Optimal Management of Forestry Biomass Affected with Toxicant

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***Abstract*: The optimum harvest strategy for forestry biomass influenced by a toxicant is suggested and studied by using a non-linear mathematical model. The nature and uniqueness of equilibrium, conditions for existence of their local and global equilibrium points, are all established. Both equilibrium levels of biomass and total sustainable yield decrease as toxicant concentrations rise, as per the analysis. The optimum harvest approach is frequently discussed using Pontryagin's Maximum principle. Numerical analysis is implemented to validate the mathematical findings.**

***Keywords*: Forestry biomass, stability, optimal harvesting, toxicant; Pontryagin's Maximum principle**

AMS subject classification: 92D25.

I. INTRODUCTION

The toxicant and contaminants releasedfrom various industries, vehicles and other man-made projects leavedeleterious effect on biological and forestry biomass. Some studies on the impact of toxicants on biological populations have been conducted in recent years. In particular, (Hallam et al., 1984) in a series of their papers studied qualitative approach oftoxicants on populations. They assumed growth rate density of singlespecies population as decreasing function of concentration oftoxicant but the corresponding carrying capacity being unaffected bythe presence of toxicant in environment. Taking this inconsideration, (Freedman and Shukla, 1991) studied the effect oftoxicant on a single species and on a predator-prey system by takinginto account the introduction of toxicant from an external source. Further (Shukla et al., 2009) studied the effects of toxicants on resource dependent population where they considered toxicant emitted from the external sources and by its precursor. The effect of environmental toxicant on resource biomass has been studied by (Gakkhar and Sahani, 2009) and (Naresh et al., 2014), and it was found that growth rate of plants was affected because of uptake of pollutants which lead to a decrease in resource biomass. (Shukla and Dubey, 1997) in their paper discussed the combined effect of population and pollution on depletion of forestry biomass. (Lata. K et al., 2016) studied the effect of industrialization on forestry resources in which they assess the effect of wood and non-wood based industries on the depletion of forestry biomass and it was found that the level of pollutants because of wood and non-wood based industries increases the metabolism of forestry resources gets affected because uptake of these pollutants by the forestryresources. Depletion and conservation of Forestry biomass in presence of industrialization was studied by (Mishra & Lata, 2015) by considering that due to forestry biomass industries migrate and due to availability of forestry biomass their growth increases. (Agarwal M. et al., 2011) studied the effect of toxicant on resource dependent competition model where toxicant emitted from external sources and formed by precursors of competing species. But the adverse effects of toxicant onharvesting of forestry biomass and that on species are yet to be studied.In these days, considerable interest has been taken in finding outthe optimal harvest policy of forestry biomass and species. Thebasic idea related to the field of Bio-economic modeling wasintroduced by (Clark, 1976, 1979, 1985). He discussed several aspects ofharvesting of renewable resources by examples from fishery. But itwas clearly mentioned in books and other references that sametechniques can be used for optimal management of otherrenewableresources (e.g. other species, forestry biomass). (Bhattacharya & Begum, 1996) discussed the feasible bioeconomic equilibrium points for alogistic growth model of two ecologically independent species, twocompeting species and Lotka-Volterra model of one prey and one predator. (Pradhan & Chaudhuri, 1999) investigated a dynamic model oftwo species fishery with tax as control variable and obtained itsoptimal policy. Later, they investigated the harvesting of a schooling fish species. (Dubey et al., 2002) in their paper discussed the dynamics offish population partially dependent on a logistically growingresource with functional response and the harvest termwas assumed tobe proportional to both stock level and effort.Dubey and Patra (2013) studied a mathematical model for the optimization and utilization of renewable resource by population. Keeping all these in mind, a model is proposed to study the harvesting of toxicantbiomass. A mathematical model is developed by means of a system ofordinarydifferential equations. Stability analysis for differentequilibrium points, local and global stability of equilibriums andalso region of attraction are discussed. MSY and optimal policy toharvest biomass are obtained and in last, results are discussed withsome numerical analysis.

II. MATHEMATICAL MODEL

A forestry biomass is considered which is affected by a toxicant present in the environment as well as by that present in itself and subjected to harvesting. The forestry biomass concentration is denoted by B(t), T(t) and U(t) are the concentrations of toxicant present in environment and in the biomass (Dubey et al., 2002) respectively. E(t) denotes the effort applied to harvest biomass. We make some assumptions in the modelling of differential equations which govern the system.These are given as follows :

H1 : Carrying capacity of forestry biomass is adversely affected by the presence of toxins in the environment i.e. if K(T) represents toxicant dependent carrying capacity, then$K(0) = K_0 > 0$, $K'(T) < 0, \forall\ T \geq 0.\exists$ a $\bar{T}$, such that $K(\bar{T}) = 0$.

