. In other words, the numbers of interpolation are subject to the order of the problem.

In this work, a linear multistep hybrid formula with two-point for direct solution of IVPs of second order ODEs is proposed. The method is implemented in block mode and problem independent. It was also shown that the block method is zero stable and consistent and therefore convergent.

II. DERIVATION OF THE METHOD

In this section, the derivation of a continuous implicit two-point hybrid method for the solution of IVP (1). Consider the equally spaced points on the integration interval given by

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b \tag{2}$$

With a specified positive integer step size given by $h = x_{n+1} - x_n,\ n = 1, \cdots, N;\ N = \dfrac{b-a}{h}$.

Assuming the polynomial function

$$y(x) = \sum_{j=0}^{2(k+1)} a_j x^j \tag{3}$$

where $a_j\,'s$ are unknown parameters to be determined, $x \in [a, b]$, the solution interval, k is the number of step.

The second derivative of (3) as compared with (1) gives

$$f\left(x,\ y,\ y'\right) = \sum_{j=2}^{2(k+1)} \left(j(j-1) a_j x^{j-2} \right) \tag{4}$$

Equations (3) and (4) are approximated at points, t_{n+j}, $j=0\left(\frac{1}{2}\right)2$ and $j=0(1)2$ respectively. A system of seven equations with seven unknowns $a_j,\ j=0,1,\cdots,6$. These may be written in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{3}{2} & \frac{9}{4} & \frac{27}{8} & \frac{81}{16} & \frac{243}{32} & \frac{729}{64} \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6 & 12 & 20 & 30 \\ 0 & 0 & 2 & 12 & 48 & 160 & 480 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_{\frac{1}{2}} \\ y_1 \\ y_{\frac{3}{2}} \\ f_0 \\ f_1 \\ f_2 \end{pmatrix} \tag{5}$$

Solving (5) above using Gaussian Elimination method in Maple soft environment gives the following values of $a_j\,'s$

$$a_0 = y_0$$

$$a_1 = -\frac{203}{75}y_0 + \frac{48}{25}y_{\frac{1}{2}} + \frac{57}{25}y_1 - \frac{112}{75}y_{\frac{3}{2}} + h^2\left(-\frac{23}{200}f_0 + \frac{41}{100}f_1 + \frac{1}{200}f_2\right)$$

$$a_2 = \frac{1}{2}h^2 f_0$$

$$a_3 = \frac{64}{15}y_0 + \frac{128}{15}y_{\frac{1}{2}} - \frac{448}{15}y_1 + \frac{256}{15}y_{\frac{3}{2}} + h^2\left(-\frac{137}{180}f_0 - \frac{203}{45}f_1 - \frac{11}{180}f_2\right)$$

$$a_4 = -\frac{16}{5}y_0 - \frac{1088}{45}y_{\frac{1}{2}} + \frac{2608}{45}y_1 - \frac{1376}{45}y_{\frac{3}{2}} + h^2\left(\frac{559}{1080}f_0 + \frac{4217}{540}f_1 + \frac{127}{1080}f_2\right) \tag{6}$$

$$a_5 = \frac{16}{25}y_0 + \frac{432}{25}y_{\frac{1}{2}} - \frac{912}{25}y_1 + \frac{464}{25}y_{\frac{3}{2}} + h^2\left(-\frac{4}{25}f_0 - \frac{114}{25}f_1 - \frac{2}{25}f_2\right)$$

$$a_6 = -\frac{32}{9}y_{\frac{1}{2}} + \frac{64}{9}y_1 - \frac{32}{9}y_{\frac{3}{2}} + h^2\left(\frac{1}{54}f_0 + \frac{32}{27}f_1 + \frac{1}{54}f_2\right)$$

