Degree of Approximation of Function in the Generalized Zygmund Class By (E, Q) $(\overline{N}, p_n)$ Means of Fourier Series

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ABSTRACT

In this paper, a theorem on degree of approximation of function in the generalized Zygmund class by (E, q) $(\overline{N}$, Pn) means of Fourier series has been established.

Keywords : Degree of approximation , Generalized Zygmund class , $(\overline{N}$, Pn) mean , (E, q) mean, (E, q) $(\overline{N}$, Pn) mean .

MSC: 41A24, 41A25, 42B05, 42B08

INTRODUCTION

The degree of approximation of function belonging to different classes like Lip α, (Lip α, p), Lip(ξ(t),p) , Lip(Lp , ξ(t)) have been studied by many mathematician using different summability means. The error estimation of function in Lipschitz and Zygmund class using different means of Fourier series and conjugate Fourier series have been great interest among the researcher. The generalized Zygmund class of functions has been widely studied in harmonic analysis and Fourier approximation because it captures functions whose smoothness is characterized by a controlled modulus of continuity. This class generalizes the well-known Zygmund class and includes functions with smoother as well as rougher behavior, making it an appropriate setting for studying precise error bounds in trigonometric approximations.

A powerful approach for enhancing convergence involves the use of product summability methods. In particular, the (E, q) means, when combined with the weighted Nörlund means $(\overline{N}$, Pn) produce a generalized summability method denoted by (E, q) $(\overline{N}$, Pn) means. The (E, q) means accelerate convergence by modifying partial sums, while the $(\overline{N}$, Pn) transformation introduces flexibility through a weight sequence {Pn}. The generalized Zygmund class was introduced by Kim [1] Leindler [2] Moricz [3], moricz and Nemeth [4]etc. Recently Singh et. al. [7] Mishra et al. [5], Pradhan et al. [6], Sinha et al. [8] find the results in Zygmund class by using different summability Means. In this paper we find the degree of approximation of function in the generalized Zygmund class by (E, q) $(\overline{N}$, Pn) means of Fourier series.

Definition

Let g be a periodic function of period 2π integrable in the sense of Lebesgue over

[π, - π]. Then the Fourier series of g given by

$$g(t) \approx \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n cos\, nx + b_n sin\, nx) \qquad \text{……..(2.1)}$$

Zygmund class z is defined as

$Z = \{g \in C[-\pi,\pi]\ \ |g(x+t)+g(x-t)-2g(x)| = O(|t|)\}$.

In this paper, we introduce a generalized Zygmund $Z^w(\alpha,\gamma)$ defined as

$$Z^w(\alpha,\gamma) = \left\{g \in C[-\pi,\pi]\left(\int_{-\pi}^{\pi}|g(x+t)+g(x-t)-2g(x)|^\gamma dx\right)^{\frac{1}{\gamma}} = O(|t|^\alpha\omega(t))\right\}$$

...........(2.2)

Where $\alpha \geq 0$, $\gamma \geq 1$ and ω is a continuous non negative and non decreasing function. If we take $\alpha = 1, \omega = constant\ and\ \gamma \to \infty$, then $Z^w(\alpha,\gamma)$ class reduces to the z class.

We write through the paper

$$\emptyset_x(\mathrm{t}) = g(x+t) - 2g(x) + g(x-t) \qquad \text{.......(2.3)}$$

$$K_n(t) = \frac{1}{2\pi(1+q)^n}\sum_{k=o}^{n}\binom{n}{k}q^{k-v}\left\{\frac{1}{P_k}\sum_{v=0}^{k}p_v\frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\} \qquad \text{.......(2.4)}$$

MAIN RESULT

In this paper we prove the following theorem.

Theorem1- Let g be a 2π periodic function ,Lebesgue integrable in $[0\,,2\pi]$ and belonging to generalized Zygmund class $Z_r^{(w)}(r \geq 1)$. Then the degree of approximation of function g by (E,q)($\overline{N}$, Pn) product mean of Fourier series is given by

$$E_n(g) = ing\ \|t_n^{NE} - g\|_r^v\ = o\left(\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{tv(t)}\,dt\right)$$

Where $\omega(t)\ and\ v(t)$ denotes the Zygmund moduli of continuity such that $\frac{w(t)}{v(t)}$ is positive and increasing.

Theorem 2- Let g be a 2π periodic function, Lebesgue integrable in $[0\,,2\pi]$ and belonging to generalized Zygmund class $Z_r^{(w)}(r \geq 1)$. Then the degree of approximation of function g by (E,q)($\overline{N}$, Pn) product mean of Fourier series is given by

$$E_n(g) = \inf\|l_n(.)\|_p^v = o\left(\left(\frac{(n+1)\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)\left(\pi - \frac{1}{n+1}\right)\right)$$

where $\omega(t)\ and\ v(t)$ denotes the Zygmund moduli of continuity such that $\frac{w(t)}{tv(t)}$

is positive and decreasing.