H2 : Growth rate function of forestry biomass decreases with increase in toxicant concentration in biomass i.e., if r(U) is growth rate function, then$r(0) = r_0 > 0$, $r'(U) < 0 \forall\ U \geq 0$ and$\exists$ a $\bar{U}$, such that $r(\bar{U}) = 0$.

H3 : Forestry biomass is harvested in direct proportion to the product of its concentration and applied effort with constant catchability coefficient q (i.e. h = qBE), where h(t) = total harvest at time t.

H4 : Forestry biomass is subjected to dynamic harvesting with a tax $\tau > 0$ which is imposed by regulatory agency in order to maintain desired level of forestry biomass.

Toxicant in the environment is introduced beyond its initial concentration with positive constant rate Q_0 and washed out with constant rate δ_0. One example of such toxicant is Chomney exhaust into the atmosphere that affects fishery. Another is of dumping of heavy metal pollutant in a lake or ground that effects marine ecosystem or forestry. α is the depletion rate coefficient of toxicant in the environment due to its intake by biomass. The toxicantconcentration in biomass may be depleted from environment with constant rate δ_0 and may also be removed from biomass in proportion to their concentrations with proportionality constant γ. The term 'γBU' corresponds to decrease in toxicant concentration in biomass due to decay of toxicated biomass. This decayed biomass is now a part of environment which may in turn pollute soil or water. So a fraction of γBU (i.e. $\pi\gamma BU$) contributes in toxicant contribution in the environment.

Above assumptions and considerations lead to the following system of differential equations :

$$\frac{dB}{dt} = r(U)B - \frac{r_0 B^2}{K(T)} - qEB$$

$$\frac{dT}{dt} = Q_0 - \delta_0 T - \alpha BT + \pi\gamma BU \quad (1)$$

$$\frac{dU}{dt} = -\delta_1 U + \alpha BT - \gamma BU$$

$$\frac{dE}{dt} = \alpha_0 E[(p-\tau)qB - c]$$

$B(0) \geq 0, \quad T(0) \geq 0, \quad U(0) \geq 0, \quad E(0) \geq 0, \quad 0 \leq \pi \leq 1.$

Here, p is the fixed price per unit of forestry biomass and c is the cost of harvesting per unit effort applied. The constant α_0 is called stiffness parameter measuring the strength of reaction of effort to the perceived rent.

III. EXISTENCE AND STABILITY ANALYSIS OF EQUILIBRIUM POINTS

The system (1) has three non–negative equilibrium points namely $P_1\left(0, \frac{Q_0}{\delta_0}, 0, 0\right)$, $P_2(\tilde{B}, \tilde{T}, \tilde{U}, 0)$ and

P*(B*, T*, U*, E*). The existence of $P_1\left(0, \frac{Q_0}{\delta_0}, 0, 0\right)$ is obvious. $P_2(\tilde{B}, \tilde{T}, \tilde{U}, 0)$ is solution of equations

$$\tilde{B} = \frac{r(\tilde{U})K(\tilde{T})}{r_0}$$

$$\alpha\tilde{T} = \left[\frac{r_0\delta_1}{r(\tilde{U})K(\tilde{T})} + \gamma\right]\tilde{U} \qquad 2(a)$$

$$\delta_0\tilde{T} = Q_0 - \left[\delta_1 + \frac{(1-\pi)\gamma r(\tilde{U})K(\tilde{T})}{r_0}\right]\tilde{U} \qquad 2(b)$$

Equation 2(a) represents $\tilde{T}$ as an increasing function of $\tilde{U}$ from zero and 2(b) represents $\tilde{T}$ as decreasing function of $\tilde{U}$ from$\frac{Q_0}{\delta_0}$to zero.Hence, two isoclines given by 2(a) and 2(b) intersect at unique point $(\tilde{T}, \tilde{U})$, provided $\tilde{T} < \bar{T}$ and $\tilde{U} < \bar{U}$, so P_2 exists.

P*(B*,T*,U*,E*) is given by equations

$$B^* = \frac{c}{(p-\tau)q} \qquad (3a)$$

$$E^* = \frac{1}{q}\left[r(U^*) - \frac{r_0 B^*}{K(T^*)}\right] \qquad (3b)$$

$$\alpha T^* = \left[\gamma + \frac{\delta_1 q(p-\tau)}{c}\right]U^* \qquad (3c)$$

$$\delta_0 T^* = Q_0 - \left[\delta_1 + \frac{(1-\pi)\gamma c}{(p-\tau)q}\right]U^* \qquad 3(d)$$

Again from 3(c), T* is increasing function of U* from zero and equation 3(d) shows that T* is decreasing function of U* from $\frac{Q_0}{\delta_0}$ to zero. So sections of the two hyper–cylinders given by 3(c)) and 3(d) in the T – U plane must intersect at unique point (T*, U*) provided T* < $\bar{T}$ and U* < $\bar{U}$.

