On The Exponential Diophantine Equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$

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Abstract: **Diophantine equations are those equations of theory of numbers which are to be solved in integers. The class of Diophantine equations is classified in two categories, one is linear Diophantine equations and the other one is non-linear Diophantine equations. Both categories of these equations are very important in theory of numbers and have many important applications in solving the puzzle problems. In the present paper, author discussed the existence of the solution of exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$, where m, n, r, ω are whole numbers.**

Keywords: **Exponential Diophantine equation; Congruence; Modulo system; Numbers.**

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I. INTRODUCTION

Nowadays, scholars are very interested to determine the solution of different Diophantine equations because these equations have many applications in the field of coordinate geometry, cryptography, trigonometry and applied algebra. Finding the solution of Diophantine equations have many challenges for scholars due to absence of generalize methods. Aggarwal et al. [1] discussed the Diophantine equation $223^x + 241^y = z^2$ for solution. Aggarwal et al. [2] discussed the existence of solution of Diophantine equation $181^x + 199^y = z^2$. Bhatnagar and Aggarwal [3] proved that the exponential Diophantine equation $421^p + 439^q = r^2$ has no solution in whole number.

Gupta and Kumar [4] gave the solutions of exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$. Kumar et al. [5] studied exponential Diophantine equation $601^p + 619^q = r^2$ and proved that this equation has no solution in whole number. Kumar et al. [6] considered the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They showed that these equations have no non-negative integer solution. Kumar et al. [7] studied the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. They determined that these equations have no non-negative integer solution.

Mishra et al. [8] studied the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ and proved that the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ has no solution in whole number. Diophantine equations help us for finding the integer solution of famous Pythagoras theorem and Pell's equation [9-10]. The Diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$ were studied by Sroysang [11, 14]. He proved that these equations have a unique solution which is given by $\{x = 1, y = 0, z = 3\}$. Sroysang [12] proved that the exponential Diophantine equation $31^x + 32^y = z^2$ has no positive integer solution. Sroysang [13] discussed the Diophantine equation $3^x + 5^y = z^2$.

Goel et al. [15] discussed the exponential Diophantine equation $M_5{}^p + M_7{}^q = r^2$ and proved that this equation has no solution in whole number. Kumar et al. [16] proved that the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$ has no solution in whole number. The exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$ has studied by Kumar et al. [17]. Aggarwal and Sharma [18] studied the non-linear Diophantine equation$379^x + 397^y = z^2$ and proved that this equation has no solution in whole number.

The main aim of this article is to discuss the existence of the solution of exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$, where m, n, r, ω are whole numbers.

II. PRELIMINARIES

Lemma: 1 The exponential Diophantine equation $(7^{2m}) + 1 = \omega^2$, where m, ω are the whole numbers, has no solution in whole number.

Proof: Since (7^{2m}) is an odd number for all whole number m.

$\Rightarrow (7^{2m}) + 1 = \omega^2$ is an even number for all whole number m.

$\Rightarrow \omega$ is an even number.

$\Rightarrow \omega^2 \equiv 0(mod3)$ or $\omega^2 \equiv 1(mod3)$ (1)

Now, $7 \equiv 1(mod3)$, for all whole number m.

$\Rightarrow (7^{2m}) \equiv 1(mod3)$, for all whole number m.

$\Rightarrow (7^{2m}) + 1 \equiv 2(mod3)$, for all whole number m.

$\Rightarrow \omega^2 \equiv 2(mod3)$ (2)

Equation (2) contradicts equation (1).

Hence the exponential Diophantine equation $(7^{2m}) + 1 = \omega^2$, where m, ω are the whole numbers, has no solution in whole number.

Lemma: 2 The exponential Diophantine equation $1 + (6^{r+1} + 1)^n = \omega^2$*, where* r, n, ω *are whole numbers, has no solution in whole number.*

Proof: Since $(6^{r+1} + 1)$ is an odd number for all whole number r so $(6^{r+1} + 1)^n$ is an odd number for all whole numbers r and n.

$\Rightarrow 1 + (6^{r+1} + 1)^n = \omega^2$ is an even number for all whole numbers r and n.

$\Rightarrow \omega$ is an even number

$\Rightarrow \omega^2 \equiv 0(mod3)$ or $\omega^2 \equiv 1(mod3)$ (3)

Now $(6^{r+1} + 1) \equiv 1(mod3)$, for all whole number r.

$\Rightarrow (6^{r+1} + 1)^n \equiv 1(mod3)$, for all whole numbers r and n.

$\Rightarrow 1 + (6^{r+1} + 1)^n \equiv 2(mod3)$, for all whole numbers r and n.

$\Rightarrow \omega^2 \equiv 2(mod3)$ (4)

Equation (4) contradicts equation (3).

Hence the exponential Diophantine equation $1 + (6^{r+1} + 1)^n = \omega^2$, where r, n, ω are whole numbers, has no solution in whole number.

Main Theorem: The exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$, where m, n, r, ω are whole numbers, has no solution in whole number.

Proof: There are four cases:

Case: 1 If $m = 0$ then the exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$ becomes$1 + (6^{r+1} + 1)^n = \omega^2$, which has no whole number solution by lemma 2.

Case: 2 If $n = 0$ then the exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$ becomes $(7^{2m}) + 1 = \omega^2$, which has no whole number solution by lemma 1.

Case: 3 If m, n are positive integers, then $(7^{2m}), (6^{r+1} + 1)^n$ are odd numbers.

$\Rightarrow (7^{2m}) + (6^{r+1} + 1)^n = \omega^2$ is an even number

$\Rightarrow \omega$ is an even number

$\Rightarrow \omega^2 \equiv 0(mod3)\ or\ \omega^2 \equiv 1(mod3)$ (5)

Now $7 \equiv 1(mod3)$

$$\Rightarrow \begin{bmatrix} (7^{2m}) \equiv 1(mod3) \\ and\ (6^{r+1} + 1) \equiv 1(mod3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (7^{2m}) \equiv 1(mod3) \\ and\ (6^{r+1} + 1)^n \equiv 1(mod3) \end{bmatrix}$$

$\Rightarrow (7^{2m}) + (6^{r+1} + 1)^n \equiv 2(mod3)$

$\Rightarrow \omega^2 \equiv 2(mod3)$ (6)

Equation (6) contradicts equation (5).

Hence the exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$, where m, n are positive integers and r, ω are whole numbers, has no solution in whole number.

Case: 4 If $m, n = 0$, then $(7^{2m}) + (6^{r+1} + 1)^n = 1 + 1 = 2 = \omega^2$, which is impossible because ω is a whole number. Hence exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$, where $m, n = 0$ and r, ω are whole numbers, has no solution in whole number.

III. CONCLUSION

In this article, authors successfully discussed the existence of the solution of exponential Diophantine equation$(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$, where m, n, r, ω are whole numbers. They determined that the exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$, where m, n, r, ω are whole numbers, has no solution in whole number.

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