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Investigation of Some Explicit Exact Solution of the Damped Forced KDV Burger Equation by Modified Exp(-$\varphi(\xi)$)-Expansion Method

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ABSTRACT

This work presented the some explicit exact solution of the damped forced KdV-Burger equation with variable coefficients. We have successfully applied the exp(-$\varphi(\xi)$)-expansion method with modification to obtain the generalized explicit exact solution of the damped forced KdV-Burger's equation. The obtained solution contains the hyperbolic function and trigonometric function. The dynamic behavior of the solution is demonstrated graphically in three dimensional and two dimensional space.

Keywords: Exp(-$\varphi(\xi)$)-expansion method, variable of separation, KdV-Burger's equation,forcing term, variable coefficients etc.

INTRODUCTION

The Korteweg-de Vries-Burgers (KdV-Burgers) equation has garnered significant attention over the past three decades due to its relevance in various physical contexts. These include the propagation of undular bores in shallow water [1], flow of liquids with gas bubbles [2], wave propagation in elastic tubes filled with viscous fluids [3], and nonlinear plasma waves with dissipative effects [4-6]. The equation also finds applications in crystal lattice theory, ferroelectricity, nonlinear circuit theory, and turbulence [7-12]. The standard form of the KdV-Burgers equation is

$$u_t + A(t)uu_x + B(t)u_{xxx} + C(t)u_{xx} = 0 \qquad (1.1)$$

It combines the Burgers equation [13] and the KdV equation [14], incorporating nonlinearity, dispersion, and dissipation terms. Its validity has been demonstrated in specific physical problems, such as wave propagation in liquid-filled elastic tubes [15], making it a fundamental model for understanding complex wave phenomena.
In general it is known that particle interactions in a medium often lead to damping effects. Various phenomena can cause damping in dynamical systems, such as resonant energy exchange between particles and electrostatic waves in plasma environments [16-17]. Experimental studies on space plasma have shown that externally applied damping significantly influences wave propagation. External forces can also arise in specific situations, like flowing water over bottom topography or waves generated by moving ships [18-19]. Considering these factors, the focus is on a KdV-Burger's equation with external forcing and damping terms, which is presented as a model to study these complex dynamics

$$u_t + A(t)uu_x + B(t)u_{xxx} + C(t)u_{xx} + D(t)u = H(t) \qquad (1.2)$$

where A= non-linearity coefficients, B= dispersion coefficients, C=dissipation coefficients, D=damping coefficients and H=forcing term. In recent years, a wide range of effective methods has been proposed to solve nonlinear equations. These include the tanh function method [20], symmetry reduction method [21], extended tanh method [22], sine-cosine method [23], homogeneous balance method [24], F-expansion method [25], exp-function method [26-27], modified simple equation method [28-29], first integral method [30], extended trial equation method [31], (G'/G)-expansion method[32], (G'/G, 1/G)-expansion method [33-34], soliton solution method [35], and auxiliary equation method [36], among others. These methods have been successfully applied to various nonlinear problems, providing valuable insights and solutions to complex equations.

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In this work we obtained the generalized exact solution of the damped forced KdV-Burger's equation with variable coefficients with the help of modified $\exp(-\boldsymbol{\varphi}(\boldsymbol{\xi}))$-expansion method. The obtain solution contains hyperbolic function and trigonometric function. The dynamics behavior of the solution is demonstrated graphically in three dimensional and two-dimensional space.

2.Preliminary of Exp(-$\boldsymbol{\varphi}(\boldsymbol{\xi})$)-expansion method

Given a non-linear partial differential equation in general form

$$Q(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots\dots\dots\dots\dots..) = 0 \qquad (2.1)$$

where $u = u(x,t), Q$ is a polynomial of u and its derivatives and the subscripts means for the partial derivatives. The method involves the following steps.

Step I: First we make the transformation $\xi = x + g(t)$, where $g(t)$ is an unknown function and $u(x,t) = u(\xi)$.Under this transformation ,Eq.(2.1) convert into an ordinary differential equation

$$\mathrm{R}(\mathrm{u}, \mathrm{u}', \mathrm{u}'', \mathrm{u}''', \dots\dots\dots\dots\dots..) = 0 \qquad (2.2)$$

where R is a polynomial of u and its derivatives and the subscripts means for ordinary derivatives w.r.t. ξ.