For the existence of P*, positivety of B* and E* implies

$$p > \tau,$$

$$r(U^*) > \frac{r_0 B^*}{K(T^*)}$$

$$\Rightarrow \tau < p - \frac{r_0 c}{qK(T^*) r(U^*)}$$

Above equation gives the uppper bound for the tax which is also affected by concentration of toxicant in the environment and in biomass. Also, it is clear, that increase in concentration of toxicant in environment or in biomass decreases the equilibrium value of the harvesting effort which may even be zero when

$$r(U^*) = \frac{r_0 B^*}{K(T^*)}$$

In that case, equilibrium point P_2 exists and we can not harvest biomass. In order to determine stability nature of equilibrium points, the corresponding variational matrices are considered. The characteristic matrix of P* can be written as

$$M^* = \begin{bmatrix} \frac{-r_0 B^*}{K(T^*)} & \frac{r_0 B^{*2} K'(T^*)}{K^2(T^*)} & r'(U^*)B^* & -qB^* \\ -\alpha T^* + \pi\gamma U^* & -\delta_0 - \alpha B^* & \pi\gamma B^* & 0 \\ \alpha T^* - \gamma U^* & \alpha B^* & -\delta_1 - \gamma B^* & 0 \\ \alpha_0 (p-\tau) q E^* & 0 & 0 & \alpha_0\{(p-\tau)qB^* - c\} \end{bmatrix}$$

From variational matrix analysis, it is clear that P_1 is a saddle point with one dimensional unstable and three-dimensional stable manifold. P_2 is a saddle point with one dimensional unstable and three-dimensional stable manifold if $\tilde{B} > c/(p-\tau)q$ and P_2 is locally asymptotically stable if $\tilde{B} < c/(p-\tau)q$. Global stability in this case can be analyzed in a similar manner as was done in Freedman and Shukla [8]. When r(U*) = r_0B*/K(T*) i.e. E* = 0, then P* is a saddle point with one dimensional unstable and three dimensional stable manifold if B* > c/(p – τ)q. Similarly when E* = 0, P* is locally asymptotically stable if B* < c/(p – τ)q. When E* ≠ 0, the characteristic equation of P* can be written as

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0 \qquad (4)$$

$$b_1 = \frac{r_0 B^*}{K(T^*)} + \delta_0 + \delta_1 + (\alpha+\gamma)B^* \text{ where}$$

$$b_2 = \frac{r_0 B^*}{K(T^*)}(\delta_0 + \delta_1 + (\alpha+\gamma)B^*) + (\delta_0 + \alpha B^*)(\delta_1 + \alpha B^*) - \pi\alpha\gamma B^{*2}$$

$$+ \frac{r_0 B^{*2} K'(T^*)}{K^2(T^*)}(\alpha T^* - \pi\gamma U^*) - r'(U^*)B^*(\alpha T^* - \gamma U^*) + q^2 B^* \alpha_0 (p-\tau)E^*$$

$$b_3 = \frac{r_0 B^*}{K(T^*)}\{(\delta_0 + \alpha B^*)(\delta_1 + \gamma B^*) - \pi\gamma\alpha B^{*2}\}$$

$$- \frac{r_0 B^* K'(T)}{K^2(T^*)}\{(\delta_1 + \gamma B^*)(-\alpha T^* - \pi\gamma U^*) + \pi\gamma B^*(-\alpha T^* - \gamma U^*)\}$$

$$- r'(U^*)B^*\{\overline{\pi - 1}\alpha\gamma B^* U^* + \delta_0(\alpha T^* - \gamma U^*)\}$$

$$b_4 = qB^*\alpha_0(p-\tau)E^*\{(\delta_0 + \alpha B^*)(\delta_1 + \gamma B^*) - \pi\alpha\gamma B^{*2}\}$$

Clearly $b_1 > 0$ and $b_4 > 0$. Now, if $b_2 > 0$, $b_3 > 0$ and $b_1 b_2 b_3 > b_1^2 b_4 + b_3^2$ then by Routh Hurwitz criterion, all roots of equation (4) will have negative real part. So, under these conditions, P* is locally asymptotically stable point.

Now, before proving theorem which gives conditions for global stability, we require a lemma which establishes a region of attraction for the system (1).

