q–Special Function and Integral Transform

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Abstract:-**In this paper we discuss about special function with q-analog and find out relation of q-Gamma function into Laplace Transform and Fourier Transform. We find out some new property and relations.**

Keywords **– Special Function, Laplace Transform, Fourier Transform, q-analog.**

I. INTRODUCTION

Quantum calculus or q- calculus is widely used in Mathematics. It is considered to be one of the most difficult subject to engage in mathematics. Quantum calculus and its application use in various fields of Physics, Mechanics and Mathematical Science. In previous years q-analogy play important role in Mathematics like q-Gamma function, q-Beta function and q-Integral Transform etc.

In this paper we present the definition of q – beta function and generalized q –gamma function and their relation and properties on q-integral.

We give notation and preliminaries of q analog in second section and discuss about q-pochhammer symbol. In third section we will define generalized q-gamma function and q beta function and their relation to integral transform and obtain some auxiliary result.

Notations and Preliminaries

q –Pochhammer symbol-First we define how to apply q-notation in factorial n! , we now that by definition of limits, for q tends to 1.

$$[n] \text{ or } [n]_q = \frac{1-q^n}{1-q}$$

$$= \frac{1-q}{1-q}\frac{1-q^2}{1-q}\frac{1-q^3}{1-q}\cdots\cdots\frac{1-q^n}{1-q}$$

$$= (1+q)(1+q+q^2)\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= 1+q+q^2+\cdots q^{n-1}+\cdots+q^n+\cdots\cdots\cdots\cdots$$

Then q-shifted factorial notation are

$$\langle a\,;q\rangle_n = \begin{cases} 1 & n=0 \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,3 \end{cases}$$

We also write this type

$$\langle a;q\rangle_n = (q^a;q)_n$$

Furthermore $0 < |q| < 1$

$$(a;q)_\infty = \prod_{m=0}^{\infty}(1-aq^m)$$

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}$$

For $|x| < 1, |q| < 1$ Power Series corresponding to

$$\sum_{k=0}^{\infty}(a)_k\frac{x^k}{k!} = \sum_{k=0}^{\infty}\frac{(q^a;q)_k}{(q;q)_k}x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

q-Exponential function –

The exponential function ex has many different q-extensions, one of them is defined as

$$E_q(x) = \sum_{k=0}^{\infty}\frac{x^k}{[k]_q!}$$

$$E_q(x) = \sum_{r=0}^{\infty}\frac{q^{k^2/2}\ x^k}{\langle q;q\rangle_k}$$

Consequently in the limitq → 1, we have $\lim_{q\to1}\big(E_q(1-q)x=ex$.

q–Gamma Function-

Jackson defined q –analogue of the gamma function by $\Gamma_q(n)$

Let us define

$$\Gamma_q(\alpha) = \int_0^{\frac{1}{1-q}} x^{\alpha-1}\,E_q^{-qx}\,d_q x$$

$|q| < 1$] $\alpha \neq 0,-1,-2\ldots\ldots$

$$\Gamma_q(\alpha) = (1-q)^{1-\alpha}\frac{(q;q)_\infty}{(q^\alpha;q)_\infty}$$

$$= (1-q)^{1-\alpha}\prod_{n=0}^{\infty}\frac{1-q^{\alpha+1}}{1-q^{n+\alpha}}$$

$$\Gamma_q(\alpha) = [\alpha-1]_q!$$

Generalized Gamma Function-

Classical gamma function extends infinitely many ways. This is useful to Mathematical problems. It is also called extension of gamma function.

Defination.1-The generalized gamma function is defined by[5]

$$\Gamma_k(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{\frac{-x^k}{k}} dx \qquad k>0,\ Re(\alpha)>0$$

Generalized gamma function Γ_k is a one parameter k-deformation, k>0 a real number of the classical gamma function it is also denoted in the form of pochhammer symbol

$$\Gamma_k \alpha = \log_{n\to\infty} \frac{n!\ k^n\ (nk)^{\frac{\alpha}{k}-1}}{(\alpha)_{n,k}}$$

q-Generalized Gamma Function-

we give Jackson integral equation for q-generalized gamma function and beta function in terms of exponential function and pochhammer symbol are given by following formula[5].

$$\Gamma_{q,k}(\alpha) = \int_0^{\frac{k}{1-q^k}} x^{\alpha-1} e_{q,k}^{\frac{-q^k x^k}{[k]_q}} d_q x$$

$$\Gamma_{q,k}(\alpha) = \frac{(1-q^k)_{q,k}^{\frac{\alpha}{k}-1}}{(1-q)^{\frac{\alpha}{k}-1}}$$