Step II: Now we consider the solution of Eq.(2.2) in the form of

$$u(\xi) = \sum_{\mathrm{i}=0}^{\mathrm{n}} \mathrm{c_i}(\exp(-\varphi(\xi)))^{\mathrm{i}} \qquad (2.3)$$

where c_i,(i=0,1…,n) are constant to be determined and $\varphi(\xi)$ satisfies the ordinary differential equation. N is a positive integer determined by the homogeneous balance principle between the nonlinear term and the highest order derivatives in Eq. (2.1)

$$\varphi'(\xi) = \exp\big(-\varphi(\xi)\big) + \mu \exp\big(\varphi(\xi)\big) + \lambda \qquad (2.4)$$

When $\mu \neq 0, \Delta = \lambda^2 - 4\mu > 0$,

$$\varphi(\xi) = \ln\left(\frac{-\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi + \mathrm{a})\right) - \lambda}{2\mu}\right) \qquad (2.5)$$

Step III: Now substitute Eq. (2.3) in Eq.(2.2) along with Eq. (2.4) and then collect the coefficients of different power of $\exp(-\varphi(\xi))$ and equating to zero. This provides a system of algebraic equations. Solving these system for $c_{i,}, \lambda, \mu$ and substitute it in Eq. (2.3) along with Eq.(2.5), we get a complete exact solution for the Eq.(2.1).

3. Some exact solution by Exp(-$\boldsymbol{\varphi}(\boldsymbol{\xi})$)-expansion method

Here we obtain some exact solution of the Eq.(1.2) with the help of this method. Under the submission of $\mathrm{u(x,t) = p(t)v(x,t) + q(t)}$ into the Eq. (1.1), we get

$$\mathrm{v_t + A(t)p(t)vv_x + A(t)q(t)v_x + B(t)v_{xxx} + C(t)v_{xx} = 0} \qquad (3.1)$$

with the choices $\mathrm{p(t) = k_1 e^{-\int D(t)dt}, q(t) = e^{\int D(t)dt} \int\big(H(t)e^{-\int D(t)dt}\big)\, dt}$,where $\mathrm{k_1}$ is an integrating constant. The variable separation solution of Eq. (1.2) is investigated by using the Exp(-$\varphi(\xi)$)- expansion method with variable separation transformation. Using the transformation $\xi = x + g(t)$ in Eq. (3.1), we get the following ODE

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$$\left(g'(t)+A(t)q(t)\right)v'+A(t)p(t)vv'+C(t)v''+B(t)v'''=0 \qquad (3.2)$$

The expression for the solution of Eq. (3.2) is given by

$$v(\xi)=c_0(x,t)+c_1(x,t)\exp\left(-\varphi(\xi)\right)+c_2(x,t)(\exp(-\varphi(\xi)))^2 \quad , \qquad (3.3)$$

Now inserting the Eq.(3.3) into Eq. (3.2) and collecting the corresponding coefficients of $\exp\left(-\varphi(\xi)\right)$, we obtained a system of algebraic equation and by symbolic calculation, we get the following parameter values

$$c_2=-\frac{12B(t)}{A(t)k_1e^{-\int D(t)dt}} \quad , \quad c_1=\frac{12\left(C(t)-5\lambda B(t)\right)}{5A(t)k_1e^{-\int D(t)dt}}$$

$$c_0=\frac{-(25B^2\lambda^2+200B^2\mu-30BC\lambda+25ABq-C^2+25Bg\prime)}{25A(t)B(t)k_1e^{-\int D(t)dt}}, \qquad (3.4)$$

with the condition $C(t)=5\sqrt{\lambda^2-4\mu}\,B(t)$ and $g(t)$ is a test function. Then the variable separation solution of the Eq.(1) is given by

$$u=k_1e^{-\int D(t)dt}\left(c_0-\frac{2\mu c_1}{\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)+\lambda}-\frac{4\mu c_2}{\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)+\lambda\right)^2}\right)+e^{\int D(t)dt}\int\left(H(t)e^{-\int D(t)dt}\right)dt$$