Lemma : The set

Ω = {(B, T, U, E) : 0 ≤ B ≤ K_0, 0 ≤ T + U ≤ 0 ≤ α_0 (p – τ) B + E ≤ A}

$$A = \frac{K_0[r_0 + \in \alpha_0(p-\tau)]^2}{4r_0 \in}, \quad 0 < \in < \alpha_0 c\Big\}$$ attracts all solutions of system (1) in the interior of the positive octant,Where

δ = min(δ_0, δ_1),

Theorem1: Let r(U), K(T) satisfy the conditions

$0 \le -r'(U) \le \rho_1$, $0 \le -K'(T) \le \rho_2 K_c \le K(T) \le K_0$, for some positive constants ρ_1, ρ_2, K_c and K_0 then if following inequalities hold

$$\left[\rho_1 + \gamma\frac{Q_0}{\delta} + \alpha T^*\right]^2 < \frac{r_0}{K(T^*)}(\delta_0 + \gamma B^*) \qquad 4(a)$$

$$\left[\frac{r_0 K_0 \rho_2}{K_c^2} + \alpha\frac{Q_0}{\delta} + \pi\gamma U^*\right]^2 < \frac{r_0}{K(T^*)}(\delta_1 + \gamma B^*) \qquad 4(b)$$

$$[\pi\gamma + \alpha]^2 K_0^2 < (\delta_0 + \alpha B^*)(\delta_1 + \gamma B^*) \qquad 4(c)$$

Then P* is globally asymptotically stable w.r.t. all solutions initiating in the positive octant.

Proof. Let us consider positive definite Liapunov function as

$$V = B - B^* - B^* \ln\frac{B}{B^*} + \frac{1}{2}(T-T^*)^2 + \frac{1}{2}(U-U^*)^2 + k\left[E - E^* - E^* \ln\frac{E}{E^*}\right]$$

$$\dot V = (B-B^*)\left[r(U) - \frac{r_0 B}{K(T)} - qE\right] + (T-T^*)[Q_0 - \delta_0 T - \alpha BT + \pi\gamma BU]$$

$+ (U - U^*) [-\delta_1 U + \alpha BT - \gamma BU] + k\alpha_0 [(p - \tau) qB - c]$

$(E - E^*)$

Choosing $$k = \frac{1}{\alpha_0(p-\tau)}$$

After some algebraic manipulations, considering the equilibrium points and functions

$$\eta_1(U) = \frac{r(U) - r(U^*)}{U - U^*} \qquad U \ne U^*$$

$$= r'(U^*) \qquad U = U^*$$

$$\eta_2(T) = \frac{1}{(T-T^*)}\left[\frac{1}{K(T)} - \frac{1}{K(T^*)}\right], \quad T \ne T^*$$

$$= -\frac{K'(T^*)}{K(T^*)^2} \qquad T = T^*$$

Now $\dot V$ take the form

$$\dot V = -\frac{1}{2}b_{11}(B-B^*)^2 + b_{12}(B-B^*)(T-T^*) - \frac{1}{2}b_{22}(T-T^*)^2$$

$$-\frac{1}{2}b_{11}(B-B^*)^2 + b_{13}(B-B^*)(U-U^*) - \frac{1}{2}b_{33}(U-U^*)^2$$

$$-\frac{1}{2}b_{22}(T-T^*)^2 + b_{23}(U-U^*)(T-T^*) - \frac{1}{2}b_{33}(U-U^*)^2$$

where, $b_{11} = \dfrac{r_0}{K(T^*)}$, ,

$b_{22} = \delta_0 + \alpha B^*$, $b_{33} = \delta_1 + \gamma B^*$ $b_{12} =$

$\eta_1(U) + \alpha T^* - \gamma U$, $b_{13} = r_0 B \eta_2(T) - \alpha T + \pi\gamma U^*$, $b_{23} = (\pi\gamma + \alpha) B$.

Then conditions for $\dot V$ to be negative definite are that following inequalities hold

$b_{12}^2 < b_{11} b_{22}$ 5 (a)

$b_{13}^2 < b_{11} b_{33}$ 5(b)

$b_{23}^2 < b_{22} b_{33}$ 5(c)

It is clear that (4(a)) $\Rightarrow$ (5(a)),(4(b)) $\Rightarrow$ (5(b)) and (4(c)) $\Rightarrow$ (5(c)).So, if inequalities given in theorem hold, then $\dot V$ is negative definite function.

Let us consider subset S of Ω as

$$S \equiv \{(B, T, U, E) \in \overline{\Omega} : V = 0\}$$

the largest invariant set in S is $\{(B, T, U, E) \in \overline{\Omega} : B = B^*, U = U^*, T = T^*, E = E^*\}$. So by LaSalle's invariance principle, P* is globally asymptotically stable equilobrium point.

