Graphical Interpretation of the Slopes Used in the Derivation of Classical Fourth Order Runge-Kutta (RK4) Formula

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***Abstract*: Many practical issues in science and engineering are formulated by ordinary differential equations (ODE) that require their own numerical solution. There are a variety of numerical approaches, e.g. the Euler method, the modified Euler method, the Heun's method, the Adam-Bashforth method, and so on, that exist in the context of numerical analysis. Amongst them, the classical fourth order Runge-Kutta (RK4) technique is the most reliable and most used. The objective of this paper is twofold. The first goal is to derive the value of different parameters in the formulation of the fourth order Runge-Kutta method, and the second goal is to give details of the geometrical interpretation of this method, principally explaining the role of the increment parameters k_i in the formula. The whole discussion will facilitate perception of the key mechanism of the Runge-Kutta method.**

***Keywords*: Runge-Kutta method, Euler method, Heun's method, Increment parameter.**

I. INTRODUCTION:

Around 1900, two German mathematicians, Curl Runge and Wilhelm Kutta, devised the Runge-Kutta techniques in numerical analysis [1]. In 1895, C. Runge presented a work that was a more complex development and was an extension of the Euler method's approximation. To determine the numerical solution of differential equations, various order Runge-Kutta techniques have been widely utilized [2-3]. To solve second-order fuzzy differential equations, a novel version of the enhanced Runge-Kutta Nystrom technique is used [4]. For the numerical solution of *n*-th order fuzzy differential equations based on the Seikkala derivative with initial value issue [5-8], the Runge-Kutta technique of order five is utilized. Also, the fourth and fifth-order Runge-Kutta techniques [9-11] are used to the specific Lorenz equation. In [12-14], implicit and multistep Runge-Kutta techniques are investigated. Euler and Coriolis [15-16] explore the fundamental concepts of differential equation theory and their numerical solution. In the articles of Runge [17], Heun [18], and Nystrom [19], the early works of the Runge-Kutta technique are examined. Adams and Bashforth, Dahlquist [21-22], and Moulton have published the foundations of multistep Runge-Kutta techniques. Recently, some work on the Runge-Kutta technique has been published, for example, Mechee and Yasen, Geeta and Varun [27-28] used extended RK integrators to solve ordinary differential equations. Vijeyata and Pankaj describe computational approaches for solving differential equations based on the Runge-Kutta method of various orders and kinds. Wusu and Akanbi investigated explicit fourth-derivative two-step linear multistep techniques for solving ordinary differential equations.

Runge-Kutta techniques as well as their many variants are used to solve differential equations in a number of areas, as this brief overview shows. The Runge-Kutta method's formulae may be found in most textbooks [24-25]. Some books and publications provide the Runge-Kutta technique in a larger framework, rather than delving into the method in its final form [2]. Butcher displays the Runge-Kutta method's parameter values in a table, but in a compact manner [3]. The goal of this study is to demonstrate how to obtain various arbitrary parameters of the most commonly used fourth order Runge-Kutta method in the numerical analysis as well as the geometrical interpretation of the slope k_i's at various stages. The majority of the information in this paper is sourced mostly from [2] and [25].

This paper is arranged as follows: An introduction is given in section 1. The determining procedure of different parameters in the formation of the fourth order Runge-Kutta is described in detail in section 2. The geometrical concept of this method is discussed sequentially in section 3 that explained with figures. Sections 4 through 5 contain the discussion and conclusion.

II. FORMULA DERIVATION OF THE FOURTH ORDER RUNGE-KUTTA (RK4) METHOD

The key idea of the fourth order Runge-Kutta method is to find the numerical solution of the first order ordinary differential equation

$$\frac{dy}{dx} = f(x,y),\ \ y(x_0) = y_0 \qquad (1)$$

In spite of the fact that the Runge-Kutta technique has several variations, it is best described as follows:

$$y_{m+1} = y_m + hf(x_m, y_m) \qquad (2)$$

where $hf(x_m, y_m)$ is an increment function and the slope $f(x_m, y_m)$ may be recast as

$$f = a_1k_1 + a_2k_2 + a_3k_3 + \cdots + a_nk_n \tag{3}$$

The a_i s and k_i s in Eq. (3) are arbitrary constants where the general form of k_i s are given by

$$\begin{aligned} k_1 &= f(x_m, y_m) \\ k_2 &= f(x_m + u_1h, y_m + v_{11}k_1h) \\ k_3 &= f(x_m + u_2h, y_m + v_{21}k_1h + v_{22}k_2h) \\ &\vdots \\ k_n &= f(x_m + u_{n-1}h, y_m + v_{n-1,1}k_1h + v_{n-2,2}k_2h + \cdots + v_{n-1,n-1}k_{n-1}h) \end{aligned} \tag{4}$$