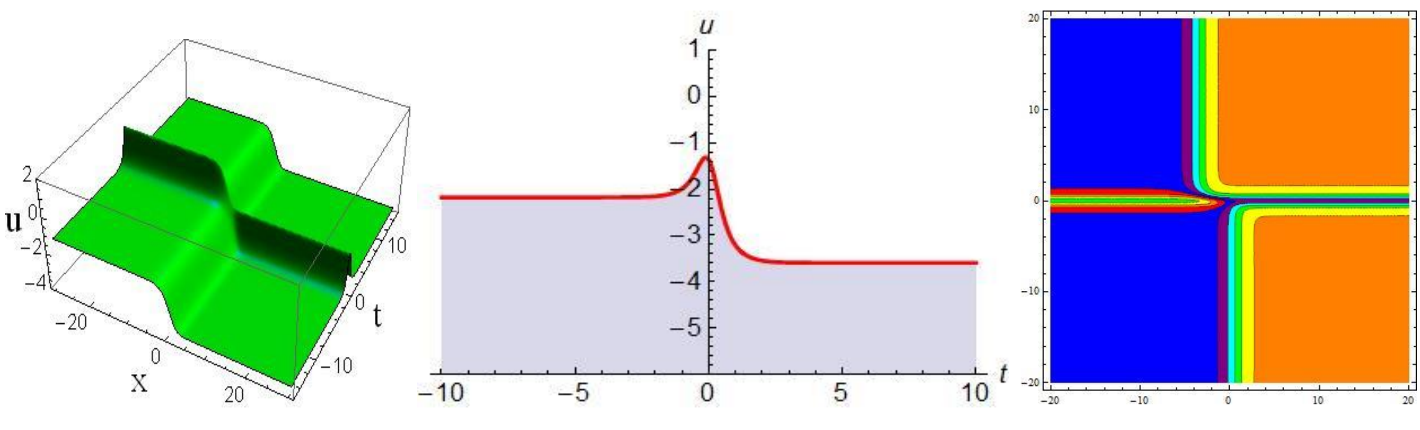
$$(3.5)$$

where k_1 and a are arbitrary constants. This solution (3.5) is clearly a singular non-traveling wave solution. Now we will introduced the different test function $g(t)$ to get different type of solution of Eq.(1.2).

Case I: Set $g(t)=\tanh(t)$.Then from Eq.(3.5),we get exact solution of Eq.(1.2) as

$$u=-\frac{25B^2\lambda^2+200B^2\mu-30BC\lambda+25ABq-C^2+25B\,\mathrm{sech}(t)^2}{25AB}-\frac{24\mu\left(C(t)-5\lambda B(t)\right)}{5A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)++\lambda\right)}+\frac{48\mu^2B}{A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)+\lambda\right)^2}+$$
$$e^{\int D(t)dt}\int\left(H(t)e^{-\int D(t)dt}\right)dt \qquad (3.6)$$

Fig.1 presented the dynamic behavior of the exact solution (3.6).It shown that shock nature of the solution is observed and a soliton in the background of shock is clearly observed. Clearly a dominance of $g(t)=\tanh(t)$ is observed on the solution. Fig.1(b) shown the two-dimensional view of the exact solution (3.6) with $x=-0.5$ which show a soliton-shock nature which is surely observed in the contour plot of the solution (3.6).