Equation (6) are then substituted into equation (3), and after some algebraic evaluations obtained a continuous scheme of the form

$$y_k(x) = \sum_{j=0}^{k-1}\alpha_j(x)y_j + (\tau_1(x)y_{\frac{1}{2}} + \tau_2(x)y_{\frac{3}{2}}) + h^2\sum_{j=0}^{k}\beta_j(x)f_j\,, \tag{7}$$

where $y(x)$ is the numerical solution of the initial value problem and $\alpha_j(x), \tau_1(x), \tau_2(x), \beta_j(t)$ are continuous coefficients.

$$\alpha_0 = \frac{1}{75}(37l - 160l^3 + 48l^5)$$

$$\alpha_1 = \frac{1}{225}(225 + 1073l - 4640l^3 - 4000l^4 + 1392l^5 + 1600l^6)$$

$$\tau_{\frac{1}{2}} = \frac{1}{225}(-928l + 3040l^3 + 2000l^4 - 912l^5 - 800l^6)$$

$$\tau_{\frac{3}{2}} = \frac{1}{225}(225 + 1073l - 4640l^3 - 4000l^4 + 1392l^5 + 1600l^6)$$

$$\beta_0 = \frac{1}{5400}(91l + 430l^3 - 25l^4 - 264l^5 + 100l^6)$$

$$\beta_1 = \frac{1}{2700}(1147l + 1350l^2 - 4960l^3 - 4960l^4 + 1488l^5 + 2300l^6) \tag{8}$$

$$\beta_2 = \frac{1}{5400}(17l - 110l^3 - 25l^4 + 168l^5 + 100l^6)$$

where $l = \frac{1}{h}(x - x_{n+k-1}),\ \frac{dl}{dx} = \frac{1}{h}.$

when (8) is evaluated at $x = x_2$ or $l = 1$, the result yields a symmetric scheme

$$y_2 = \frac{256}{39}y_{\frac{5}{3}} - \frac{395}{39}y_{\frac{4}{3}} + \frac{395}{39}y_{\frac{2}{3}} - \frac{256}{39}y_{\frac{1}{3}} + y_0 + \frac{h^3}{9477}(28f_2 - 1856f_1 + 28f_0). \tag{9}$$

The first and second derivatives of (8) are

$$\alpha_0' = \frac{1}{75}(37 - 480l^2 + 240l^4)$$

$$\alpha_1' = \frac{1}{225}(1073 - 13920l^2 - 16000l^3 + 6960l^4 + 9600l^5)$$

$$\tau'_{\frac{1}{2}} = \frac{1}{225}(-928+9120l^2+8000l^3-4560l^4-4800l^5)$$

$$\tau'_{\frac{3}{2}} = \frac{1}{225}(1073-13920l^2-16000l^3+6960l^4+9600l^5)$$

$$\beta'_0 = \frac{1}{5400}(91+1290l^2-100l^3-1320l^4+600l^5)$$

$$\beta'_1 = \frac{1}{2700}(1147+2700l-14880l^2-19840l^3+7440l^4+13800l^5)$$

$$\beta'_2 = \frac{1}{5400}(17-330l^2-100l^3+840l^4+600l^5) \tag{10}$$

$$\alpha''_0 = \frac{192}{15}(l+l^3)$$

$$\alpha''_1 = \frac{1}{45}(5568l-9600l^2+5568l^3+9600l^4)$$

$$\tau''_{\frac{1}{2}} = \frac{1}{45}(3648l+4800l^2-3648l^3-4800l^4)$$

$$\tau''_{\frac{3}{2}} = \frac{1}{45}(5568l-9600l^2+5568l^3+9600l^4)$$

$$\beta''_0 = \frac{1}{270}(129l-15l^2-264l^3+150l^4)$$

$$\beta''_1 = \frac{1}{135}(135-1488l-2976l^2+1488l^3+3450l^4)$$

$$\beta''_2 = \frac{1}{270}(33l-15l^2+168l^3+150l^4) \tag{11}$$

To get additional discrete schemes to form the block, the first and second derivatives (10) and (11) are evaluated at points $x = x_i,\ i = 0\left(\frac{1}{2}\right)2$ and $\left(\frac{1}{2},\frac{3}{2}\right)$ respectively give

