k-Factors of k-Factorization of $K_{2^r,2^r,2^r,\cdots,2^r}$ with n-Partite Sets for $k = 1,2$ and $n \geq 2,\ n,r \in \mathbb{Z}^+$

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Abstract: **Graph factorization is one of the most flourishing areas in Graph Theory. Most of the research work on factorization is on complete graphs and complete bipartite graphs. In this research, complete k- partite graphs are considered. By considering degree factorization, two theorems have been proved to obtain factors of factorization of the complete multipartite graphs $K_{2,2,2,\cdots,2}$ and $K_{2^r,2^r,\ldots,2^r}$ with n partite sets where $n \geq 2$ and $n,r \in \mathbb{Z}^+$. Moreover, when n is even, 2-factors for 2-factorization of a complete multipartite graph of the form $K_{2,2,2,\cdots,2}$ have been obtained using the tournament scheduling technique by considering n partite sets as n teams.**

Key words: **Complete multipartite graphs, Factorization, Tournament scheduling technique**

I. INTRODUCTION

Graph Theory has many applications in all disciplines. A graph consists of vertices and edges. A simple graph G is a pair $(V(G), E(G))$ where $V(G)$ is a non empty finite set of elements called vertices and $E(G)$ is a set of unorded pair of distinct elements of $V(G)$ called edges [4]. A factor of a graph G is a spanning sub-graph of G which is not totally disconnected and a graph factorization of G is a partition of edges of G into disjoint factors. There are two different types of factors; degree-factors and component-factors. The notion of component-factors was introduced recently whereas graph factorization with respect to the degree has a history of more than one century and is one of the most active research areas in Graph theory. Applications of graph factorization are involved in Travelling salesman problem, Round-Robing tournaments, Kirkman's school girl problem etc. The notion of factorization was introduced by Kirkman in 1847 and a result on a factorization of 1-factors was obtained by Reiss in 1859. A recent survey paper of factors and factorizations can be found in [3].

Definition 1

A graph is *multipartite* or *k-partite* if its vertices can be partitioned into k sets (called partite sets) in such a way that no edge joins vertices in the same set. A *complete k-partite graph* is a simple k-partite graph in which each vertex in one partite set is adjacent to all the vertices in the other partite sets. A complete k-partite graph is denoted by $K_{m_1,m_2,m_3,\ldots,m_k}$ where m_i is the cardinality of the i^{th} partite set.

Definition 2 [2, p 403]

A factor that is n-regular is called an n-*factor*.

Definition 3 [2, p 403]

If a graph G can be represented as the edge-disjoint union of factors $F_1, F_2, \ldots, F_k$, $\{F_1, F_2, \ldots, F_k\}$ is refered to as a *factorization* of a graph G.

Definition 4

A tournament schedule is an arrangement of all pairs of teams into the minimum number of rounds. Suppose that there are n teams in a competition. In a season, every team plays against every other team once. Then, the number of matches to be played is ${}^{n}C_2 = \frac{n(n-1)}{2}$.

II. MATERIALS AND METHODS

Theorem 1

A complete multipartite graph with n partite sets of the form $K_{2,2,2,\ldots,2}$ has $2(n-1)$, 1-factors for a 1-factorization for $n \geq 2$.

Proof:

Here, a combinatorial proof has been introduced.

First, consider the multipartite graph $K_{2,2,2,\ldots,2}$ with n partite sets. The total number of vertices is $2n$ and the total number of edges is ${}^{2n}C_2 - n$. The number of edges in a one 1-factor of 1-factorization is equal to the number of partite sets of the complete multipartite graph. By dividing the ${}^{2n}C_2 - n$ edges by the number of partite sets, number of 1-factors in 1-factorization can be obtained.

i.e. Number of 1-factors in 1- factorization $= \frac{{}^{2n}C_2 - n}{n}; n \geq 2$

$$= \frac{\frac{(2n)!}{2!(2n-2)!} - n}{n}$$

$$= 2(n-1)$$

When $n = 2$; $K_{2,2}$

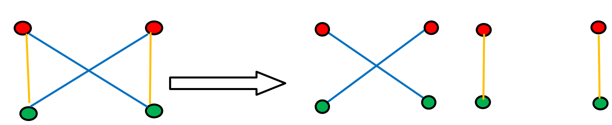


Figure 1: A 1-factorization of $K_{2,2}$ graph.