IV. MAXIMUM SUSTAINABLE YIELD

The Maximum Sustainable Yield (MSY) of forestry biomass may be defined as the maximum rate at which it can be harvested even after maintaining biomass concentration at constant level and any larger harvest rate will lead to the depletion of biomass eventually to zero.The sustainable yield is given by

$$h = qE^*B^* = \left[r(U^*) - \frac{r_0 B^*}{K(T^*)}\right]B^*$$

$$\frac{\partial h}{\partial B^*} = 0 \Rightarrow B^* = \frac{r(U^*)K(T^*)}{2r_0}$$

Also,

Hence, $$h^0_{MSY} = \frac{r_0 K_0}{4}.$$

$$h_{MSY} = \frac{r^2(U^*)K(T^*)}{4r_0}$$

It is clear that $h^0_{MSY} > h_{MSY}$, so MSY also decreases with increase in toxicant concentration level and it is maximum in the absence of toxicant.

V. OPTIMAL HARVESTING POLICY

In this section, the optimal harvesting policy is discussed which plans to maximize the total discounted net revenue from the harvesting using taxation as control instrument.The net economic revenue to society, π(B, T, U, E, τ)= net revenue to regulatory agency + net revenue to the harvestor

= τqBE + [(p – τ) qB – c] E

= (pqB – c) E

Our objective is to solve the problem

$$\max \int_0^\infty e^{-st}(pqB-c)Edt \qquad (5)$$

subject to state equations of (1) and control constraint,

$$\tau_{min} \le \tau \le \tau_{max} \qquad (6)$$

where s is the instantaneous annual rate of discount.

Using Pontryagin's principle,for solving problem (5) for which asociated Hailtonian is given by

$$H(B,T,U,E,\tau,t) = e^{-st}(pqB-c)E + \lambda_1(t)\left[r(U)B - \frac{r_0B^2}{K(T)} - qBE\right]$$

$+ \lambda_2(t)\ [Q_0 - \delta_0T - \alpha BT + \pi\gamma BU] + \lambda_3(t)\ [-\delta_1U + \alpha BT - \gamma BU] + \lambda_4(t)\alpha_0E[(p-\tau)qB - c]$

where λ_1, λ_2, λ_3 and λ_4 are adjoint variables.

For H to be maximum on control set, we must have

$$\frac{\partial H}{\partial \tau} = 0 \Rightarrow \lambda_4(t) = 0 \qquad (7)$$

Now, as per maximum principle rule, we have

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial B} \quad , \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial T} \qquad (8)$$

Now use of eq.(7) and last relation of eq.(8) give

$$\lambda_1(t) = e^{-st}\left(p - \frac{c}{qB}\right) \qquad (9)$$

In order to obtain optimal equilibrium, we use relations given by eq.(8) and eq.(9)with interior equilibrium point and obtain the solution

$$\lambda_2(t) = \frac{A_2e^{-st}}{A_1+s} \qquad (10)$$

where, $A_1 = \delta_0 + \alpha B^* - \frac{(\alpha T^* - \pi\gamma U^*)}{\alpha T^* - \gamma U^*}\alpha B^*$

$$A_2 = \left(p - \frac{c}{qB^*}\right)\left\{\frac{r_0B^{*2}K'(T^*)}{K(T^*)^2} + \frac{\alpha B^*}{\alpha T^* - \gamma U^*}\left(s + \frac{r_0B^*}{K(T^*)}\right)\right\} - \frac{\alpha pqB^*E^*}{\alpha T^* - \gamma U^*}$$

,

And , $\lambda_3(t) = \frac{B_2e^{-st}}{B_1+s}$ (11)

Where, $B_1 = \delta_1 + \gamma B^*$ and

$$B_2 = r'(U^*)B^*\left(p - \frac{c}{qB^*}\right) + \frac{A_2\pi\gamma B^*}{A_1+s}$$

Also,

$$\lambda_1 = \frac{C_1e^{-st}}{C_1+s} \qquad (12)$$

Where,

$C_1 = \frac{r_0B*}{K(T*)}$, $C_2 = \frac{B_2}{B_1+s}(\alpha T^* - \gamma U^*) - \frac{A_2}{A_1+s}(\alpha T^* - \pi\gamma U* + pqE*$.

Equations (9) and (12) gives,

$$\frac{C_2}{C_1+s} = p - \frac{c}{qB^*} \qquad (13)$$

The above equation gives the optimal equilibrium level of forestry biomass along with toxicant concentration in environment and in biomass i.e.