$$y'_0 = \frac{1}{h}\left(-\frac{112}{75}y_{\frac{3}{2}}+\frac{57}{25}y_1+\frac{48}{25}y_{\frac{1}{2}}-\frac{203}{75}y_0\right)+\frac{h}{200}\left(f_2+82f_1-23f_0\right)$$

$$y'_{\frac{1}{2}} = \frac{1}{h}\left(\frac{259}{225}y_{\frac{3}{2}}-\frac{272}{225}y_1+\frac{217}{225}y_{\frac{1}{2}}-\frac{68}{75}y_0\right)+\frac{h}{21600}\left(-f_2-7214f_1+571f_0\right)$$

$$y'_1 = \frac{1}{h}\left(-\frac{256}{225}y_{\frac{3}{2}}+\frac{1073}{225}y_1-\frac{928}{225}y_{\frac{1}{2}}+\frac{37}{75}y_0\right)+\frac{h}{5400}\left(17f_2+2294f_1-91f_0\right)$$

$$y'_{\frac{3}{2}} = \frac{1}{h}\left(\frac{653}{75}y_{\frac{3}{2}}-\frac{408}{25}y_1+\frac{213}{25}y_{\frac{1}{2}}-\frac{68}{75}y_0\right)+\frac{h}{800}\left(-f_2-982f_1+23f_0\right) \tag{12}$$

$$y'_2 = \frac{1}{h}\left(\frac{6064}{225}y_{\frac{3}{2}}-\frac{12287}{225}y_1+\frac{6832}{225}y_{\frac{1}{2}}-\frac{203}{75}y_0\right)+\frac{h}{5400}\left(f_2-27386f_1+379f_0\right)$$

$$y''_{\frac{1}{2}} = \frac{1}{5h^2}\left(-4y_{\frac{3}{2}}+32y_1-52y_{\frac{1}{2}}+24y_0\right)+\frac{1}{240}\left(f_2+22f_1-23f_0\right)$$

$$y''_{\frac{3}{2}} = \frac{1}{5h^2}\left(204y_{\frac{3}{2}}-432y_1+252y_{\frac{1}{2}}-24y_0\right)+\frac{h}{80}\left(f_2-654f_1+11f_0\right)$$

III. IMPLEMENTATION OF THE BLOCK METHOD

The general block formula with modification for the implementation of the new block method in the normalized form is as given below

$$A^0Y_m = A^iy_m + B_iF_m. \tag{13}$$

Solving equations (9) and (12) using matrix inversion to obtain the coefficients of (13) of the new block method as

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}\begin{pmatrix} y_{\frac{1}{2}}\\ y_1\\ y_{\frac{3}{2}}\\ y_2\\ y'_{\frac{1}{2}}\\ y'_1\\ y'_{\frac{3}{2}}\\ y'_2 \end{pmatrix} = \begin{pmatrix} 0&0&0&1&0&0&0&\frac{1}{2}h\\ 0&0&0&1&0&0&0&h\\ 0&0&0&1&0&0&0&\frac{3}{2}h\\ 0&0&0&1&0&0&0&2h\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{pmatrix}\begin{pmatrix} y_{-\frac{1}{2}}\\ y_{-1}\\ y_{-\frac{3}{2}}\\ y_0\\ y'_{-\frac{1}{2}}\\ y'_{-1}\\ y'_{-\frac{3}{2}}\\ y'_0 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{367h^2}{5760} & \frac{3h^2}{32} & -\frac{47h^2}{960} & \frac{29h^2}{1440} & -\frac{7h^2}{1920}\\ \frac{53h^2}{360} & \frac{2h^2}{5} & -\frac{h^2}{12} & \frac{2h^2}{45} & -\frac{h^2}{120}\\ \frac{147h^2}{640} & \frac{117h^2}{160} & \frac{27h^2}{320} & \frac{3h^2}{32} & -\frac{9h^2}{640}\\ \frac{14h^2}{45} & \frac{16h^2}{15} & \frac{4h^2}{15} & \frac{16h^2}{45} & 0\\ \frac{251h}{1440} & \frac{323h}{720} & -\frac{11h}{60} & \frac{53h}{720} & -\frac{19h}{1440}\\ \frac{29h}{180} & \frac{31h}{45} & \frac{2h}{15} & \frac{h}{45} & -\frac{h}{180}\\ \frac{27h}{160} & \frac{51h}{80} & \frac{9h}{20} & \frac{21h}{80} & -\frac{3h}{160}\\ \frac{7h}{45} & \frac{32h}{45} & \frac{4h}{15} & \frac{32h}{45} & \frac{7h}{45} \end{pmatrix}\begin{pmatrix} f_0\\ f_{\frac{1}{2}}\\ f_1\\ f_{\frac{3}{2}}\\ f_2 \end{pmatrix} \tag{14}$$

