Alternative method for construction of Steiner Triple Systems of order n; $n \equiv 1$ or $3(mod\ 6)$and $n > 12$

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Abstract:-**Construction of Steiner Triple System is well-known. In this work, an alternative construction is given for the construction of $STS(n)$; $n \equiv 1(mod\ 6)$ and$n > 1$. Basic blocks have been used for this construction and these blocks have special properties. Starting with these blocks $STS(13)$, $STS(19)$ and $STS(25)$ have been constructed. Furthermore, generalizations of this work for $STS(3n)$ and $STS(n^2)$have been given by introducing Cartesian Products of two sets.**

Keywords: **Steiner system, Steiner triple system, Basic Blocks**

I. INTRODUCTION

Steiner Systems were introduced by the mathematician Steiner in 1853 and are widely used in constructing designs. A pair $(X, \mathcal{B})$ where X is a n-set and $\mathcal{B}$ is a family of m-subsets that any l-set lies in exactly one number of $\mathcal{B}$ is called a *Steiner System*(l, m, n). A Steiner System $S(2,3,n)$ is called a *Steiner Triple System* of order n and is denoted by $STS(n)$. Construction of $STS(n)$for $n \equiv 1$ or $3(mod\ 6)$are well known. One such construction method is using complete graphsK_n. This work gives a recursive construction method of $STS(n)$ using basic blocks as an alternative method. Arecursive construction is given for the construction of $STS(13), STS(19), STS(25)$; $n \equiv 1(mod\ 6)$. Main focus of this research is to construct triples(blocks) of size three so that each pair of elements are in exactly one block. For this construction, basic blocks B_1, B_2, B_3 etc were constructed by taking the set X of n elements as the additive group $\mathbb{Z}_n$ ={0, 1, 2,…, $n-2, n-1$}.

Definition 1

Let G be an additive group of order v and D is a subset of G of cardinality k. If the set of differences $d_i - d_j$ where d_i, $d_j \in D$; $i \neq j$contains every non-zero element of G exactly λ times, then D is called a (v, k, λ)-*difference set*.

Further, if D_i is a difference set then $D_i + g; g \in \mathbb{Z}_n$ is also a difference set.

Definition 2

Number of blocks of a Steiner System (l, m, n) is $|B| = \frac{{}^nC_l}{{}^mC_l}$.

Number of blocks of a Steiner Triple System $(2,3,n)$ is given by $|B| = \frac{{}^nC_2}{{}^3C_2} = \frac{n(n-1)}{6}$.

Further, if B_i is a block, then $B_i + g; g \in \mathbb{Z}_n$ is also a block,

II. METHODOLOGY

A recursive construction of $STS(13)$, $STS(19)$ and $STS(25)$ using basic blocks are given below.

First, basic blocks of relevant $STS(n)$were constructed such that their differences collectively give each non-zero element of $\mathbb{Z}_n$ exactly once. Using the property that if B_i is a block, then $B_i + g; g \in \mathbb{Z}_n$ is also a block, all blocks of $STS(n)$ have been constructed.

For example in$STS(13)$,

Total number of blocks = $|B| = \frac{{}^{13}C_2}{{}^3C_2} = 26.$

Consider $B_1 = \{0,1,4\}$ and $B_2 = \{0,2,8\}$ as basic blocks where $X = \mathbb{Z}_{13} = \{0,1,2, \ldots, 11,12\}$. For any non-zero $z \in X$ there is a unique way to write $z = u - v$ with u, v chosen from the same set B_i $(i = 1,2)$.

We claim that(X, B) is a $STS(13)$.

Clearly, X is a 13- set and B is a family of 3-subsets of X. If$x, y \in B_i + z$ then,

$x - z, y - z \in B_i$ and $(x - z) - (y - z) = x - y$.

A unique choice is there fori, u, v such that $x - y = u - v$ where $u, v \in B_i$.

Thus, a unique triple containing x and y can be obtained.

Basic blocks are $B_1 = \{0,1,4\}$ and $B_2 = \{0,2,8\}$.

Differences of elements of the blocks give all the non-zero elements of $\mathbb{Z}_{13}$modulo 13 exactly once.

$B_1 = \{0,1,4\}$ gives

$$0-1=12$$
$$1-0=1$$
$$4-0=4$$
$$0-4=9$$
$$1-4=10$$
$$4-1=3$$

$B_2 = \{0,2,8\}$ gives

$$0-2=11$$
$$2-0=2$$
$$7-0=7$$
$$0-7=6$$
$$2-7=8$$
$$7-2=5$$

Let $B = \{B_1 + z, B_2 + z | z \in X\}$ where $B_i + z = \{t + z/t \in B_i, z \in X\}; i = 1,2$.

