"Harmonic Forms and Killing Tensor Fields on Almost Grayan Manifold"

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***Abstract:*-In this paper we have studied different aspect of para complex and almost para complex manifold which is similar to almost Grayan manifold.**

I. INTRODUCTION

An almost para complex structure F is intregrable if an only if $N_F=0$.

Proof:

Consider the two projections $\pi \pm : Tm \to T \pm M,$

$$\pi \pm := \frac{1}{2}\left(Fd \pm F\right)$$

Then by the Frobeneus theorem, the integrability of T+M and T-M is equivalent to respectively

$$\left.\begin{array}{l} \pi - \left[\pi + X, \pi + Y\right] = 0 \ \ and \\ \pi + \left[\pi - X, \pi - Y\right] = 0 \end{array}\right\}$$

For all vector fields X and Y. The sum and the difference of these expressions are proportional to $N_F\left(X,Y\right)$ and $FN_F\left(X,Y\right)$

Ex.1 Any para complex vector space $\left(V,F\right)$ can be considered as a Para complex manifold, with constant Para-complex structure.

Ex.2 The certesian product MXN of Two Para-complex manifolds $\left(M,F_M\right)$ and $\left(N,F_N\right)$ is a Para complex manifold with the Para complex structure $F_{MXN} := F_M \oplus F_N$. Here we have used the identification $T\left(MXN\right) = T_M \oplus T_N$.

Ex.3 Let $M = M_+ X\ M_-$ be the certesian product of two smooth manifolds M_+ and M_- of same dimension. We can identify $T_{(p+,p-)}M = T_{P+}M_+ \oplus T_P - M$ and define a para complex structure F on $FT_P \pm M \pm := \pm Fd$. The next result show that any para complex manifolds is locally of this from.

II. EIGEN VALUE OF *F* ON PARA

F has M eigen values $+i$ and M eigen value $-i$.

Solution:

I be a eigen values of F and the corresponding eigen value vector P then

$$\bar{P} = IP$$

Conversely

$$-P = \bar{\bar{P}}$$
$$= I\bar{P}$$
$$= I^2 P$$
$$\because I^2 = -1$$

Since $I$is a real and of rank 2m.Then M pairs of complex conjugate eigen value (i, $-i$)

Theorem-1.1:

The necessary and sufficient condition that -V_n be a almost para complex manifold is that it contains a tangent bundle π_M of dimension M and a tangent bundle $\tilde{\pi}_M$ conjugate to $\pi_M S \in \tilde{\pi}_M \cap \pi_M = \phi$.And they span together a tangent bundle of dimension 2m, projections on π_M and $\tilde{\pi}_M$ being L and$M$ given by

(3.1)a, $$2L \underline{\underline{def}}\ I_n - I_R$$

b) $$2M \underline{\underline{def}}\ I_n + IF$$

$$L = \frac{I - F}{2}$$

$$M = \frac{I+F}{2}$$

Solution:

$$L^2 = \frac{(I-F)^2}{4}$$

$$= \frac{I^2 + F^2 - 2iF}{4}$$

$$= \frac{I - 2F + I}{4}$$

$$= \frac{2I-2F}{4} \; \frac{I-F}{4} m^2 = \frac{(I+M)^2}{4} \Rightarrow \frac{I^2+M^2+2iF}{4}$$

$$M^2 = \frac{(I+M)^2}{4} \Rightarrow \frac{I^2+M^2+2iF}{4}$$

$$= \frac{I^2+M^2+2iF}{4} \Rightarrow \frac{I+I+2F}{4} \Rightarrow \frac{2I+2F}{4}$$

$$= \frac{I+F}{2}$$

$$\because LM = ML \Rightarrow \frac{I^2-F^2}{4} \Rightarrow \frac{I^2-F2}{4} = \frac{I-I}{4} 0$$