Writing equation (14) explicitly gives:

$$y_{\frac{1}{2}} = y_n + \frac{1}{2}hy'_n + \frac{h^2}{5760}(367 f_n + 540 f_{\frac{1}{2}} - 282 f_1 + 116 f_{\frac{3}{2}} - 21 f_2)$$

$$y_1 = y_n + hy'_n + \frac{h^2}{360}(53 f_n + 144 f_{\frac{1}{2}} - 30 f_1 + 16 f_{\frac{3}{2}} - 3 f_2)$$

$$y_{\frac{3}{2}} = y_n + \frac{3}{2}hy'_n + \frac{h^2}{640}(147 f_n + 468 f_{\frac{1}{2}} + 54 f_1 + 60 f_{\frac{3}{2}} - 9 f_2)$$

$$y_2 = y_n + 2hy'_n + \frac{h^2}{45}(314 f_n + 48 f_{\frac{1}{2}} + 12 f_1 + 16 f_{\frac{3}{2}})$$

$$y'_{\frac{1}{2}} = hy'_n + \frac{h}{1440}(251 f_n + 646 f_{\frac{1}{2}} - 264 f_1 + 106 f_{\frac{3}{2}} - 19 f_2)$$

$$y'_1 = hy'_n + \frac{h}{180}(29 f_n + 124 f_{\frac{1}{2}} + 24 f_1 + 4 f_{\frac{3}{2}} - f_2)$$

$$y'_{\frac{3}{2}} = hy'_n + \frac{h}{160}(27 f_n + 102 f_{\frac{1}{2}} + 72 f_1 + 42 f_{\frac{3}{2}} - 3 f_2)$$

(15)

$$y'_2 = hy'_n + \frac{h}{45}(7 f_n + 32 f_{\frac{1}{2}} + 12 f_1 + 32 f_{\frac{3}{2}} + 7 f_2)$$

IV. ANALYSIS OF THE PROPERTIES OF THE METHOD

4.1 Order and error constant of the block method

Following the multistep collocation method (7), the associated linear difference operator ℓ defined by

$$\ell[y(x);h] = \sum_{j=0}^{r}\phi_j(x)y(x+jh) + (\zeta(x)y(x+uh) + \psi(x)y(x+vh)) + h^2\sum_{j=0}^{s}\gamma_j(x)y''(x+jh) \quad (16)$$

where $y(x)$ is an arbitrary function, continuously differentiable on $[a,b]$. Following [6], equation (16) can be written as Taylor series expansion about point x to obtain

$$\ell[y(x);h] = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^p(x) + \cdots, \quad (17)$$

where the constant coefficients. Expanding the block method (14) by Taylor series and combining coefficients of like terms in h^n give the order of the proposed block methods as $[5, 5, 5, 5, 5, 5, 5, 5]^T$ and error constants as

$$\begin{bmatrix} 8.2930\times10^{-5}, 1.9841\times10^{-4}, 3.1390\times10^{-4}, 3.9683\times10^{-4}, 2.9297\times10^{-4}, 1.7371\times10^{-4}, 2.9297\times10^{-4}, \\ -6.6138\times10^{-5} \end{bmatrix}^T$$