$B^* = B_s$, $T^* = T_s$, $U = U_s$

Then, optimal levels of effort and tax are given as,

$$E_s = \frac{1}{q}\left[r(U_s) - \frac{r_0B_s}{K(T_s)}\right] \qquad \tau_s = p - \frac{c}{qB_s}$$

From equation (13)

$$pqB^* - c = \frac{C_2qB^*}{C_1+s} \to 0 \quad \text{as } s \to \infty$$

This shows that net economic revenue to society vanishes when discount rate is infinite and hence the harvesting of biomass remains closed.Also, zero discount rate gives the maximum value of net revenue to the society for definite effort.

VI. NUMERICAL ANALYSIS

Let us choose most general form of growth rate and carrying capacity of biomass as,

$r(U) = r_0 - r_{01}U$, $K(T) = K_0 - K_{01}T$

Taking$r_0 = 15$, $\alpha = 0.6$, $s = 1$, $K_0 = 20$, $Q_0 = 35$, $r_{01} = .02$, $q = .1$, $K_{01} = .02$, $p = 25$, $\delta_0 = 10$, $c = 8$, $\delta_1 = 12$, $\alpha_0 = 1$,

$\pi = 0.5$ and $\gamma = .05$

With different values of tax , we obtain equilibrium points as

τ	B*	T*	U*	E*
5	4.0000	2.8271	0.5560	119.8038
10	5.3333	2.6586	0.6934	109.7597
15	8.0000	2.3773	0.9203	89.6730
20	16.0000	1.8131	1.3606	29.5100
22	19.9276	1.6278	1. 4981	0.0000

It can be verified that conditions for local stability are satisfied in each case. Also by choosing $\rho_1 = \rho_2 = .02$ and Kc = 2 from theorem (3.1), it can be checked that inequalities for global stability are satisfied so P* is stable equilibrium point in each case. There exist a value of tax$\tau = 22$for which E*=0, after that harvesting remains closed and biomass attains its maximum carrying capacity.

Figures have been plotted between dependent variables and time for different parameter values to show changes occurring in parameters with time under different conditions. The results of numerical simulation are displayedgraphically. From figure it is noted for given initial values the forestry biomass concentration, toxicants present in environment as well as biomass and effort applied to harvest biomass tend to their corresponding value of equilibrium point$P^*$and hence coexist in the form of steady state assuring global stability ofP^*.

In Figure 1, we have considered the four different initial values of the forestry biomass concentration and toxicant concentration present in biomass . All trajectories starting from different initial values approach to(B^*, U^*). This point is independent of the initial status. This shows that (B^*, U^*) is globally asymptotically stable in B U-plane.

In Figure 2, we have considered the four different initial values of the forestry biomass concentration and toxicant concentration present in environment. All trajectories starting from different initial values approach to (T^*, U^*). This point is independent of the initial status. This shows that (T^*, U^*) is globally asymptotically stable in T U-plane.

In Figure 3, we have considered the four different initial values of the toxicant present in biomass and effort applied to harvest biomass. All trajectories starting from different initial values approach to (U^*, E^*). This point is independent of the initial status. This shows that (U^*, E^*) is globally asymptotically stable in U E-plane.

From figure 4-7, we can depict that Forestry biomass and toxicant present in biomass increases and toxicant present in environment decreases with the increase in the value of tax τ, and finally attain their equilibrium levels. This is obvious as the tax increases harvesting effort for biomass decreases and becomes closed for $\tau = 22$.