$\because$ LM is complementary projection on π_M

$$a^x P_x = 0 \Rightarrow a^x = 0 \quad \forall x$$

$$b^x Q_x = 0 \Rightarrow b^x = 0$$

$$c^x P_x + d^x Q_x = 0 \qquad \text{(i)}$$

$$cF^x P_x + d^x F\, Q_x = 0$$

$$c^x P_x - d^x Q_x = 0 \qquad \text{(ii)}$$

Adding 1 + 2

$$2c^x P_x = 0 \quad \Rightarrow \quad c^x,\; d^x = 0$$

$\therefore P_x, Q_x$ is linearly independent

$$LP_x = P_x$$

$$L_{Px} = \frac{1}{2}(I+F)\,P_x$$

$$= \frac{1}{2}(P_x + P_x)$$

$$= P_x$$

$$LQ_x = \frac{1}{2}(I+F)\,Q_x$$

$$= \frac{1}{2}(Q_x - Q_x) = 0$$

$$MP_x \Rightarrow \frac{1}{2}(I-F)P_x \Rightarrow \frac{1}{2}(P_x - P_x) = 0$$

$$MQ_x \Rightarrow \frac{1}{2}(I-F)Q_x$$

$$\Rightarrow Q_x$$

L		M
π_M	$\cap$	$\tilde{\pi}_M = \phi$
$(P_x\, Q_x)^{-1}$		$(P^x\, Q^x)$

$$P_x P_x = 0$$

$$P_x Q^x = 0$$

Similarity -

$$Q_x P^x = 0$$

$$Q_x Q^x = 1$$

$$I = P^x \otimes P_x + q^x \otimes Q_x$$

$$= P_x P_y = \delta_4^x$$

and

$$q^x Q_y = \delta_y^x$$

$$P^x Q_y = 0$$

$$F = 1\left\{P^x \otimes P_x - q^x \otimes Q_x\right\}$$

$$F^2 = FF = \left\{P^x \otimes P_x - q^x \otimes OF \otimes Q_x\right\}$$

$$= P^x O\, F \otimes P_x - q^x\, O\, F \otimes Q_x$$

$$= P_x P_x - q^x Q_x$$ Proved.

Definition 2.1:

A vector field V is said to be contravariant almost para if it satisfies.

$$L_V F = 0$$

A vector field V said to be strictly contravariant almost para and if both V and $\bar{V}$ are contravariant almost para analytic.

$$L_x Y = [X, Y]$$

$$(L_x F)(Y) = L_x(FY) - FL_x Y$$

$$(L_x F)(Y) = L_x(FY) - FL_x Y$$

$$(L_x F)(Y) = L_x \bar{Y} - FL_x Y$$

$$(L_x F)(Y) = [X, \bar{Y}] - F[X, Y]$$

$$(L_x F)(Y) = [X, \bar{Y}] - \overline{[X, Y]}$$

If

$$(L_x F)(Y) = 0$$

$$[X, \bar{Y}] - [X, Y] = 0$$

Barring $X=V$ in (1)

$$(L_V F)(Y) = [V, \bar{Y}] - \overline{[V, Y]}$$

$$(V, \bar{Y}) - \overline{[V, Y]} = 0$$

$\because V$ is contravariant para complex.

$$(L_V F)(X) = [\bar{V}, \bar{X}] - [\bar{V}, X] \qquad \text{(A)}$$

$$\overline{(L_{\bar{V}} F)(X)} = \overline{[\bar{V}, \bar{X}]} - \overline{\overline{[V, X]}} \qquad \text{(B)}$$

(A) – (B)

$$(L_V F)(X) - \overline{(L_V F)(X)} = [\bar{V}, \bar{X}] - \overline{[\bar{V}, X]} - [V, \bar{X}] + \overline{\overline{[V, Y]}}$$

$$= [\bar{V}, \bar{X}] - [\bar{V}, X] - [V, \bar{X}] + [V, X]$$

$$= N[V, X]$$

$$(L_V F)(X) - \overline{(L_V F)(X)} = N(V, Y)$$

$$(L_V F)(X) = \overline{(L_V F)}(X) + N(V, Y)$$

$$\overline{(L_V F)(X)} = \overline{\overline{[L_v, F]}}(X) + N\overline{(V, Y)}$$

$$\overline{(L_V F)(X)} = \overline{[L_v, F]}(X) + N(V, Y)$$

$$\overline{(L_V F)(X)} - (L_V F)(X) = N(V, X)$$

Necessary and sufficient condition on Para complex manifold:

$$N[V, X] = 0$$

$$\overline{(L_V F)(X)} = (L_V F)(X)$$

$$\overline{\overline{(L_V F)}}(X) = \overline{(L_V F)}(X) \Rightarrow \boxed{(L_{\bar{V}} F)(X) = \overline{(L_V F)}(X)}$$

(1)

$$(L_{\bar{\bar{V}}} F)(X) = \overline{(L_{\bar{V}} F)(X)}$$

$$(L_V F)(X) = \overline{(L_{\bar{V}} F)(X)}$$

$$(L_V F)(X) - (L_{\bar{V}} F)(X) = 0$$

$$(L_{\bar{V}} F)(X) - \overline{(L_{\bar{\bar{V}}} F)(X)} = 0$$

$$(L_{\bar{V}} F)(X) - \overline{(L_V F)(X)} = 0$$

Adding (1) and (2) we get

$$(L_V F)(X) = 0$$

Theorem-2.3:

A necessary and sufficient condition that a vector field *V* on an almost para complex manifold be contra variant almost analytic.