4.2 Zero stability of the block method

The new block method (14) is said to be zero stable if the first characteristic polynomial $\rho(\lambda) = \left|\sum_{i=0}^{k} a^{(i)}\lambda^{k-1}\right| = 0$ and satisfies $|\lambda_j| \le 1,\ j = 1, \cdots, k$ and for those roots with $|\lambda_j| = 1,$ the multiplicity does not exceed two, [6]. The characteristics function of the new block method is as follows:

$$\rho(\lambda) = \left| \lambda \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&0&0&1&0&0&0&\frac{1}{2}h \\ 0&0&0&1&0&0&0&h \\ 0&0&0&1&0&0&0&\frac{3}{2}h \\ 0&0&0&1&0&0&0&2h \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1 \end{pmatrix} \right| = \lambda^7(\lambda - 1) = 0$$

and the solution is given as $\lambda = 0,0,0,0,0,0,0,1$. The roots of the characteristic polynomial are $\lambda_i = 0,$ for $i = 1, \ldots, 7$ and $\lambda_8 = 1$. Therefore, the method is zero-stable, since the roots of the characteristic polynomial are all zero except one, whose absolute value is one (see [10]) .

4.3 Consistency

The hybrid block method is said to be consistent if the order of the individual method that make up the block is greater than or equal to one. It is therefore clear from subsection 4.1 that the new hybrid block method is consistent.

4.4 Convergence

The necessary and sufficient conditions for the hybrid block method (13) to be convergent are that it must be consistent and zero-stable (see [2]). Therefore, since the new derived hybrid block method is consistent and zero-stable, then the method is convergent.

4.5 Region of Absolute Stability

In this section, the regions of absolute stability of the new methods are determined in order to guide the choice of the step size for the method. In doing this, let the test problem for the methods be given as

$$y'' = -\lambda^2 f \text{ where } f = f(x, y, y') \text{ and } \lambda \text{ is complex.} \quad (18)$$

The stability polynomial of the derived continuous block method (7) given by

$$\pi(r, \bar{h}) = \rho(r) - \bar{h}\sigma(r) = 0 \qquad (19)$$

where $\rho(r)$ and $\sigma(r)$ are the first and second characteristic polynomials respectively, $\bar{h} = -\lambda^2 h^2$ and $\lambda = \dfrac{d^2 f}{dy^2}$.

Using the test problem in (18) for the block mode (13) the method gives

$$\bar{h}(r) = -\left(\frac{A^0 Y_m(r) - A^i y_m(r)}{B_i F_m(r)} \right) \qquad (20)$$

since $\bar{h}$ is given as $\bar{h} = h^2 \lambda^2$ and $r = e^{i\theta}$, James *et al.* (2013).

Equation (20) is adopted to determine the region of absolute stability for the new hybrid block method.

Adopting the method of [1], the method was reformulated as

$$\begin{bmatrix} Y \\ \cdot \\ \cdot \\ \cdot \\ Y_{i+1} \end{bmatrix} = \begin{bmatrix} A & & U \\ \cdot & & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & & \cdot \\ B & & V \end{bmatrix} \begin{bmatrix} h^2 f(y) \\ \cdot \\ \cdot \\ \cdot \\ f_{i-1} \end{bmatrix} \qquad (21)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{367}{5760} & \frac{3}{32} & -\frac{47}{960} & \frac{29}{1440} & -\frac{7}{1920} \\ \frac{53}{360} & \frac{2}{5} & -\frac{1}{12} & \frac{2}{45} & -\frac{1}{120} \\ \frac{147}{640} & \frac{117}{160} & \frac{27}{320} & \frac{3}{32} & -\frac{9}{640} \\ \frac{14}{45} & \frac{16}{15} & \frac{4}{15} & \frac{16}{45} & 0 \end{bmatrix}, \quad f(y) = \begin{bmatrix} f_n \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}, \quad Y = \begin{bmatrix} y_n \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{367}{5760} & \frac{3}{32} & -\frac{47}{960} & \frac{29}{1440} & -\frac{7}{1920} \\ \frac{147}{640} & \frac{117}{160} & \frac{27}{320} & \frac{3}{32} & -\frac{9}{640} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad V \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

M and I are identity matrix of dimension 7 and 3 respectively.