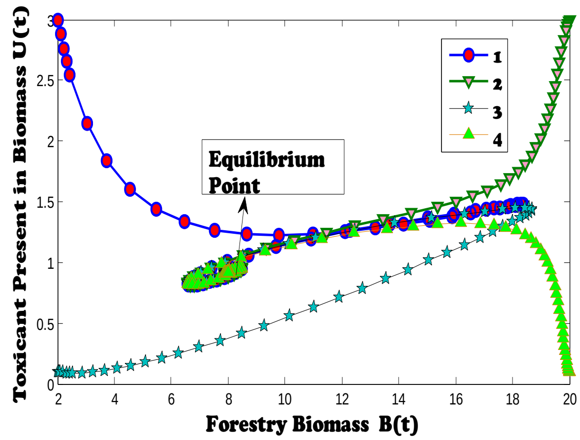

Figure 1. Global stability in B and U

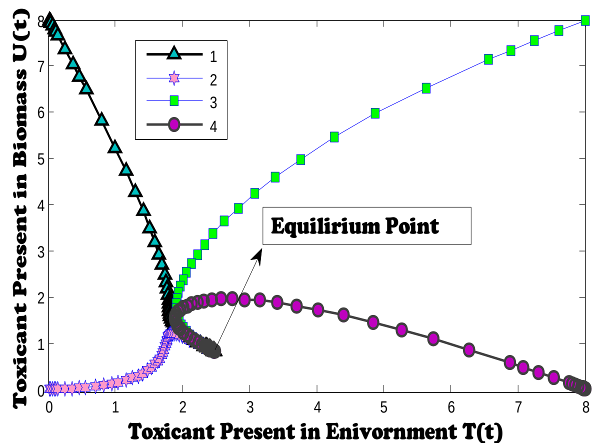

Figure 2. Global stability in T and U

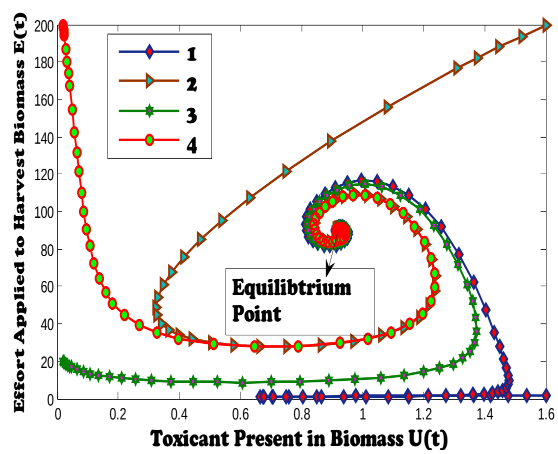

Figure 3. Global stability in U and E

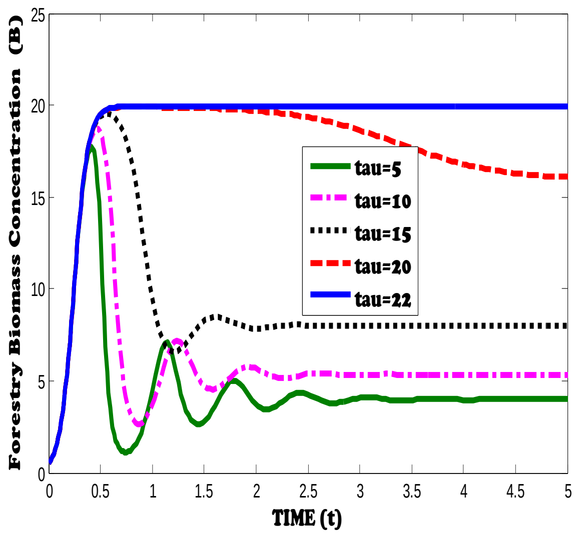

Figure4.Variation of B with time for different values of taxes.

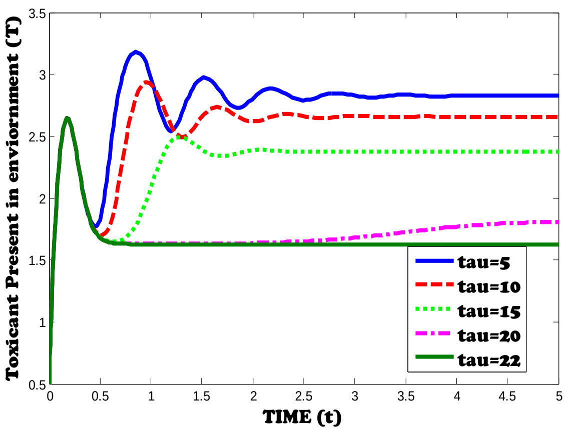

Figure 5. Variation of T with time for different values of taxes.

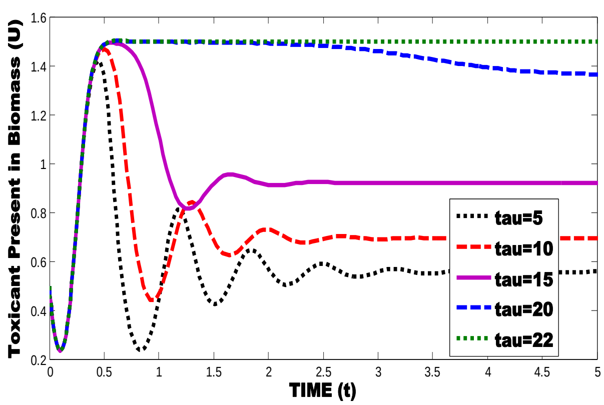

Figure 6.Variation of U with time for different values of taxes.