Number of 1-factors for one 1-factorization = $2(n-1) = 2$

When $n = 3$; $K_{2,2,2}$

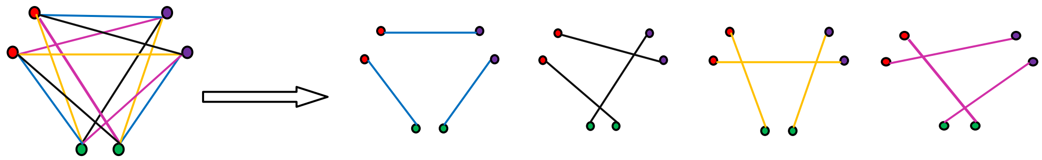


Figure 2: A 1- factorization of $K_{2,2,2}$ graph.

Number of 1-factors for one 1-factorization = $2(n-1) = 4$

When $n = 4$; $K_{2,2,2,2}$

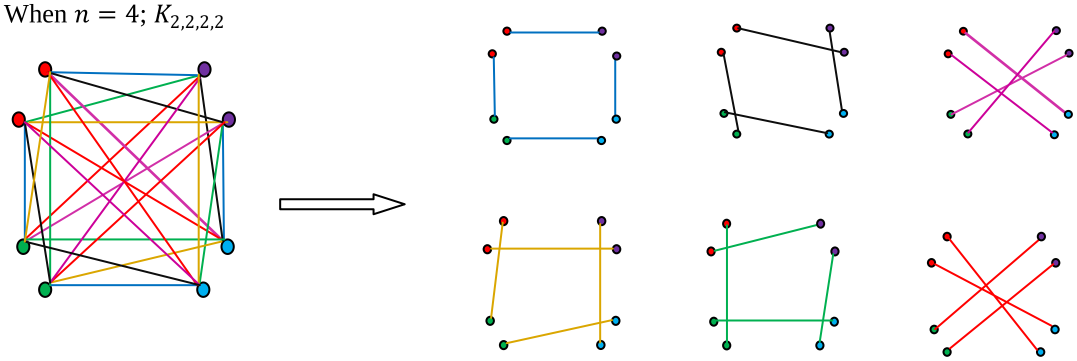


Figure 3: A 1- factorization of $K_{2,2,2,2}$ graph.

Number of 1-factors for one 1-factorization = $2(n-1) = 6$

Theorem 2

A complete multipartite graph with n partite sets of the form $K_{2^r,2^r,\ldots,2^r}$ has $2^r(n-1)$, 1-factors for a 1-factorization for $n \geq 2$ and $r \in \mathbb{Z}^+$.

Proof:

Consider the multipartite graph $K_{2^r,2^r,\ldots,2^r}$ with n partite sets. Then the total number of vertices is $2^r n$ and the total number of edges is ${}^{2^r n}C_2$. By removing edges between vertices of the same partite sets, ${}^{2^r n}C_2 - \sum_{k=1}^{2^r-1} kn$ edges can be obtained and the resulting graph is the complete multipartite graph $K_{2^r,2^r,\ldots,2^r}$. Dividing ${}^{2^r n}C_2 - \sum_{k=1}^{2^r-1} kn$ by $2^{r-1}n$, where n is denoted by the number of partite sets and 2^{r-1} is denoted by the pairs of vertices in one partite set, the number of 1-factors in 1-factorization can be obtained.

i.e. Number of 1-factors in 1- factorization= $\dfrac{{}^{2^r n}C_2 - \sum_{k=1}^{2^r-1} kn}{2^{r-1}n}$

$= 2^r(n-1).$

2-Factors of 2-Factorization

When $n = 2$, $K_{2,2}$

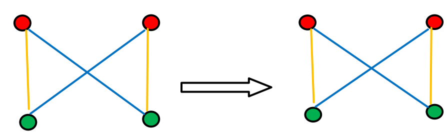


Figure 4: A 2-factorization of $K_{2,2}$ graph.