Eq. (4) clearly exhibits that each k is a functional evaluation and k_i s is in recurrence relationship. A more used an alternative form to the fourth order Runge-Kutta method described in Equations (3) and (4) by

$$y(x+h) \approx y(x) + ak_1 + bk_2 + ck_3 + dk_4 \tag{5}$$

where k_i s are

$$\begin{aligned} k_1 &= hf(x, y) \\ k_2 &= hf(x + mh, y + mk_1) \\ k_3 &= hf(x + nh, y + nk_2) \\ k_4 &= hf(x + ph, y + pk_3) \end{aligned} \tag{6}$$

We derive the arbitrary constants a, b, c, d, m, n, p such that Eq. (5) is consistent with Taylor series solution up to term h^4.

From Eq. (1)

$$y' = \frac{dy}{dx} = f(x, y) = f \tag{7}$$

Differentiating the above, we derive

$$y'' = \frac{\partial f}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} \tag{8}$$

which implies that

$$y'' = f_x + f.f_y = G_1 \tag{9}$$

Differentiating (9) another times

$$\begin{aligned} y''' &= \frac{d}{dx}\left(f_x(x,y) + f(x,y).f_y(x,y)\right) \\ &= \frac{\partial f_x}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial f_x}{\partial y} \cdot \frac{dy}{dx} + f.\left[\frac{\partial f_y}{\partial x}.\frac{dx}{dx} + \frac{\partial f_y}{\partial y} \cdot \frac{dy}{dx}\right] + f_y\left(f_x + f.f_y\right) \\ &= f_{xx} \cdot 1 + f_{xy} \cdot f + f\left[f_{yx} \cdot 1 + f_{yy} \cdot f \right] + f_y.f_x + f.f_y^2 \end{aligned}$$

which implies that

$$y''' = (f_{xx} + 2f.f_{xy} + f^2.f_{yy}) + f_y(f_x + f.f_y) \tag{10}$$

Let us suppose that $G_2 = f_{xx} + 2f.f_{xy} + f^2.f_{yy}$, thus (10) becomes $y''' = G_2 + f_yG_1$ (11)

A more time derivative of (10) yields

$$\begin{aligned} y^{(iv)} &= \frac{d}{dx}\left(y'''\right) = \frac{d}{dx}[(f_{xx} + 2f.f_{xy} + f^2.f_{yy}) + f_y(f_x + f.f_y)] \\ &= \frac{d}{dx}(f_{xx} + 2f.f_{xy} + f^2.f_{yy} + f_x.f_y + f.f_y^2) \\ &= \frac{d}{dx}\left(f_{xx}\right) + 2\frac{d}{dx}\left(f.f_{xy}\right) + \frac{d}{dx}\left(f^2.f_{yy}\right) + \frac{d}{dx}\left(f_y.f_x\right) + \frac{d}{dx}\left(f.f_y^2\right) \\ &= \frac{\partial f_{xx}}{\partial x}\frac{dx}{dx} + \frac{\partial}{\partial y}\left(f_{xx}\right).\frac{dy}{dx} + 2\left[f_{xy}.\frac{df}{dx} + f.\frac{d}{dx}\left(f_{xy}\right)\right] + \left[f_{yy}.2f.f' + f^2.\frac{d}{dx}\left(f_{yy}\right)\right] \\ &\quad + \left[f_y\frac{d}{dx}(f_x) + f_x\frac{d}{dx}\left(f_y\right)\right] + \left[f_y^2.\frac{df}{dx} + 2f.f_y\frac{d}{dx}\left(f_y\right)\right] \end{aligned}$$