(a) (b) (c)

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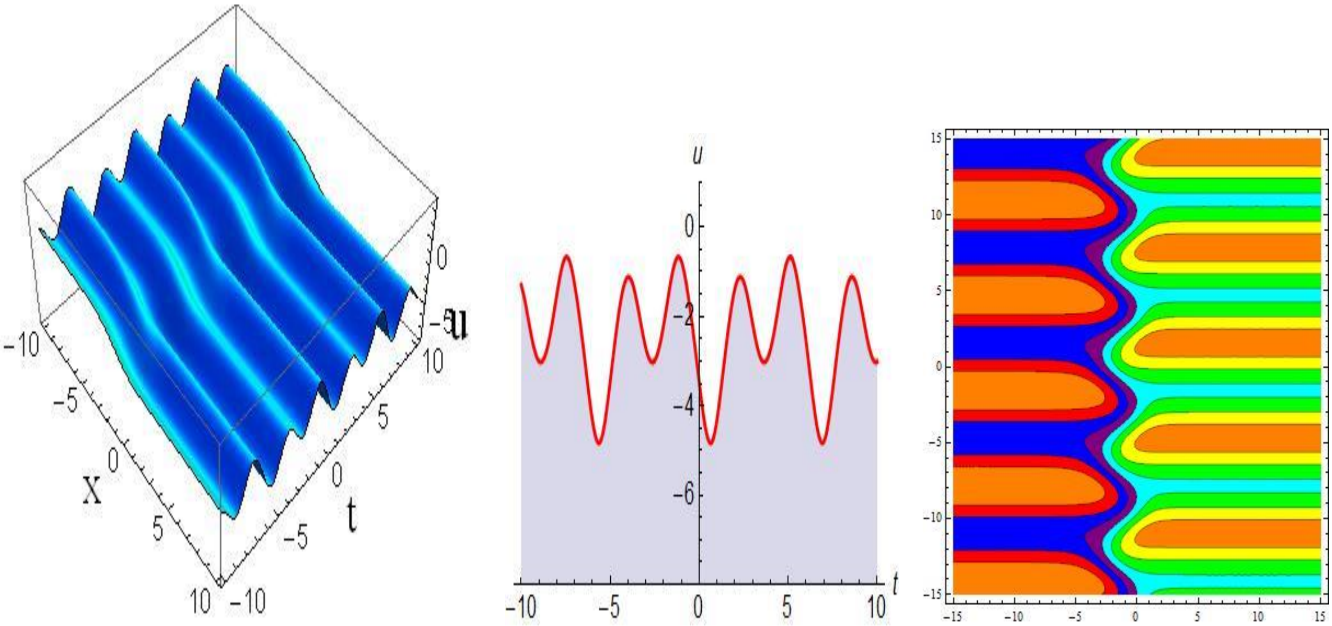
Fig. 1: Three dimensional profile, two dimensional profile and contour plot of the exact solution (3.9) is presented with the following numeric values of the parameters $A(t) = -1.5, B(t) = 0.1, D(t) = 1, H(t) = \sin(t), \lambda = 3, \mu = 1, k_1 = 1, a = 2$

Case II: Set $g(t) = \cos(t)$.Then from Eq.(3.5),we get exact solution of Eq.(1.2) as

$$u = -\frac{25B^2\lambda^2+200B^2\mu-30BC\lambda+25ABq-C^2+25B\cos(t)}{25AB} - \frac{24\mu\left(C(t)-5\lambda B(t)\right)}{5A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)++\lambda\right)} + \frac{48\mu^2 B}{A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)+\lambda\right)^2} + e^{\int D(t)dt}\int\left(H(t)e^{-\int D(t)dt}\right)dt \qquad (3.7)$$

Fig.2(a) presented the three dimensional dynamic behavior of the solution (3.7).It shown the shock nature of the solution (3.7) in the periodic background $g(t) = \cos(t)$. Obviously a dominance of $g(t) = \cos(t)$ is observed in the solution. Fig.2(b) depicted the two dimensional profile of this solution with $x = -1.5$ and Fig.2(c) is the contour plot of the solution (3.7) shown the strong periodic behavior the solution with shock nature.