So, the following26 blocks can be obtained.

$\{0,1,4\}$,$\{1,2,5\}$, $\{2,3,6\}$, $\{3,4,7\}$, $\{4,5,8\}$, $\{5,6,9\}$, $\{6,7,10\}$, $\{7,8,11\}$, $\{8,9,12\}$, $\{9,10,0\}$, $\{10,11,1\}$, $\{11,12,2\}$, $\{12,0,3\}$, $\{0,2,8\}$, $\{1,3,9\}$, $\{2,4,10\}$, $\{3,5,11\}$, $\{4,6,12\}$, $\{5,7,0\}$, $\{6,8,1\}$, $\{7,9,2\}$,$\{8,10,3\}$, $\{9,11,4\}$, $\{10,12,5\}$, $\{11,0,6\}$, $\{12,1,7\}$

Using a similar recursive construction, $STS(19)$ and $STS(25)$ can be obtained.

For $STS(19)$, basic blocks are $B_1 = \{0,1,6\}, B_2 = \{0,2,10\}$ and $B_3 = \{1,5,17\}$.

For $STS(25)$, basic blocks are $B_1 = \{0,1,7\}, B_2 = \{1,4,12\}, B_3 = \{2,6,11\}$ and $B_4 = \{3,5,15\}$.

III. RESULTS AND DISCUSSION

The above constructions can be generalized as follows.

(a) Construction of $STS(3n)$:

Number of triples or blocks $= \dfrac{{}^{3n}C_2}{{}^{3}C_2} = \dfrac{n(3n-1)}{2}$

$3n$points can be divided into three sets with n points each and then three $STS(n)$ can be constructed. Denote the three $STS(n)$ as X, Y and Z, where $X = \{x_0, x_1, \dots, x_{n-1}\}$, $Y = \{y_0, y, \dots, y_{n-1}\}$ and $= \{z_0, z_1, \dots, z_{n-1}\}$.

Total number of triples constructed is $3 \times \dfrac{n(n-1)}{6} = \dfrac{n(n-1)}{2}$.

Then remaining number of triples are given by $\dfrac{n(3n-1)}{2} - \dfrac{n(n-1)}{2} = n^2$.

The Cartesian product of two sets has been used to construct those remaining triples as follows:

Let $Y \times Z = \{(y_j, z_k) | y_j \in Y, Z_k \in Z; j,k = 0,1,\dots,n-1\}$. These n^2 pairs (y_j, z_k) can be converted to triples with $x_i \in X$, such that $j + k \equiv 2i (mod\, n)$. So, n^2 triples can be constructed.

Similarly, using $STS(3n)$, $STS(9n)$,$STS(27n)$ and all higher multiples of $3n$ can be obtained.

(b) Construction of $STS(n^2)$ using $STS(n)$:

Number of blocks = $\dfrac{{}^{n^2}C_2}{{}^{3}C_2} = \dfrac{n^2(n^2-1)}{6}$.

First, n^2 points will be divided into n sets with n pointseach and then construct$STS(n)$using each set with n points.

Denote each $STS(n)$as a point and $newSTS$can be construct denoting each $STS(n)$as points $A_0, A_1, \dots, A_{n-2}, A_{n-1}$. This gives, $\dfrac{n(n-1)}{6} \times n = \dfrac{n^2(n-1)}{6}$ number of triples.

Then the remaining number of triples $\dfrac{n^2(n^2-1)}{6} - \dfrac{n^2(n-1)}{6} = \dfrac{n^3(n-1)}{6}$ can be constructed as follows.

For each new triple $A_i A_j A_k$ where $i, j, k = 0,1, \dots, n-1$, using Cartesian product construction as above n^2 triples can be constructed.

Since $\dfrac{n(n-1)}{6}$ triples are there, $\dfrac{n(n-1)}{6} \times n^2 = \dfrac{n^3(n-1)}{6}$ triples can be constructed.

IV. CONCLUSION

In this work, an alternative constructions are given for $STS(13)$, $STS(19)$ and $STS(25)$. Generalization of this work for $STS(3n)$ is given when$n \equiv 3(mod\, 6)$ and further generalization for $STS(n^2)$ also given when$n \equiv 1(mod\, 6)$.

These generalizations could be repeatedly apply to construct $STS(n)$ when n takes higher values.

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