On The Cycle Indices of Cyclic and Dihedral Groups Acting on X^2

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Abstract: **An effective method of deriving the cycle indices of cyclic and dihedral groups acting on X^2, where $X = \{1, 2, \dots, n\}$ is provided. This paper extents some results of $Harary\ and\ Palmer$(1973); $Krishnamurthy$ (1985) and $thoka\ et.al.$(2015).**

I. INTRODUCTION

The concept of cycle index was first done by Howard Redfield in 1927, however, his paper was overlooked but came to attention of Mathematicians long after his death in 1944. The concept was later rediscovered by Pölya in an independent study and applied his results to solve interesting combinatorial problems in chemistr as outlined by Donald Woods(1979).

The cycle index of Cyclic and Dihedral groups acting on X can be found in various books (Krishnamurthy, 1985; Harary and Palmer, 1973) respectively.

The cycle index of cyclic and dihedral groups acting on ordered pairs was computed by Muthoka et. al. in 2015.

II. DEFINITIONS AND PRELIMINARY RESULTS

This section outlines some definitions and established results that will be used throughout this paper.

Definition 1. A dihedral group is the group of symmetries of a regular n-gon. It has degree n and order $2n$.

Definition 2. A cyclic group is a group of order $n$that can be generated by a single element.

Definition 3. Suppose $g \in G$ has $\beta_1$1-cycles, $\beta_2$2-cycles… β_nn-cycles, we say that β and hence $g \in G$ has cycle type $(\beta_1, \beta_2, \dots, \beta_n)$

Definition 4. Let G be a finite group acting on a set X with $|X| = n$, and suppose $\sigma \in G$ has cycle type$(\beta_1, \beta_2, \dots, \beta_n)$, we shall define the monomial of σ to be $mon(\sigma) = s_1^{\beta_1}.s_2^{\beta_2} \dots .s_n^{\beta_n} = \prod_{j=1}^{n} s_j^{\beta_j}$, where $s_1, s_2, \dots s_n$ are distinct commuting indeterminates.

Definition 5. Let G be a finite group acting on a finite set say X, then we define the cycle index of the action of G on X as the polynomial $Z(G, X)$(say over a rational field) in indeterminates $s_1, s_2, \dots s_n$ by $Z(G, X) = \frac{1}{|G|}\sum_{g \in G} mon(g)$.

Theorem 1. (Krishnamurthy, 1985) The cycle index of Cyclic group(C_n) acting on X is given as;

$$Z(C_n, X) = \frac{1}{n}\sum_{t/n} \emptyset(t) s_t^{\frac{n}{t}}.$$

Theorem 2. ($Harary\ and\ Palmer$, 1973) The cycle index of D_n acting on X is given as follows;

$$Z(D_n, X) = \frac{1}{2n}\left[\sum_{t/n} \emptyset(t) s_t^{\frac{n}{t}} + \frac{n}{2} s_1^2 s_2^{\frac{n-2}{2}} + \frac{n}{2} s_2^{\frac{n}{2}}\right],$$

when n is even, and

$$Z(D_n, X) = \frac{1}{2n}\left[\sum_{t/n} \emptyset(t) s_t^{\frac{n}{t}} + n s_1 s_2^{\frac{n-1}{2}}\right],$$ when n is odd.

We shall next highlight some results obtained by ($Harary$, 1955) when he was deriving the cycle index of symmetric group acting on the ordered pairs. These results were employed by Muthoka et. al when they were computing the cycle index of cyclic and dihedral group acting on the ordrerd2 element subsets.

(a)Pairs of points coming from a common cycle of σ. In this case, we have the following contributions;

$f_k^{\beta_k} \rightarrow s_k^{(k-1)\beta_k}$, for β_k cycles of length$k$.

(b) Pairs of points coming from distinct cycles of σ. If there are β_v and $\beta_w$cycles of lengths v and w respectively, then we have the following contributions;

(i) If $v \neq w$, then we have the following;

$$f_v^{\beta_v} f_w^{\beta_w} \rightarrow s_{lcm\ of\ (v,w)}^{2\beta_v\beta_w.\text{gcd}\ of\ (v,w)}$$

(ii) If $v = w = k$, then we have the following;

$$f_k^{\beta_k} \rightarrow s_k^{2k\binom{\beta_k}{2}}$$

The cycle index of C_n acting on X^2

We shall need the following lemma in this paper

***Lemma* 1.** Let $[a, a, \dots, a] \in X^r$ and $\sigma \in G$ be a permutation of the group G of length k, Then σ' permutes one cycle of length 1.