Number of 2-factors of a 2-factorization =1

When n is large, tournament scheduling technique can be applied.

When $n = 6$, $K_{2,2,2,2,2,2}$

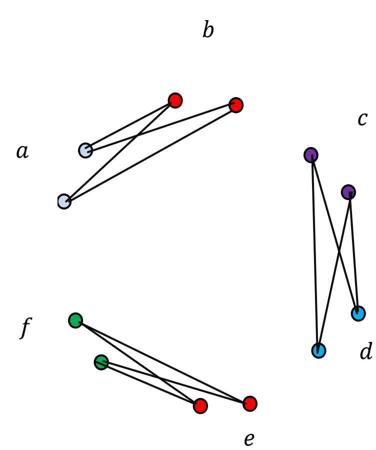


Figure 5: A 2- factor of $K_{2,2,2,2,2,2}$ graph.

$K_{2,2,2,2,2,2}$ has ${}^{2n}C_2 - n$ = ${}^{12}C_2 - 6 = 60$ edges. One can construct a 2-factor between any pair of partite sets. By considering all disjoint pairs of partite sets, one can construct the graph $K_{2,2,2,2,2,2}$ which has ${}^{n}C_2 = {}^{6}C_2 = 15$, 2-factors. Considering 6 partite sets as 6-teams of a tournament, using tournament scheduling technique, 2-factorization can be obtained. According to the tournament scheduling technique number of matches can be played between 6 teams is ${}^{6}C_2 = 15$ and 3 matches will be played in each round and 5 rounds are required. So, labeling 6 partite sets as a, b, c, d, e, f (as given in the above figure) can schedule ab, cd, ef as first round matches which corresponds to 2-factors of a 2-factorization.

Consider a n-partite set where n is even,

Number of edges is ${}^{2n}C_2 - n = 2n(n-1)$.

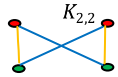
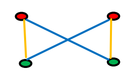
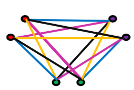
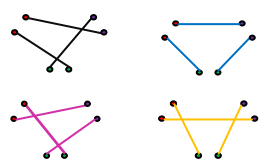
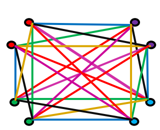
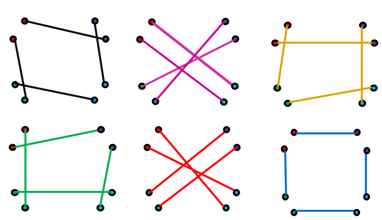
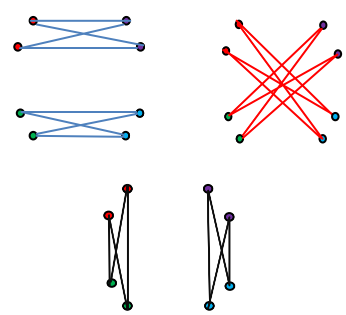
By considering different partite sets, $\frac{2n(n-1)}{4}$ of 2-factors can be obtained, which is equal to $\frac{n(n-1)}{2} = {}^{n}C_2$.

That is the number of matches between n-teams of a tournament in which every team plays against every other team once. Since n is even, $\frac{n}{2}$ matches will be played in each round and $(n-1)$ rounds are required.

So, labeling partite sets as $a_1, a_2, \cdots, a_n$, then 2-factors of a 2-factorization can be obtained as $a_i a_{i+1}$ for $i = 1,2, \cdots, n-1$. Further, $(n-1)$ different 2-factorizations can be obtained by using this construction.