(a) $L_V \bar{X} = \overline{L_V X}$

$\left[V, \bar{X}\right] = \overline{\left[V, X\right]}$

(b) $\overline{L_V X} + L_V X = 0$

$\overline{\left[V, X\right]} + \left[V, X\right] = 0$

From (a)

$$\left(L_V F\right)\left(X\right) = \left[V, \bar{X}\right] - \left[V, X\right]$$

$$\left(L_V F\right)\left(X\right) = \left[L_V, \bar{X}\right] - \overline{L_V X}$$

If $\left(L_V F\right)\left(X\right) = 0$ then

$$L_V\left(FY\right) = \left(L_V F\right)\left(X\right) + FL_V Y$$
$$D_x\left(FY\right) = \left(D_x F\right)\left(Y\right) + FD_x Y$$

Theorem 2.4:

Nijenhuis tensor w.r.t. a contravariant Almost Para analytic vector *V* is Lie constant i.e. Lee derivative of Nijenhuis tensor with *r* to *V* vanishes.

Proof:

$$N\left[X,Y\right] = \left[\bar{X},\bar{Y}\right] + \left[X,Y\right] - \overline{\left[\bar{X},Y\right]} - \overline{\left[X,\bar{Y}\right]}$$

$$L_V N\left[X,y\right] = \left[L_V N\right]\left(X,Y\right) + N\left(L_V X,Y\right) + N\left(X,L_V Y\right)$$

Where v is contravariant almost para analytic

$$L_V\left\{\left[\bar{X},\bar{Y}\right] + \left[X,Y\right] - \left[\bar{X},Y\right] - \left[X,\bar{Y}\right]\right\} = \left(L_V N\right)\left(X,Y\right) + N\left(\left(V,X\right)Y\right)$$
$$+ N\left(X,\left(V,Y\right)\right)$$

$$\left[V,\left(\bar{X},\bar{Y}\right)\right] + \left[V,\left[X,Y\right]\right] - \left[V,\overline{\left[\bar{X},Y\right]}\right] - \left[V_1\overline{\left[X,\bar{Y}\right]}\right]$$

$$= \left(L_V N\right)\left[X,Y\right] + N\left[\left[V,X\right]Y\right] + N\left[X,\left[V,Y\right]\right]$$

$$\left[V,\left[\bar{X},\bar{Y}\right]\right] + \left[V,\left[X,Y\right]\right] - \left[V,\overline{\left[\bar{X},\bar{Y}\right]}\right] - \left[V,\overline{\left[X,\bar{Y}\right]}\right]$$

$$= L_V N\left(X,Y\right)\Big] + \left[\overline{\left[V,X\right]},\bar{Y}\right] + \left[\left(V,X\right),Y\right] - \left[\overline{\left(V,X\right)},Y\right]$$

$$- \left[\left(V,X\right),\bar{Y}\right] + \left[\bar{X},\overline{\left(V,Y\right)}\right] + \left[X,\left(V,Y\right)\right]$$

$$- \left[\bar{X},\left(V,Y\right)\right] - \left[X,\overline{\left(V,Y\right)}\right]$$

$$L_V N\left(X,Y\right) + \left[\left(V,\bar{X}\right),\bar{Y}\right] + \left[\left(V,X\right)Y\right] - \left[\overline{\left(V,X\right)},Y\right] - \overline{\left(V,X\right)},Y +$$

$$\left[X,\left[V,\bar{X}\right] + \left[X,\left[V,Y\right] - \left[\bar{X},\left[V,Y\right] - \left[X,\left(\overline{V,Y}\right)\right]\right]\right]\right] - V\left[\bar{X},\bar{Y}\right] +$$

$$\left[V,\left(X,Y\right)\right] - \left[V,\left(\bar{X},Y\right)\right] - V\overline{\left(X,\bar{Y}\right)} = 0$$

By the Jacobi's identies, this equation assume the from

$$L_V N = 0$$

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