***Proof*.** The proof of this lemma follows from the fact that $\frac{k!}{k!} = 1$, since the 2-element subset under consideration contains the same repeated elements.Thus we have the following contribution; $f_k^{\beta_k} \to s_k^{\beta_k}$, for any β_k cycles of length k. Thus for any cycle of the type $[a, a, \ldots, a] \in X^r$ fixes itself when acting on a set, say X. ⊠

Now let C_n act on X so that C_n^2 acts on X^2 by the rule that if $\sigma' \in C_n^2$ is the permutation induced by $\sigma \in C_n$, then for each $[a_1, a_2] \in X^2$ we have $\sigma(a_1, a_2) = (\sigma a_1, \sigma a_2)$. If $mon(\sigma) = \prod_k f_k^{\beta_k}$ in C_n, we need to find the corresponding $mon(\sigma')$ which we shall denote by $\left(\prod_k s_k^{\beta_k}\right)^2$ in C_n^2. To do this we consider the following contributions from σ to the induced permutation σ'. We shall consider the following cases:

Case 1. Contributions from pairs of the form $[a, a]$. In this case, using result $Lemma$ 1 above , we have the following contribution.

$$f_t^{\frac{n}{t}} \to s_t^{\frac{n}{t}} \qquad (1)$$

Case 2. Contributions from pairs of the form $[a, b]$ where $a \neq b$ and

(i) a and b come from common cycles of length t. In this case, from $(b)(i)$, the results of $Harary$ (1955), we have the following overall contribution;

$$f_t^{\frac{n}{t}} \to s_t^{(t-1)\frac{n}{t}} = s_t^{n-\frac{n}{t}} \qquad (2)$$

$(\boldsymbol{ii})$ a and b come from distinct cycles of length t. In this case, from result $(b)(ii)$we have the following contribution;

$$f_t^{\frac{n}{t}} \to s_t^{2t\binom{\frac{n}{t}}{2}} = s_t^{\frac{n^2}{t}-n} \qquad (3)$$

Collecting and multiplying the the contributions on the right hand side of (1), (2) and (3), we have the following overall contribution;

$$f_t^{\frac{n}{t}} \to s_t^{\frac{n^2}{t}} \qquad (4)$$

Therefore, from$Theorem$ 1 and (4)we have that;

$$Z(C_n, X^2) = \frac{1}{n}\sum_{t/n} \emptyset(t) s_t^{\frac{n^2}{t}}. \qquad (5)$$

$Example$ 1

Let $n = 8$, then $t = 1,2,4,8$ and $X = \{1,2,3,4,5,6,7,8\}$

And so $\emptyset(1) = 1, \emptyset(2) = 1, \emptyset(4) = 2, \emptyset(8) = 4$

Therefore $Z(C_n, X^2) = \frac{1}{8}[s_1^{64} + s_2^{32} + 2s_4^{16} + 4s_8^{8}]$

$Example$ 2

Let $n = 17$, then $t = 1$ and 17, $X = \{1,2,3, \ldots, 17\}$

And so $\emptyset(1) = 1$ *and* $\emptyset(17) = 16$

Therefore $Z(C_n, X^2) = \frac{1}{17}[s_1^{289} + 16s_{17}^{17}]$

The cycle index of D_n acting on X^2

The dihedral group has two parts, the cyclic part and the reflections. We have already considered the cyclic part in the previous section, therefore in this section we need to consider the reflections. We shall consider for cases when n is even and when n is odd as indicated in $theorem$ 2.

Now let D_n act on X so that D_n^2 acts on X^2 by the rule that if $\sigma' \in D_n^2$ is the permutation induced by $\sigma \in D_n$, then for each $[a_1, a_2] \in X^2$ we have $\sigma(a_1, a_2) = (\sigma a_1, \sigma a_2)$. If $mon(\sigma) = \prod_k f_k^{\beta_k}$ in D_n, we need to find the corresponding $mon(\sigma')$ which we shall denote by $\left(\prod_k s_k^{\beta_k}\right)^2$ in D_n^2. To do this we consider the following contributions from σ to the induced permutation σ'. We shall consider the following cases:

$(\boldsymbol{A})$ When n is even.

Here we shall consider the reflections with cycle types $f_1^2 f_2^{\frac{n-2}{2}}$ and $f_2^{\frac{n}{2}}$. We start with $f_1^2 f_2^{\frac{n-2}{2}}$

***Case* 1**Contributions by elements of the form $[a, a]$. From $Lemma$ 1 we have the following;

$$f_1^2 f_2^{\frac{n-2}{2}} \to s_1^2 s_2^{\frac{n-2}{2}} \qquad (6)$$

***Case* 2**Contribution by elements of the form $[a, b]$, where $a \neq b$ and

(i) a and b come from distinct cycles of distinct lengths. From result $b(i)$, we have the following contribution;

$$f_1^2 f_2^{\frac{n-2}{2}} \to s_2^{\frac{4(n-2)}{2}} = s_2^{2n-4} \qquad (7)$$

(ii) each of a and $b$come from distinct cycles of length one. From result $b(ii)$, we have that;

$$f_1^2 \to s_1^2 \qquad (8)$$

(iii)each of the points a and b come from distinct cycles of length two. From result $b\,(ii)$, we have the following contribution;

$$f_2^{\frac{n-2}{2}} \to s_2^{4\binom{\frac{n-2}{2}}{2}} = s_2^{\frac{n^2}{2}-3n+4} \qquad (9)$$

(iv) both of a and b come from common cycle of length two.. From result (a), we have that;

$$f_2^{\frac{n-2}{2}} \to s_2^{\frac{n-2}{2}} \qquad (10)$$

Collecting and multiplying the coefficients on the right hand side of (6), (7), (8), (9)and(10), We have the following induced monomial;

$$mon(\sigma') = s_1^4 s_2^{\frac{n^2-4}{2}} \quad (\mathbf{11})$$

Next we shall consider the reflection with the monomial $f_2^{\frac{n}{2}}$. In this case, we have the following cases.