Substituting these entries A, B, U, V, M and I into the stability matrix given as

$$M(z) = V + zB(M - zA)^{-1} U \qquad (22)$$

and then into the stability function given as

$$\rho(\eta, z) = \det(\eta I - M(z)) \qquad (23)$$

to give stability polynomial using Maple software. This is plotted in MATLAB (R2013a) environment to produce the required region of absolute stability of the method to generate the required region of absolute stability of the new hybrid block method shown in figure 1.

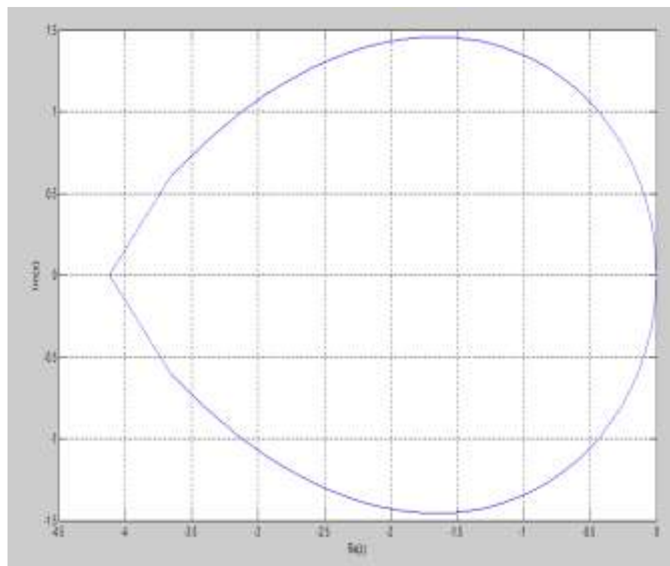

Figure 1: Region of Absolute Stability of 2-step 4-point hybrid method. The Figure 1 shows the region cover the complex plane $z \in C^n$ hence the method is A-Stable.

V. NUMERICAL EXPERIMENTS

Some numerical examples are presented to show the accuracy of the developed Hybrid Block Method (HBM). In the examples considered the absolute errors were obtained as $Err = |y_i - y(x_i)|$, where y_i is the approximate solution obtained using the new method BHM and $y(x_i)$ is the exact solution of the problem considered at the grid points.

Problem 1. Cooling of a body

The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3y'' + y' = 0$. By using the substitution $z = y'$, or otherwise, find y in terms of t given that $y = 60$ when $t = 0$ and $y = 35$ when $t = 6\ln 4$. Find after how many minutes the rate of cooling of the body will have fallen bellow one degree per minute, giving your answer correct to the nearest minute.

Problem formulation 1

$$y'' = -\frac{y'}{3},\ y(0) = 60,\ y'(0) = -\frac{80}{9},\ \ h = 0.1$$

Theoretical solution is

$$y(t) = \frac{80}{3} e^{-(\frac{1}{3})t} + \frac{100}{3}$$

Table 1: Shows the numerical solution of our method compared with the method of [8].

t	y_{exact}	$y_{computed}$	Error in New Scheme, $p=5, k=2.$	Error in [8] $p=5, k=3.$
0.10	59.125762679520165000	59.125762679520399000	2.344791e-13	3.55e-11
0.20	58.280186267509812000	58.280186267510032000	2.202682e-13	4.58e-11
0.30	57.462331147625591000	57.462331147625719000	3.935749e-12	7.00e-11
0.40	56.671288507811937000	56.671288507811006000	2.704951e-12	6.50e-12
0.50	55.906179330416379000	55.906179330416211000	7.599112e-12	3.33e-11
0.60	55.166153415412850000	55.166153415412338000	1.569518e-12	4.20e-11
0.70	54.450388435647511000	54.450388435644114000	2.756872e-12	4.38e-11
0.80	53.758089023057302000	53.758089023093056000	4.375392e-11	1.07e-10
0.90	53.088485884845809000	53.088485884802636000	6.474571e-11	6.58e-11
1.00	52.440834948634382000	52.440834948611451000	9.100178e-11	1.69e-10