II. MAIN RESULT

Properties of q-Generalized Gamma Function

Theorem1.1.The generalized gamma function satisfy the following equation

$$\Gamma_{q,k}(\alpha+k) = [\alpha]_{q,k}\,\Gamma_{q,k}(\alpha)$$

k >0 and q tends to 1

Proof- We know that generalized gamma function is

$$\Gamma_{q,k}(\alpha+k) =$$

$$= \frac{(1-q^{\alpha+k})_{q,k}^{\frac{\alpha+k}{k}-1}}{(1-q)^{\frac{\alpha+k}{k}-1}}$$

$$= \frac{(1-q^k)_{q,k}^{\frac{\alpha}{k}-1}}{(1-q)^{\frac{\alpha}{k}-1}} . \frac{(1-q^{\alpha})_{q,k}}{(1-q)}$$

$$\Gamma_{q,k}(\alpha+k) = [\alpha]_{q,k}\,\Gamma\Gamma_{\Gamma q,k}(\alpha)$$

Theorem-1.2 (log –convex property) Let 1 < a < ∞ and $\frac{1}{a}+\frac{1}{b}$ =1,($k>0, b>0$ then

$$\Gamma_{q,k}\left(\frac{x}{a}+\frac{y}{b}\right) \le \left(\Gamma_{q,k}(x)\right)^{1/a}\left(\Gamma_{q,k}(y)\right)^{1/b}$$

Proof- By defination of q generalized gamma function

$$\Gamma_{q,k}(\alpha) = \int_0^{\frac{k}{1-q^k}} t^{\alpha-1} e_{q,k}^{\frac{-q^k t^k}{[k]_q}} d_q t$$

Let $\alpha = \frac{x}{a}+\frac{y}{b}$ then

$$\Gamma_{q,k}\left(\frac{x}{a}+\frac{y}{b}\right) = \int_0^{\frac{k}{1-q^k}} (t)^{\frac{x}{a}+\frac{y}{b}-1} e_{q,k}^{\frac{-q^k t^k.1}{[k]_q}} d_q t^1$$

$$= \int_0^{\frac{k}{1-q^k}} (t)^{\frac{x}{a}+\frac{y}{b}-(\frac{1}{a}+\frac{1}{b})} e_{q,k}^{\frac{-q^k t^k.\left(\frac{1}{a}+\frac{1}{b}\right)}{[k]_q}} d_q t^{\left(\frac{1}{a}+\frac{1}{b}\right)}$$

$$= \int_0^{\frac{k}{1-q^k}} (t)^{\frac{x}{a}-\frac{1}{a}} e_{q,k}^{\frac{-q^k t^k.\left(\frac{1}{a}\right)}{[k]_q}} d_q t^{\left(\frac{1}{a}\right)} (t)^{\frac{y}{b}-\frac{1}{b}} e_{q,k}^{\frac{-q^k t^k.\left(\frac{1}{b}\right)}{[k]_q}} d_q t^{\left(\frac{1}{b}\right)}$$

$$= \int_0^{\frac{k}{1-q^k}} \left\{(t)^{x-1} e_{q,k}^{\frac{-q^k t^k}{[k]_q}} d_q t\right\}^{1/a} \left\{(t)^{y-1} e_{q,k}^{\frac{-q^k t^k}{[k]_q}} d_q t\right\}^{1/b}$$

By Holder Inequality

$$\Gamma_{q,k}\left(\frac{x}{a}+\frac{y}{b}\right)$$

$$\le \left(\int_0^{\frac{k}{1-q^k}} t^{x-1} e_{q,k}^{\frac{-q^k t^k}{[k]_q}} d_q t\right)^{1/a} \left(\int_0^{\frac{k}{1-q^k}} t^{y-1} e_{q,k}^{\frac{-q^k t^k}{[k]_q}} d_q t\right)^{1/b}$$

$$\Gamma_{q,k}\left(\frac{x}{a}+\frac{y}{b}\right) \le \left(\Gamma_{q,k}(x)\right)^{1/a}\left(\Gamma_{q,k}(y)\right)^{1/b}$$