$$\begin{aligned} &= [f_{xxx}.1 + f_{xxy}.f] + 2\left[f_{xy}(f_x + f.f_y) + f\left(\frac{\partial f_{xy}}{\partial x}.\frac{dx}{dx} + \frac{\partial f_{xy}}{\partial y}.\frac{dy}{dx}\right)\right] + \left[2f(f_x + f.f_y) + f^2\left(\frac{\partial f_{yy}}{\partial x}.\frac{dx}{dx} + \frac{\partial f_{yy}}{\partial y}.\frac{dy}{dx}\right)\right] \\ &\quad + \left[f_y\left(\frac{\partial f_x}{\partial x}.\frac{dx}{dx} + \frac{\partial f_x}{\partial y}.\frac{dy}{dx}\right) + f_x\left(\frac{\partial f_y}{\partial x}.\frac{dx}{dx} + \frac{\partial f_y}{\partial y}.\frac{dy}{dx}\right)\right] + \left[f_y^2\left(f_x + f.f_y\right) + 2f.f_y\left(\frac{\partial f_y}{\partial x}.\frac{dx}{dx} + \frac{\partial f_y}{\partial y}.\frac{dy}{dx}\right)\right] \\ &= [f_{xxx} + f.f_{xxy}] + \left[2f_xf_{xy} + 2f.f_y.f_{xy} + 2f(f_{xxy} + f_{xyy}.f)\right] \\ &\quad + \left[(2ff_x + 2f^2.f_y)f_{yy} + f^2(f_{xyy} + f.f_{yyy})\right] + \left[f_y(f_{xx} + f_{xy}.f) + f_x(f_{xy} + f_{yy}.f)\right] \\ &\quad + \left[f_x.f_y^2 + f.f_y^3 + 2f.f_y(f_{xy} + f.f_{yy})\right] \\ &= f_{xxx} + 3f.f_{xxy} + 3f^2f_{xyy} + f^3f_{yyy} + f_yf_{xx} + 2f.f_yf_{xy} + f^2f_yf_{yy} + 3f_xf_{xy} + 3f.f_y.f_{xy} + 3f.f_x.f_{yy} \\ &\quad + 3f^2f_yf_{yy} + f_xf_y^2 + f.f_y^3 \\ &= \left(f_{xxx} + 3f.f_{xxy} + 3f^2f_{xyy} + f^3f_{yyy}\right) + f_y\left(f_{xx} + 2f.f_{xy} + f^2f_{yy}\right) + 3\left(f_x + f.f_y\right)\left(f_{xy} + f.f_{yy}\right) + f_y^2\left(f_x + f.f_y\right) \end{aligned}$$

Therefore,

$$\begin{aligned} y^{(iv)} &= \left(f_{xxx} + 3f.f_{xxy} + 3f^2f_{xyy} + f^3f_{yyy}\right) + f_y\left(f_{xx} + 2f.f_{xy} + f^2f_{yy}\right) + 3\left(f_x + f.f_y\right)\left(f_{xy} + f.f_{yy}\right) \\ &\quad + f_y^2\left(f_x + f.f_y\right) \end{aligned} \tag{12}$$

Again consider $G_3 = \left(f_{xxx} + 3f.f_{xxy} + 3f^2f_{xyy} + f^3f_{yyy}\right)$, thus Eq. (12) becomes

$$y^{(iv)} = G_3 + f_yG_2 + 3G_1\left(f_{xy} + f.f_{yy}\right) + f_y^2G_1 \tag{13}$$

The Taylor's series in single variable is,

$$y\left(x+h\right) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(iv)}(x) + O\left(h^5\right) \tag{14}$$

Substituting the values of $y'(x), y''(x), y'''(x)$ and $y^{(iv)}(x)$ in equation (14), we get

$$y(x+h)=y(x)+hf+\frac{h^2}{2}G_1+\frac{h^3}{6}\left(G_2+f_yG_1\right)+\frac{h^4}{24}\left[G_3+f_yG_2+3G_1\left(f_{xy}+ff_{yy}\right)+f_y^{\,2}G_1\right]+\cdots$$

$$=y(x)+hf+\frac{h^2}{2}G_1+\frac{h^3}{6}G_2+\frac{h^4}{24}G_3+\frac{h^3}{6}f_yG_1+\frac{h^4}{24}f_yG_2+\frac{1}{8}h^4\left(f_{xy}+ff_{yy}\right)G_1+\frac{h^4}{24}f_y^{\,2}G_1+\cdots \quad (15)$$