Problem 2

$$y'' = \frac{(y')^2}{2y} - 2y,\ \ y\left(\frac{\pi}{6}\right) = \frac{1}{4},\ y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2},\ h = 0.01$$

Theoretical solution is

$$y(t) = \sin^2 t.$$

Table 2: Shows the numerical solution of our method compared with the method of [9].

t	y_{exact}	$y_{computed}$	Error in New Scheme, $p=5, k=2.$	Error in [9] $p=7, k=3.$
0.5440	0.267515862977780850	0.267515862981737800	3.956946e-12	4.0400e-10
0.5540	0.276415041478145830	0.276415041451067880	2.702922e-11	1.1000e-09
0.5640	0.285403650980826370	0.285403650932723310	5.491897e-11	2.0200e-09
0.5740	0.294478096161868210	0.294478096155077210	1.088321e-11	3.1700e-09
0.5840	0.303634747364189770	0.303634747303433520	4.643924e-11	4.5500e-09
0.5940	0.312869942049397220	0.312869942850434640	2.450104e-10	6.1500e-09
0.6040	0.322179986262750740	0.322179986054477070	3.279173e-10	7.9700e-09
0.6140	0.331561156110697200	0.331561156599896380	4.348920e-10	9.9900e-09
0.6240	0.341009699250378160	0.341009699720449840	5.447007e-10	1.2200e-09

Problem 3: Stiff second order

$$y'' = -\lambda^2 y,\ \lambda = 2,\ y(0) = 1,\ y'(0) = 2,\ h = 0.01$$

Theoretical solution is

$$y(t) = \cos 2t + \sin 2t$$

Table 3: Shows the numerical solution of our method compared with the method of [8].

t	y_{exact}	$y_{computed}$	Error in New Scheme, $p=5, k=2.$	Error in [8] $p=5, k=3.$
0.010	1.019798673359910900	1.019798673359911100	2.220446e-16	3.409e-11
0.020	1.039189440847612100	1.039189440847612600	4.440892e-16	3.239e-11
0.030	1.058164546414648700	1.058164546414646600	2.014302e-15	3.465e-11
0.040	1.076716400271792200	1.076716400271791000	1.419108e-15	2.400e-13
0.050	1.094837581924853900	1.094837581924851400	2.006848e-15	1.780e-12
0.060	1.112520843142785500	1.112520843142773000	1.606479e-14	7.467e-11
0.070	1.129759110856873600	1.129759110856839000	4.544461e-14	3.904e-11
0.080	1.146545489989872800	1.146545489989899500	2.503516e-14	4.132e-11
0.090	1.162873266213945600	1.162873266213345900	6.782759e-13	1.197e-10
1.000	1.178735908636302700	1.178735908636520200	5.427678e-13	8.342e-11

Problem 4: Nonlinear second order

$$y'' = x(y')^2, \ \ y(0)=1, \ \ y'(0)=\frac{1}{2}, \ \ h=\frac{1}{30}$$

Theoretical solution is

$$y(t) = 1+\frac{1}{2}\ln\left(\frac{2+t}{2-t}\right)$$

Table 4: Shows the numerical solution of our method compared with the method of [2].