(a) (b) (c)

Fig. 2: Three dimensional profile, two dimensional profile and contour plot of the exact solution (3.9) is presented with the following numeric values of the parameters $A(t) = -1.5, B(t) = 0.1, D(t) = 1, H(t) = \sin(t), \lambda = 3, \mu = 1, k_1 = 1, a = 2$

Case III: Set $g(t) = \operatorname{sech}(t)$.Then from Eq.(3.5),we get exact solution of Eq.(1.1) as

$$u = -\frac{25B^2\lambda^2+200B^2\mu-30BC\lambda+25ABq-C^2-25B\,\operatorname{sech}(t)\tanh(t)}{25AB} - \frac{24\mu\left(C(t)-5\lambda B(t)\right)}{5A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)++\lambda\right)} + \frac{48\mu^2 B}{A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)+\lambda\right)^2} + e^{\int D(t)dt}\int\left(H(t)e^{-\int D(t)dt}\right)dt \qquad (3.8)$$

Fig.3(a) presented the three dimensional dynamic behavior of the exact solution (3.8).It shown the shock nature with dominance $g(t) = \operatorname{sech}(t)$ of the solution (3.9).Fig.3(b) depicted the two dimensional visualization of this solution with $x = -1.5$ which shown peak in one side and deep in other side. Fig.2(c) is the contour plot of the solution (3.8) which ensures our observation.

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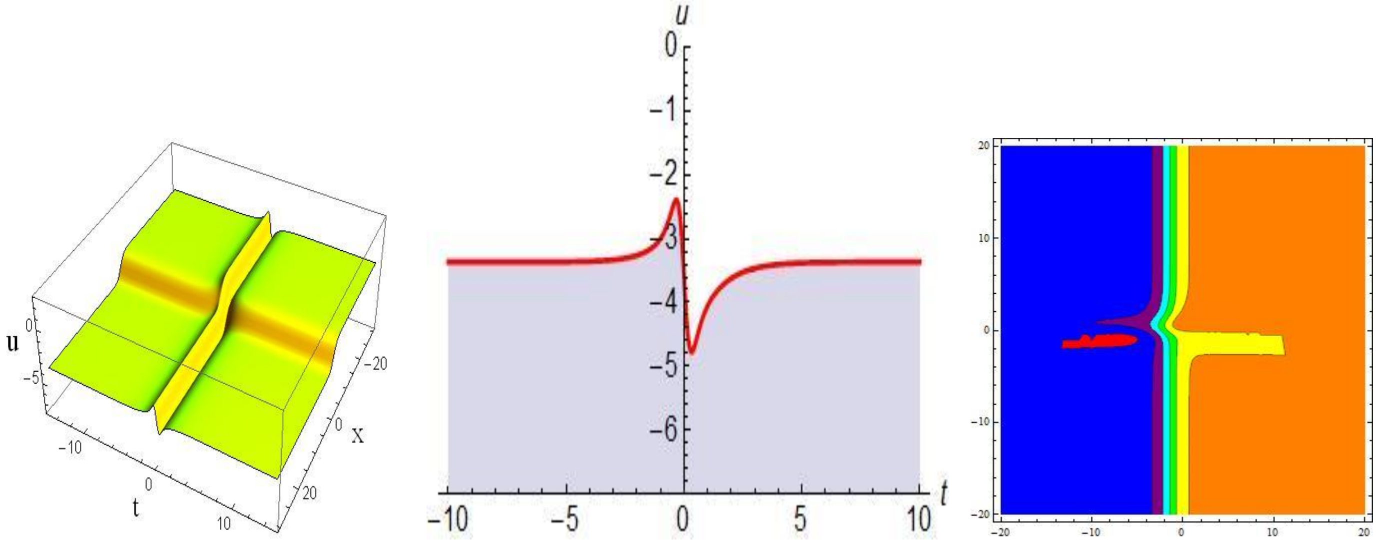