Here,

$$k_1=h.f(x,y)=h.f$$

$$k_2=hf(x+mh,y+mk_1)$$

Now, expanding the double variable function $f(x+mh,y+mk_1)$ by Taylor series,

$$f(x+mh,y+mk_1)=f(x,y)+\left(mh\frac{\partial}{\partial x}+mk_1\frac{\partial}{\partial y}\right)f(x,y)+\frac{1}{2!}\left(mh\frac{\partial}{\partial x}+mk_1\frac{\partial}{\partial y}\right)^2f(x,y)+$$

$$\frac{1}{3!}\left(mh\frac{\partial}{\partial x}+mk_1\frac{\partial}{\partial y}\right)^3f(x,y)+\cdots$$

$$=f+\left(mhf_x+mk_1f_y\right)+\frac{1}{2}\left(m^2h^2f_{xx}+2m^2hk_1f_{xy}+m^2k_1^2f_{yy}\right)$$

$$+\frac{1}{6}\left(m^3h^3f_{xxx}+3m^2h^2mk_1f_{xxy}+3mhm^2k_1^2f_{xyy}+m^3k_1^3f_{yyy}\right)+\cdots$$

Setting $k_1=h.f$ in the above equation

$$=f+mh\left(f_x+f.f_y\right)+\frac{1}{2}m^2h^2\left(f_{xx}+2f.f_{xy}+f^2f_{yy}\right)$$

$$+\frac{1}{6}m^3h^3\left(f_{xxx}+3f.f_{xxy}+3f^2f_{xyy}+f^3f_{yyy}\right)+\cdots$$

$$=f+mhG_1+\frac{1}{2}m^2h^2G_2+\frac{1}{6}m^3h^3G_3+\cdots$$

Therefore,

$$f(x+mh,y+mk_1)=f+mhG_1+\frac{1}{2}m^2h^2G_2+\frac{1}{6}m^3h^3G_3+\cdots \quad (16)$$

Hence,

$$k_2=h\left[f+mhG_1+\frac{1}{2}m^2h^2G_2+\frac{1}{6}m^3h^3G_3+\cdots\right] \quad (17)$$

Again, given that

$$k_3=hf(x+nh,y+nk_2)$$

Further expanding $f(x+nh,y+nk_2)$ in Taylor series

$$f(x+nh,y+nk_2)=f(x,y)+\left(nh\frac{\partial}{\partial x}+nk_2\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(nh\frac{\partial}{\partial x}+nk_2\frac{\partial}{\partial y}\right)^2f+\frac{1}{3!}\left(nh\frac{\partial}{\partial x}+nk_2\frac{\partial}{\partial y}\right)^3f+\cdots$$

$$=f+nhf_x+nk_2f_y+\frac{1}{2}\left(n^2h^2f_{xx}+2n^2hk_2f_{xy}+n^2k_2^2f_{yy}\right)$$

$$+\frac{1}{6}\left(n^3h^3f_{xxx}+3n^2h^2nk_2f_{xxy}+3nhn^2k_2^2f_{xyy}+n^3k_2^3f_{yyy}\right)+\cdots$$

Now, substituting this value of k_2 in the above equation we get

$$f(x+nh,y+nk_2)=f+nhf_x+\left(nhff_y+mnh^2f_yG_1+\frac{1}{2}m^2nh^3f_yG_2+\frac{1}{6}m^3nf_yh^4G_3+\cdots\right)$$

$$+\frac{1}{2}\left[n^2h^2f_{xx}+2n^2hf_{xy}\left(hf+mh^2G_1+\frac{1}{2}m^2h^3G_2+\frac{1}{6}m^3h^4G_3+\cdots\right)\right.$$

$$\left.+n^2f_{yy}\left(hf+mh^2G_1+\frac{1}{2}m^2h^3G_2+\frac{1}{6}m^3h^4G_3+\cdots\right)^2\right]$$