In above Theorem some special case can be gained. If a = b =2, x, y > 0, Then result is

$$\Gamma_{q,k}\left(\frac{x}{2}+\frac{y}{2}\right) \le \Gamma_{q,k}\left(\frac{x}{2}+\frac{y}{2}\right) \le \left(\Gamma_{q,k}(x)\right)^{1/2}\left(\Gamma_{q,k}(y)\right)^{1/2}$$

$$\le \frac{1}{2}\left\{\Gamma_{q,k}(x)+\Gamma_{q,k}(y)\right\}$$

Defination.2. The q-generalized beta function is defined by

$$B_{q,k}(m,n) = [k]_q^{-\frac{m}{k}} \int_0^{[k]_q^{1/k}} t^{m-1}\left(1+\frac{-q^k t^k}{[k]_q}\right)_{q,k}^{\frac{n}{k}-1} d_q t \qquad m,n>0$$

$$B_{q,k}(m,n) = \frac{(1-q)(1-q^k)_{q,k}^{\frac{n}{k}-1}}{(1-q^m)_{q,k}^{\frac{n}{k}}}$$

Theorem 1.3 (Relation between generalized beta and gamma function)

$$B_{q,k}(m,n,p) = \frac{\Gamma_{q,k} m \Gamma_{q,k} n}{\Gamma_{q,k} m+n}$$

Proof-By property of q-generalized gamma function

$$\Gamma\Gamma_{\Gamma q,k}(n) = \int_0^{\frac{k}{1-q^k}} t^{n-1}\, e_{q,k}^{\frac{-q^k t^k}{[k]_q}}\, d_q t$$

Substitute t = x2 then $d_q t = [2]_q x\, d_q x$

$$\Gamma\Gamma_{\Gamma q,k}(n) = [2]_q \int_0^{\left(\frac{k}{1-q^k}\right)^{1/2}} x^{2n-2+1}\, e_{q,k}^{\frac{-q^k x^{2k}}{[k]_q}}\, d_q x$$

$$\Gamma\Gamma_{\Gamma q,k}(n) = [2]_q \int_0^{\left(\frac{k}{1-q^k}\right)^{1/2}} x^{2n-1}\, e_{q,k}^{\frac{-q^k x^{2k}}{[k]_q}}\, d_q x$$

Both side multiply by $\Gamma_{q,k}(m)$

$$\Gamma_{q,k}(m)\ \Gamma_{q,k}(n) =$$

$$= [2]_q[2]_q \int_0^{\left(\frac{k}{1-q^k}\right)^{1/2}} x^{2n-1}\, e_{q,k}^{\frac{-q^k x^{2k}}{[k]_q}}\, d_q x \int_0^{\left(\frac{k}{1-q^k}\right)^{1/2}} y^{2m-1}\, e_{q,k}^{\frac{-q^k y^{2k}}{[k]_q}}\, d_q y$$

Assume $x = r\cos_{q,k}\theta$ and $y = r\sin_{q,k}\theta$

$$d_q x d_q y = [k]_q [r] d_q r d_q \theta$$

$$= [2]_q[2]_q \int_0^{\left(\frac{2k}{1-q^k}\right)^{1/2}} \int_0^{\frac{\pi}{2}} (r\sin_{q,k}\theta)^{2m-1}\, e_{q,k}^{\frac{-q^k (r\sin_{q,k}\theta)^{2k}}{[k]_q}}\, (r\cos_{q,k}\theta)^{2n-1}\, e_{q,k}^{\frac{-q^k (r\cos_{q,k}\theta)^{2k}}{[k]_q}}$$

$$[k]_q [r] d_q r d_q \theta$$

$$= [2]_q[2]_q \int_0^{\left(\frac{2k}{1-q^k}\right)^{1/2}} [r]^{2(m+n)-1} e_{q,k}^{\frac{-q^k (r)^{2k}}{[k]_q}}\, d_q r)^{2n-1}$$

$$\int_0^{\frac{\pi}{2}} (\sin_{q,k}\theta)^{2m-1}\ (\cos_{q,k}\theta[k]_q d_q\theta$$

Rewrite the above equation

$$= [2]_q \int_0^{\left(\frac{2k}{1-q^k}\right)^{1/2}} [r]^{2(m+n)-1} e_{q,k}^{\frac{-q^k (r)^{2k}}{[k]_q}}\, d_q r$$

$$[2]_q [k]_q^{\frac{-(m+n)}{k}} \int_0^{\left(\frac{\pi}{2[k]_q}\right)^k} \left(\sin_{q,k}\theta\right)^{2m-1}\ (\cos_{q,k}\theta)^{2n-1} d_q\theta$$

$= \Gamma_{q,k}(m+n)\ B_{q,k}(m,n)$

Relation Between q-Gamma Function and Integral Transform-

We introduce a new concept, namely q-Laplace transform of a function, which will play a similar role in mathematical analysis as well as mathematical physics. we generalized the function in terms of q –analog and find relation between generalized gamma function.