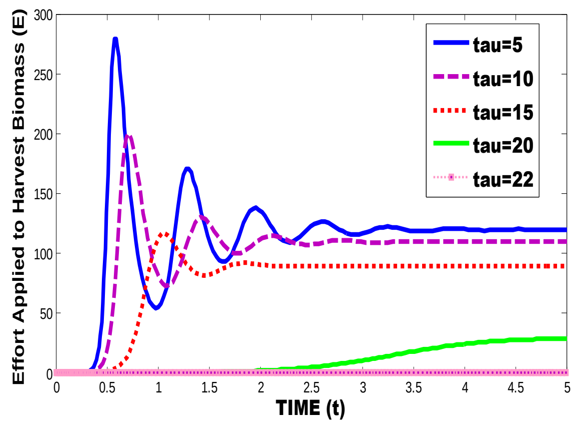

Figure 7.Variation of E with time for different values of taxes.

VII. CONCLUSION

In the present paper, a mathematical model to study the optimal harvest policy for toxicant effected forestry bionass has been discussed. Constant introduction of toxicant into the environment and dynamic harvesting effort of biomass with tax as control instrument have been taken.

The existence and uniqueness of equilibrium points, their local and global stability have been discussed. It has been shown that system is uniformly bounded, which implies that system is biologically well behaved. In last MSY and optimal harvest policy for forestry biomass have been obtained. The following results are clear from the discussion :

i. Increase in toxicant concentration, lowers the equilibrium level of the system.
ii. It has been noted that if toxicant level is sufficiently high, then biomass can not reproduce or grow and infact it will extinct.
iii. Applied harvesting effort and upper bound for tax are also affected by toxicant concentration. Even if concentration of toxicant increases without any control, harvesting must be stopped.
iv. The value of maximum yield for forestry biomass can be increased by lowering toxicant concentration.
v. If price per unit of biomass increases faster than the cost involved in harvesting, forestry biomass settles down to lower equilibrium level.
vi. If annual rate of discount is sufficiently high, then net revenue to the society vanishes and zero discount rate gives the maximum net revenue to the society for definite effort.

Introduction of periodic influx and effort dependent toxicant can be discussed in similar pattern. This model can be generalized for two or more interacting species harvested in a toxicated environment with mode of interaction being competition, cooperation or predation. This is left for future research.

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Appendix:

Proof of Lemma: From eq.(1)

$$\frac{dB}{dt} \le r_0 B - \frac{r_0 B^2}{K_0}$$

$$\Rightarrow B \le \frac{K_0}{1 + \left[\frac{K_0}{B(0)} - 1\right] e^{-r_0 t}}$$

$\Rightarrow B \le K_0$ as $t \to \infty$.

Again fron eq.(1)

$$\frac{d}{dt}(U + T) = Q_0 - \delta_0 T - \delta_1 U + (\pi - 1)\gamma BU$$

$$\le Q_0 - \delta(T + U), \quad \delta = \min(\delta_0, \delta_1).$$

$U + T \le \frac{Q_0}{\delta}$. It is positive constant $e^{-\delta t}$ $as\ t \to \infty,$

$$U(t) + T(t) \le \frac{Q_0}{\delta}.$$

Also taking $\in > 0$

$$\frac{d}{dt}[\alpha_0(p - \tau)B + E] + \in [\alpha_0(p - \tau)B + E]$$

On integrating and taking limit $t \to \infty$, we get,

$$\alpha_0(p - \tau)B + E \le \frac{\alpha_0(p - \tau)K_0(r_0 + \in)^2}{4r_0 \in}$$

Hence lemma follows.

$$= \left[r(U)B - \frac{r_0 B^2}{K(T)}\right]\alpha_0(p - \tau) - \alpha_0 CE + \in \alpha_0(p - \tau)B + \in E$$

$$= \alpha_0(p - \tau)B\left\{r(U) + \in - \frac{r_0 B}{K(T)}\right\} + (\in - \alpha_0)E$$

Choosing $\in < \alpha_0 C$, since $r(0) = r_0$, $K(0) = K_0$ and $r'(U) < 0$, $K'(T) < 0$.

$$\frac{d}{dt}[\alpha_0(p - \tau)B + E] + \in [\alpha_0(p - \tau)B + E] \le \alpha_0(p - \tau)\left\{(r_0 + \in)B - \frac{r_0 B^2}{K_0}\right\}$$

$$= -\alpha_0(p - \tau)\left[\left\{\sqrt{\frac{r_0 B}{K_0}} - \frac{(r_0 + \in)}{2}\sqrt{\frac{K_0}{r_0}}\right\}^2 - \frac{(r_0 + \in)^2 K_0}{4r_0}\right]$$

$$\le \frac{\alpha_0(p - \tau)K_0(r_0 + \in)^2}{4r_0}$$