$$+\frac{1}{6}\left[n^3h^3f_{xxx}+3n^3h^2f_{xxy}\left(hf+mh^2G_1+\frac{1}{2}m^2h^3G_2+\frac{1}{6}m^3h^4G_3+\cdots\right)\right.$$

$$+3n^3hf_{xyy}\left(hf+mh^2G_1+\frac{1}{2}m^2h^3G_2+\frac{1}{6}m^3h^4G_3+\cdots\right)^2$$

$$\left.+n^3f_{yyy}\left(hf+mh^2G_1+\frac{1}{2}m^2h^3G_2+\frac{1}{6}m^3h^4G_3+\cdots\right)^3\right]$$

Collecting the orders of $O(h^3)$

$$f(x+nh,y+nk_2)=f+nhG_1+\frac{1}{2}h^2\left(n^2G_2+2mnf_yG_1\right)+$$
$$\frac{1}{6}h^3\left(n^3G_3+3m^2nf_yG_2+6mn^2\left(f_{xy}+ff_{yy}\right)G_1\right)+\cdots \quad (18)$$

Therefore,

$$k_3=h\left[f+nhG_1+\frac{1}{2}h^2\left(n^2G_2+2mnf_yG_1\right)+\frac{1}{6}h^3\left(n^3G_3+3m^2nf_yG_2+6mn^2\left(f_{xy}+ff_{yy}\right)G_1\right)+\cdots\right] \quad (19)$$

Again, given that

$$k_4=hf\left(x+ph,y+pk_3\right)$$

Expanding $f\left(x+ph,y+pk_3\right)$ by Taylor series,

$$f(x+ph,y+pk_3)=f(x,y)+\left(ph\frac{\partial}{\partial x}+pk_3\frac{\partial}{\partial y}\right)f+\frac{1}{2!}\left(ph\frac{\partial}{\partial x}+pk_3\frac{\partial}{\partial y}\right)^2f+\frac{1}{3!}\left(ph\frac{\partial}{\partial x}+pk_3\frac{\partial}{\partial y}\right)^3f+\cdots$$

$$=f+phf_x+pk_3f_y+\frac{1}{2}p^2\left(h^2f_{xx}+2hk_3f_{xy}+k_3^2f_{yy}\right)+$$

$$\frac{1}{6}p^3\left(h^3f_{xxx}+3h^2k_3f_{xxy}+3hk_3^2f_{xyy}+k_3^3f_{yyy}\right)+\cdots$$

Substituting the value of k_3 in the above equation

$$f(x+ph,y+pk_3)=f+phf_x+pf_y\left[hf+nhG_1+\frac{1}{2}h^3\left(n^2G_2+2mnf_yG_1\right)+\frac{1}{6}h^4\left(n^3G_3+3m^2nf_yG_2+6mn^2\left(f_{xy}+ff_{yy}\right)G_1\right)+\cdots\right]$$
$$+\frac{1}{2}\left[p^2h^2f_{xx}+2p^2h\left(hf+nhG_1+\frac{1}{2}h^3\left(n^2G_2+2mnf_yG_1\right)+\frac{1}{6}h^4\left(n^3G_3+3m^2nf_yG_2+6mn^2\left(f_{xy}+ff_{yy}\right)G_1\right)+\cdots\right)\right]$$
$$+\frac{1}{2}\left[p^2f_{yy}\left(hf+nhG_1+\frac{1}{2}h^3\left(n^2G_2+2mnf_yG_1\right)+\frac{1}{6}h^4\left(n^3G_3+3m^2nf_yG_2+6mn^2\left(f_{xy}+ff_{yy}\right)G_1\right)+\cdots\right)^2\right]$$
$$+\frac{1}{6}\left(p^3h^3f_{xxx}+3p^2h^2pf_{xxy}\left(hf+nhG_1+\frac{1}{2}h^3\left(n^2G_2+2mnf_yG_1\right)+\frac{1}{6}h^4\left(n^3G_3+3m^2nf_yG_2+6mn^2\left(f_{xy}+ff_{yy}\right)G_1\right)+\cdots\right)+\cdots\right)$$