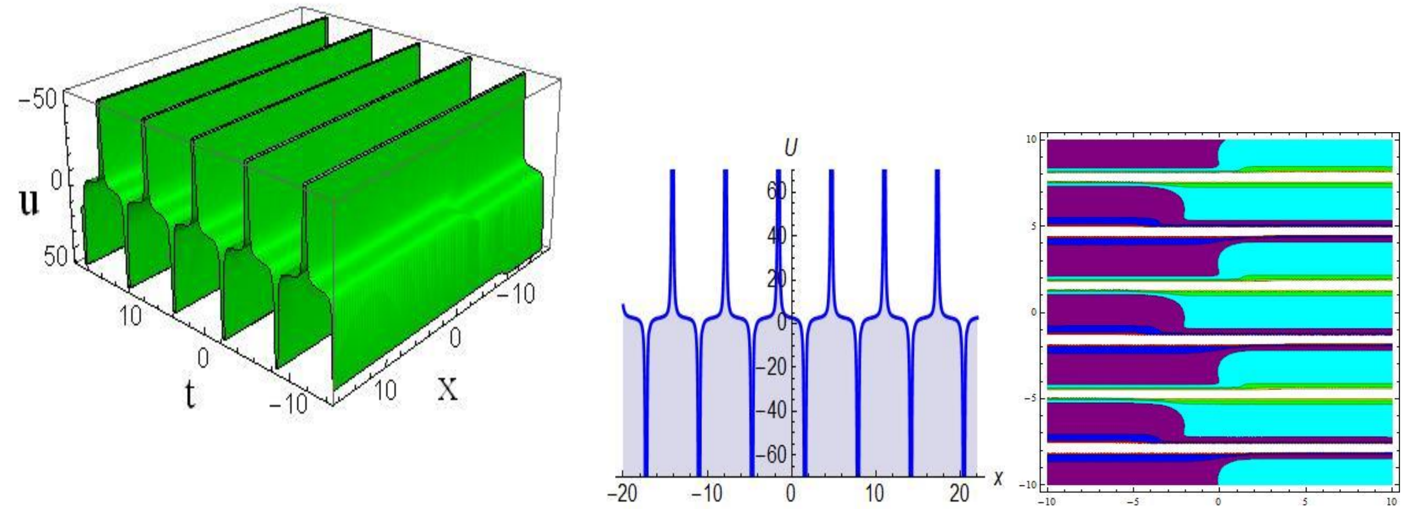
(a) (b) (c)

Fig. 3: 3D profile,2D profile and contour plot of the exact solution (3.8) is presented with the following numeric values of the parameters $A(t) = -1.5, B(t) = 0.1, D(t) = 1, H(t) = \sin(t), \lambda = 3, \mu = 1, k_1 = 1, a = 2.$

Case IV: Set $g(t) = \text{Sec}(t)$.Then from Eq.(3.5),we get exact solution of Eq.(1.2) as

$$u = -\frac{25B^2\lambda^2+200B^2\mu-30BC\lambda+25ABq-C^2-25B\sec(t)\tan(t)}{25AB} - \frac{24\mu(C(t)-5\lambda B(t))}{5A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)++\lambda\right)} + \frac{48\mu^2 B}{A\left(\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{2}(\xi+a)\right)+\lambda\right)^2} +$$
$$e^{\int D(t)dt}\int\left(H(t)e^{-\int D(t)dt}\right)dt \qquad (3.9)$$

Fig.3(a) demostrated the three dimensional dynamic behavior of the exact solution (3.8).It shown the singular periodic exact solution with dominance $g(t) = \sec(t)$.Fig.3(b) depicted the two dimensional visualization of this solution with $t = 1$ which shown singular shock periodic. Fig.2(c) is the contour plot of the solution (3.8) which ensures our observation



(a) (b) (c)

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Fig. 4: 3D profile,2D profile and contour plot of the exact solution (3.9) is presented with the following numeric values of the parameters $A(t) = -1.5, B(t) = 0.1, D(t) = 1, H(t) = \sin(t), \lambda = 3, \mu = 1, k_1 = 1, a = 2$

CONCLUSION

The Exp(-φ(ξ))-expansion method, with modifications, was successfully applied to the damped forced KdV-Burger's equation with variable coefficients, resulting in the derivation of generalized exact solutions. The effectiveness of this method was demonstrated through its ability to yield precise solutions for this complex equation. Given its success, the Exp(-φ(ξ))-expansion method holds promise for solving other nonlinear partial differential equations, offering a powerful tool for researchers to explore and analyze various nonlinear phenomena in physics and mathematics. Its potential applications span a wide range of fields, including fluid dynamics, plasma physics, and nonlinear optics, where nonlinear PDEs play a crucial role in modeling real-world systems.

Conflicts of interest

The authors declare that they have no conflicts of interests

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