q-Laplace Transform is

$$L_q\{f(x)\}s = \int_0^{1/_{1-q}} \left[e_q^{-sx} f(x)dx\right] \text{ for } \Re(s) > 0$$

Above Laplace Transform can be written in this form

$$L_q\{f(x)\}s = \begin{cases} \int_0^{1/(1-q)s} [1-(1-q)sx]^{\frac{1}{1-q}} [f(x)dx] \text{ for } \Re(1-(1-q)sx) > 0 \\ \int_0^{1/_{1-q}} \left[[1+(q-1)sx]^{-\frac{1}{1-q}} f(x)dx\right] \text{ for } \Re(s) > 0 \end{cases}$$

Generalized q-Laplace Transform is

$$L_{q,k}\{f(x)\}s = \int_0^{k/_{1-q^k}} \left[e_q^{-\frac{q^k (sx)^k}{[k]_q}} f(x) d_q x \right] \text{ for } \Re(s) > 0$$

Theorem1.4-Substitute f(x) =xn-1 in above equation, then result is

$$L_{q,k}\{x^{n-1}\}s = \int_0^{k/_{1-q^k}} e_q^{-\frac{q^k (t)^k}{[k]_q}} \left(\frac{t}{s}\right)^{n-1} d_q x \quad \text{for } \Re(s) > 0$$

Put sx = t Then x = $\frac{t}{s}$

$$= \int_0^{k/_{1-q^k}} e_q^{-\frac{q^k (t)^k}{[k]_q}} \left(\frac{t}{s}\right)^{n-1} \left(\frac{1}{s^2}\right) d_q t$$

$$= \frac{1}{s^{n+1}} \int_0^{k/_{1-q^k}} e_q^{-\frac{q^k (t)^k}{[k]_q}} (t)^{n-1} d_q t$$

$$= \frac{\Gamma_{q,k}(n)}{s^{n+1}}$$

$$L_{q,k}\{x^{n-1}\}s = \frac{\Gamma_{q,k}(n)}{s^{n+1}}$$

Theorem 1.5

Relation between Gamma function and Fourier transform. if s>0 then

$$F_{q,k}[u(t)t^{n-1}] = i^n \frac{\Gamma_{q,k}(n)}{s^n}$$

Consider, $F_{q,k}[u(t)t^{n-1}] = \int_{-\infty}^{\frac{k}{1-q^k}} u(t)t^{n-1}\, e^{\frac{q^k (ist)^k}{[k]_q}}\, d_q t$

where u(t) is Heaviside's unit function is defined as follows

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

$$F_{q,k}[u(t)t^{n-1}] = \int_{-\infty}^{\frac{k}{1-q^k}} u(t)t^{n-1}\, e^{\frac{q^k (ist)^k}{[k]_q}}\, d_q t$$

$$= \int_{-\infty}^{0} u(t)t^{n-1}\, e^{\frac{q^k (ist)^k}{[k]_q}}\, d_q t + \int_{0}^{\frac{k}{1-q^k}} u(t)t^{n-1}\, e^{\frac{q^k (ist)^k}{[k]_q}}\, d_q t$$

$$= \int_{0}^{\frac{k}{1-q^k}} t^{n-1}\, e^{\frac{q^k (ist)^k}{[k]_q}}\, d_q t$$ [Using definition of Heaviside unit function]

put ist =-x then $t = \frac{-x}{is} \Rightarrow d_q t = -\frac{d_q x}{is}$

$$= \int_{0}^{\frac{k}{1-q^k}} \left(\frac{-x}{is}\right)^{n-1} e^{\frac{(-1)^k q^k (x)^k}{[k]_q}} \left(\frac{-1}{is}\right) d_q x$$

$$= \frac{-1}{is} \int_{0}^{\frac{k}{1-q^k}} \left(\frac{-xi}{i^2 s}\right)^{n-1} e^{\frac{(-1)^k q^k (x)^k}{[k]_q}}\, d_q x$$

$$= \frac{-i.\, i^{n-1}}{i^2 s^{1+n-1}} \int_{0}^{\frac{k}{1-q^k}} (x)^{n-1}\, e^{\frac{q^k (x)^k}{[k]_q}}\, d_q x$$

$$= \frac{i^n}{s^n} \int_{0}^{\frac{k}{1-q^k}} (x)^{n-1}\, e^{\frac{q^k (x)^k}{[k]_q}}\, d_q x$$

$$= \frac{i^n}{s^n} \Gamma_{q,k}(n)$$

$$F_{q,k}[u(t)t^{n-1}] = i^n \frac{\Gamma_{q,k}(n)}{s^n}$$

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