Simplifying the above equation

$$f(x+ph,y+pk_3)=f+phG_1+\frac{1}{2}h^2\left(p^2G_2+2pnf_yG_1\right)+$$
$$\frac{1}{6}h^3\left(p^3G_3+3n^2pf_yG_2+6np^2\left(f_{xy}+ff_{yy}\right)G_1+6mnpf_y^2G_1\right)+\cdots$$

Now, substituting this value in the $k_4=hf\left(x+ph,y+pk_3\right)$ equation we get,

$$k_4=h\left[f+phG_1+\frac{1}{2}h^2\left(p^2G_2+2npf_yG_1\right)+\frac{1}{6}h^3\left(p^3G_3+3n^2pf_yG_2+6np^2\left(f_{xy}+ff_{yy}\right)G_1+6mnpf_y^2G_1\right)+\cdots\right] \quad (21)$$

Substituting the values of k_1,k_2,k_3,k_4 in equation (5) we get,

$$y(x+h)\approx y(x)+(a+b+c+d)hf+(bm+cn+dp)hG_1+\frac{1}{2}\left(bm^2+cn^2+dp^2\right)h^3G_2$$
$$+\frac{1}{6}\left(bm^3+cn^3+dp^3\right)h^4G_3+\left(cmn+dnp\right)h^3f_yG_1+\frac{1}{2}\left(cm^2n+dn^2p\right)h^4f_yG_2 \quad (22)$$
$$+\left(cmn^2+dnp^2\right)h^4\left(f_{xy}+ff_{yy}\right)G_1+dmnph^4f_y^2G_1+O(h^5)$$

Comparing the equations (15) and (22) we get

$a+b+c+d=1$	$cmn+dnp=\frac{1}{6}$
$bm+cn+dp=\frac{1}{2}$	$cmn^2+dnp^2=\frac{1}{8}$
$bm^2+cn^2+dp^2=\frac{1}{3}$	$cm^2n+dn^2p=\frac{1}{12}$
$bm^3+cn3+dp^3=\frac{1}{4}$	$\text{dmnp}=\frac{1}{24}$

The above set of equations is an overdetermined system of eight equations with seven variables. Solving the set of algebraic equations using the graphing utility like as Maple or Mathematica, the classical solution becomes

$$a=\frac{1}{6},b=\frac{1}{3},c=\frac{1}{3},d=\frac{1}{6},m=\frac{1}{2},n=\frac{1}{2},p=1 \quad (23)$$

Putting these values in Eqs. (5) and (6) we get,

$$y\left(x+h\right)=y(x)+\frac{1}{6}(k_1+2k_2+2k_3+k_4), \quad (24)$$

where

$$k_1=hf\left(x,y\right)$$
$$k_2=hf\left(x+\frac{h}{2},y+\frac{1}{2}k_1\right)$$
$$k_3=hf\left(x+\frac{h}{2},y+\frac{1}{2}k_2\right) \quad (25)$$
$$k_4=hf\left(x+h,y+k_3\right).$$

III. GEOMETRICAL INTERPRETATION OF SLOPE PARAMETER k_i 'S:

Before illustrating the geometric interpretation (20) of the RK4 method, we provide a useful explanation of the most naive Euler method. In the figure 1, the exact solution y in the graph is a curve C in the xy plane. Here, $P(x_0,y_0)$ is the initial point and T be the tangent to the curve at P. Let N and M be the points where the line $x=x_0+h$ meets the curve and the tangent respectively. Thus the exact value of y_1 at $x_1=x_0+h$ is SN and the approximated value of y_1 at $x_1=x_0+h$ is represented by SM since

$$SM=SQ+MQ$$
$$y_1\approx y_0+PQ\tan\theta$$
$$y_1\approx y_0+h\,f(x_0,y_0)$$

Suppose consider $k=h\,f(x_0,y_0)$ therefore $y_1=y_0+k$ where the value of k is the slope of the curve at the point (x_0,y_0) multiplied by increment h of x.