***Case* 1.** Contributions by elements of the form $[a,a]$. From *Lemma* 1, we have the following contribution;

$$f_2^{\frac{n}{2}} \rightarrow s_2^{\frac{n}{2}} \quad (\mathbf{12})$$

***Case* 2.**Contributions by pairs of the form $[a,b]$ with $a \neq b$ and

$(i)a$ and b come from common cycles of length two. From result a we have the following;

$$f_2^{\frac{n}{2}} \rightarrow s_2^{\frac{n}{2}} \quad (\mathbf{13})$$

$(ii)a$ and b come from distinct cycles of length two. Using result $b(ii)$ we have that;

$$f_2^{\frac{n}{2}} \rightarrow s_2^{4\binom{\frac{n}{2}}{2}} = s_2^{\frac{n^2}{2}-n} \quad (\mathbf{14})$$

Collecting and multiplying the contributions on the right hand side of $(12),(13)$and (14), we have the following induced monomial;

$$mon(\sigma') = s_2^{\frac{n^2}{2}}. \quad (\mathbf{15})$$

Therefore from$(5),(11)$ and (15), we have the following;

$$Z(D_n, X^2) = \frac{1}{2n}\left[\sum_{t/n} \emptyset(t) s_t^{\frac{n^2}{t}} + \frac{n}{2} s_1^4 s_2^{\frac{n^2-4}{2}} + \frac{n}{2} s_2^{\frac{n^2}{2}}\right] \quad (\mathbf{16})$$

$(\boldsymbol{B})$When n is odd. From *Theorem* 1, we shall consider the reflection with the monomial $f_1 f_2^{\frac{n-1}{2}}$. We have the following cases.

Case 1. Contribution by pairs of the form $[a,a,]$. From *Lemma* 1, we have the following;

$$f_1 f_2^{\frac{n}{2}} \rightarrow s_1 s_2^{\frac{n-1}{2}} \quad (\mathbf{17})$$

Case 2. Contribution elements of the form$[a,b]$ where $a \neq b$, and;come from common c

$(i)a$ and b come from common cycle of length two. From result (a), we have the following;

$$f_2^{\frac{n-1}{2}} \rightarrow s_2^{\frac{n-1}{2}} \quad (\mathbf{18})$$

(ii) both a and b come from distinct cycles of length two. From result $(b)(ii)$, we have the following contribution;

$$f_2^{\frac{n-1}{2}} \rightarrow s_2^{4\binom{\frac{n-1}{2}}{2}} = s_2^{\frac{n^2}{2}-2n+\frac{3}{2}} \quad (\mathbf{19})$$

(iii)each *of* a and b come from distinct cycles with one coming from a cycle of length one while the other comes from a cycle of length two. Using the result in $(b)(i)$, we have that;

$$f_2^{\frac{n-1}{2}} \rightarrow s_2^{n-1} \quad (\mathbf{20})$$

CollectIng and multiplying the coefficients on the right hand side of $(17),(18),(19)$and (20), we have the following induced monomial;

$$mon(\sigma') = s_1 s_2^{\frac{n^2-1}{2}} \quad (\mathbf{21})$$

Therefore using *Theorem* 2, (5) and (21), we have that;

$$Z(D_n, X^2) = \frac{1}{2n}\left[\sum_{t/n} \emptyset(t) s_t^{\frac{n^2}{t}} + n s_1 s_2^{\frac{n^2-1}{2}}\right], \text{ when } n \text{ is odd.} \quad (\mathbf{22})$$

Example 3

Let $n = 6$, then $t = 1,2,3,6$ and $X = \{1,2,3,4,5,6\}$,

Then $\emptyset(1) = 1, \emptyset(2) = 1, \emptyset(3) = 2, \emptyset(6) = 2$,

Therefore from (16), $Z(D_6, X^2) = \frac{1}{16}[s_1^{36} + 4s_2^{18} + 2s_3^{12} +$ *2s66+3s14s216.*

Example 4

Let $n = 11$, *then* $t = 1,11$, and $X = \{1,2,\ldots,11\}$,

Then $\emptyset(1) = 1, \emptyset(11) = 10$,

Therefore from (22), we have$Z(D_{11}, X^2) = \frac{1}{22}[s_1^{121} +$ *10s1111+11s1s260.*

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