t	y_{exact}	$y_{computed}$	Error in New Scheme, $p=5, k=2.$	Error in [2] $p=5, k=1.$
0.10	1.050041729278491400	1.050041729278497300	6.594725e-15	1.310063e-14
0.20	1.100335347731075800	1.100335347731074500	1.229683e-15	3.974598e-14
0.30	1.151140435936466800	1.151140435936466600	7.095879e-15	1.021405e-14
0.40	1.202732554054081900	1.202732554054084600	3.544276e-15	3.304024e-13
1.00	1.549306144334053000	1.549306144334071300	2.290128e-14	1.292744e-12

Problem 5

$$y'' = y', \ \ y(0)=0, \ \ y'(0)=-1, h=0.1$$

Theoretical solution is

$$y(t) = 1-e^{t}$$

Table 5: Shows the numerical solution of our method compared with the method of [8].

t	y_{exact}	$y_{computed}$	Error in New Scheme, $p=5, k=2.$	Error in [8] $p=5, k=3.$
0.10	-0.105170918075647710	-0.105170918070019560	5.628151e-12	7.56500e-11
0.20	-0.221402758160169850	-0.221402758133080300	2.708955e-11	1.60170e-10
0.30	-0.349858807576003180	-0.349858807508350580	6.765261e-11	1.76000e-10
0.40	-0.491824697641270350	-0.491824697510172710	1.310976e-10	6.07843e-10
0.50	-0.648721270700128190	-0.648721270478339050	2.217891e-10	1.47289e-09
0.60	-0.822118800390509330	-0.822118800045752550	3.447568e-10	2.53363e-09
0.70	-1.013752707470477100	-1.013752706964691400	3.447568e-10	4.78762e-09
0.80	-1.225540928492468300	-1.225540927780946100	7.115222e-10	7.27701e-09
0.90	-1.459603111156950700	-1.459603110187362100	9.695886e-10	1.01696e-08
1.00	-1.718281828459046400	-1.718281827170330000	1.288716e-09	1.48265e-08

Problem 6: (Two body Problem)

$$y_1'' = -y_2' + \cos t,\ \ y_1(0) = -1,\ \ y_1'(0) = -1,$$
$$y_2'' = y + \sin t,\ \ y_2(0) = 1,\ \ y_2'(0) = 0$$

Theoretical solution is $y_1(t) = -\cos t - \sin t;\ \ y_2(t) = \cos t$

The maximum errors $\left|y_{exact} - y_{computed}\right|$ obtained with the method for Problem 6, the execution time in microseconds t_e and the total steps taken are compared with that of [9] two step two point method.

Table 6: Shows the numerical solution of our method compared with the method of [9].

TOL	Results of [9]				Results in New Scheme			
	MTD	**TS**	**MAXE**	**TIME**	**NMTD**	**TS**	**MAXE**	**TIME**
10^{-2}	2PHM	33	2.768463E-10	119	2PHBM	33	2.1345E-12	85
10^{-4}	2PHM	55	1.275646E-13	213	2PHBM	55	3.7642E-15	102
10^{-6}	2PHM	74	3.519407E-14	262	2PHBM	74	1.5218E-16	222
10^{-8}	2PHM	130	7.510659E-13	447	2PHBM	130	4.7135E-16	350
10^{-10}	2PHM	278	3.088640E-13	922	2PHBM	278	6.5937E-17	689

VI. CONCLUSION

This paper has produced a new hybrid block method for direct solution of general second order initial value problems. The method is developed in such a way that the hybrid points are at $y - function$ which enhanced the reduction of function evaluation. The derived method was implemented in block mode with the merits of being self-starting and uniform order of accuracy. It should be noted that this particular block method is problem independent and as such there is freedom of choice of numbers of interpolation points. Furthermore, the new derived method is preferable for solving most of the problems used in this paper. Finally, the region of absolute stability of the block method is presented in Figure 1. All codes are designed using Maple and Matlab software package to generate the schemes and results.

VII. ABBREVIATIONS

TOL – Tolerance, MTD – Method Employed, TS – Total Steps taken, MAXE – Magnitude error of the computed solution, TIME – The execution time taken in microseconds, NMTD – New method employed, 2PHBM – 2-Point Hybrid Block Method

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