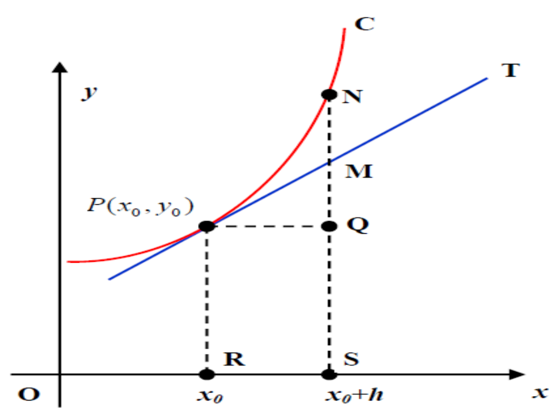

Fig. 1: The basic geometry of the Euler method

Now let us back to the RK4 formula Eq. no. (24). Rewrite this equation as

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

The first approximated value y_1 of y is evaluated by the weighted average slopes at different points. From Eq.(25), the slope of the function are evaluated at the different four points viz. (x, y), $(x + h/2, y + k_1/2)$, $(x + h/2, y + k_1/2)$ and $(x + h, y + k_3)$ (Fig. 2)

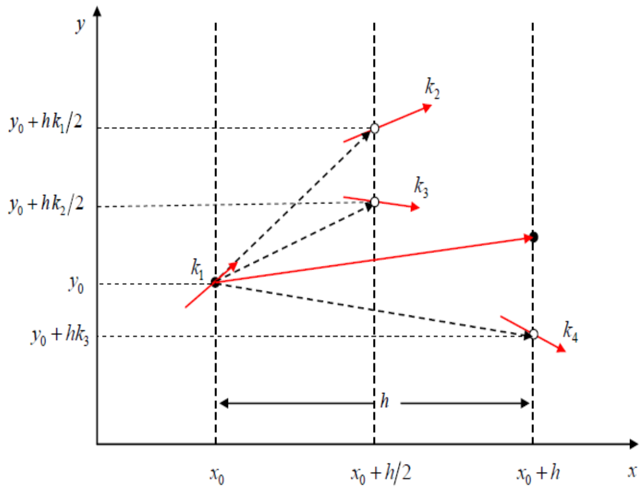

Fig. 2: Slopes used by the classical Runge-Kutta method

From the Fig. 2, and Eq. (25) it is clearly seen that $k_1 = hf(x, y)$ is the slope of the curve at the beginning of the interval which is evaluated by the Euler's method. $k_2 = hf(x + h/2, y + k_1/2)$ is the slope of the curve at the midpoint of the interval by using y and k_1. Therefore the recurrence relation is used i.e. the slope function k_2 is evaluated with the help of first slope k_1. Again, the slope $k_3 = hf(x + h/2, y + k_2/2)$ is evaluated at the midpoint but this time using y and k_2. Finally, the slope $k_4 = hf(x + h, y + k_3)$ is derived at the end of the interval using y and k_3. In averaging the four slopes, it is seen from Eq. (24) that greater weight is given to the slopes at the midpoint.

IV. DISCUSSION

A number of different Runge-Kutta methods have been proposed and developed. Differential equations of various orders have been the subject of numerous theoretical and numerical researches that have been published in the literature which includes a collection of implicit and explicit methods for estimating the solutions of ODEs. In order to better understand and use the RK techniques, first we must study and derive them more directly and then it dire needs to understand them graphically which is shown in the earlier section. Notably, the increment functions $hf(x_m, y_m)$ are used in Eq. (2) whereas the k's are related by the Eq. 4. The recurrence relation holds in here. The Eq. (5) and (6) exhibits of using a weighted approximations of where the weights are a's and m, n, p are arbitrary constants. These constants' derivation techniques are given in a clear, step-by-step manner. The overdetermined equation system is solved using Maple, a mathematical computing tool.

V. CONCLUSION

The main goal of this work is, first, to provide more details on how to formulate the fourth order Runge-Kutta formula, and secondly, to provide a geometrical interpretation of the weighted slopes which are applied at four stages. This study aims to present a deeper knowledge of the RK4 method's fundamental principles, as well as to improve its analytical capabilities in order to encourage further research into this formula. The study is particularly essential for comprehending the overall formulation of the Runge-Kutta technique of fourth order in general. Our main concern is to obtain familiarity with the procedure itself and to develop skills in applying it.

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