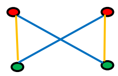
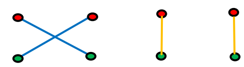
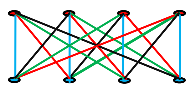
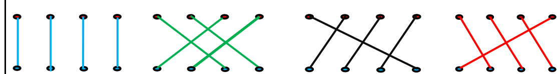
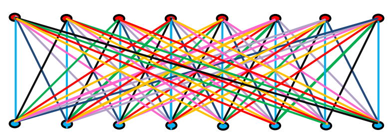
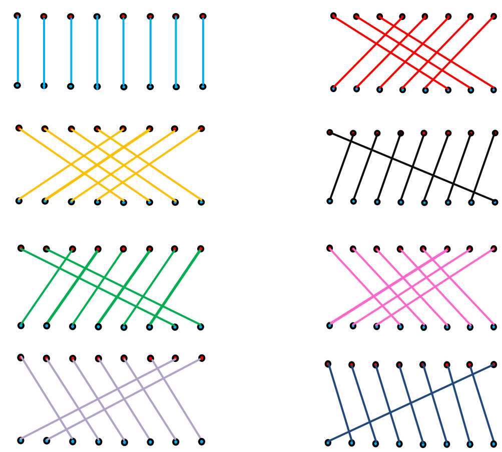
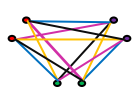
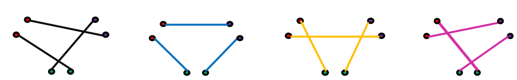
III. RESULTS AND DISCUSSIONS

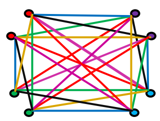
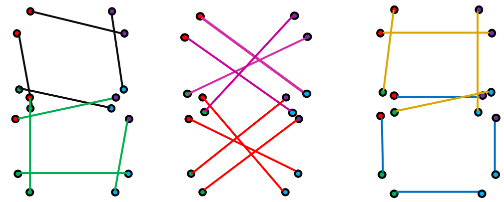
Table 1: A 1- factorization and 2- factorization of $K_{2,2,2,\cdots,2}$ graph for different values of n.

When $n = 2$,	Number of 1-factors	Number of 2-factors
$K_{2,2}$ 	$2(2-1) = 2$ 	 1
When $n = 3$,		
$K_{2,2,2}$ 	$2(3-1) = 4$ 	None
When $n = 4$,		
$K_{2,2,2,2}$ 	$2(4-1) = 6$ 	3 

This can be generalized for $K_{2^r,2^r,\ldots,2^r}$.

Table 2: A 1- factorization of $K_{2^r,2^r,\ldots,2^r}$ graph for different n and r values

Case 1: For $n = 2$ and different value of r. When $r = 1$,	1-factors, Number of 1-factors $2^r(n-1)$.
$K_{2,2}$ 	$2^1(2-1) = 2$ 
When $r = 2$,	
$K_{4,4}$ 	$2^2(2-1) = 4$ 
When $r = 3$,	
$K_{8,8}$ 	$2^3(2-1) = 8$ 
Case 2: For $n = 3$ and different value of r. When $r = 1$,	
$K_{2,2,2}$ 	$2^1(3-1) = 4$ 
When $r = 2$,	
$K_{4,4,4}$	$2^2(3-1) = 8$
When $r = 3$,	
$K_{8,8,8}$	$2^3(3-1) = 16$
Case 3: For $n = 4$ and different value of r. When $r = 1$,	

$K_{2,2,2,2}$ 	$2^1(4-1)=6$ 
When $r=2$,	
$K_{4,4,4,4}$	$2^2(4-1)=12$
When $r=3$,	
$K_{8,8,8,8}$	$2^3(4-1)=24$

IV. CONCLUSIONS

Most of the research work on factorization is on complete graphs and complete bipartite graphs. In our research, complete multipartite graphs $K_{2,2,2,\cdots,2}$ with n partite sets have been used to constructed 1-factors and 2-factors of their factorizations for different values of n. Moreover, 1-factors of $K_{2^r,2^r,2^r,\cdots,2^r}$ have been constructed using a Theorem and illsustrated by using a table.

As future work, we are trying to generalized this work for higher order factors of $K_{2^r,2^r,2^r,\cdots,2^r}$ with n partite sets for integers $r, n \geq 2$.

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