Lemma–To prove the theorem we need the following Lemma.

Lemma 4(a) - For $0 \leq t \leq \frac{\pi}{n+1}$ we have $\sin nt = n\, sint$

$$|K_n(t)| = o(n) \qquad \text{.......(4.1)}$$

Proof - For $0 \leq t \leq \frac{\pi}{n+1}$ and $\sin nt = n\, sint$ then

$$|K_n(t)| = \left|\frac{1}{2\pi(1+q)^n}\sum_{k=o}^{n}\binom{n}{k} q^{k-v}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_v \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=o}^{n}\binom{n}{k} q^{k-v}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_v \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=o}^{n}\binom{n}{k} q^{k-v}(2k+1)\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_v\right\}\right|$$

$$\leq \frac{(2n+1)}{2\pi(1+q)^n}\left|\sum_{k=o}^{n}\binom{n}{k} q^{k-v}\right|$$

$$= o(n)$$

Lemma 4(b) - For $\frac{\pi}{n+1} \leq t \leq \pi$, $sin\frac{t}{2} \geq \frac{t}{\pi}$ and $sin\, nt \leq 1$ we have

$$|\, K_n(t)| = o\left(\frac{1}{t}\right) \qquad \text{……….(4.2)}$$

Proof - For $\frac{\pi}{n} \leq t \leq \pi$, $sin\frac{t}{2} \geq \frac{t}{\pi}$ and $sin\, nt \leq 1$

$$|\, K_n(t)| = \left|\frac{1}{2\pi(1+q)^n}\sum_{k=o}^{n}\binom{n}{k} q^{k-v}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_v \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}\right|$$

$$\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=o}^{n}\binom{n}{k} q^{k-v}\left\{\frac{1}{P_k}\sum_{v=0}^{k} p_v \frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}\right|$$

$$\leq \frac{1}{2t(1+q)^n}\left|\sum_{k=o}^{n}\binom{n}{k} q^{k-v}\right|$$

$$= o\left(\frac{1}{t}\right)$$

Lemma 4(c) – Let $g \in Z_p^{(w)}$ then for $0 < t \leq \pi$

(i) $\|\phi(.,t)\|_p = o(w(t))$

(ii) $\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \begin{cases} o(w(t) \\ o(w(y) \end{cases}$

(iii) If $\omega(t)$ and $v(t)$ are defined as in theorem then

$$\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \left\{v(y)\frac{\omega(t)}{v(t)}\right.$$

where $\phi(x,t) = g(x+t) + g(x-t) - 2g(x)$.

Proof of Theorem 1

Let $s_n(x)$ denotes the partial sum of fourier series given in (2.1) then we have

$$s_n(x) - g(x) = \frac{1}{2\pi}\int_0^{\pi} \emptyset(t)\frac{\sin(n+\frac{1}{2})t}{sin\frac{t}{2}}dt. \qquad \text{….(5.1)}$$

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The (E,q) transform E_n^q of s_n is given by

$$E_n^q - g(x) = \frac{1}{2\pi(1+q)^n}\int_0^\pi \emptyset(t)\left\{\sum_{k=0}^n \binom{n}{k} q^{n-k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right\}dt\,. \qquad \ldots\ldots(5.2)$$

The (E,q)($\overline{N}$, Pn) transform of $s_n(x)$ is given by

$$t_n^{NE}(g) - g(x) = \frac{1}{2\pi(1+q)^n}\int_0^\pi \emptyset(x,t)\sum_{k=o}^n \binom{n}{k} q^{k-v}\left\{\frac{1}{P_k}\sum_{v=0}^k p_v\frac{\sin(v+\frac{1}{2})t}{\sin(\frac{t}{2})}\right\}dt \qquad \ldots\ldots(5.3)$$

$$= \int_0^\pi \emptyset(x,t)\ k_n(t)dt\,. \qquad \ldots\ldots\ldots(5.4)$$

Let $l_n(x) = t_n^{NE} - g(x) = \int_0^\pi \emptyset(x,t)\ k_n(t)dt$ then

$$l_n(x+y) + l_n(x-y) - 2l_n(x) = \int_0^\pi [\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)\,dt$$

using the generalized Minkowaski's inequality we get

$$\|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |l_n(x+y) + l_n(x-y) - 2l_n(x)|^p\,dx\right\}^{\frac{1}{p}}$$

$$= \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^\pi [\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)\,dt\right|^p dx\right\}^{\frac{1}{p}}$$

$$\le \int_0^\pi \left\{\frac{1}{2\pi}\int_0^{2\pi} |[\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]\ k_n(t)|^p\,dx\right\}^{\frac{1}{p}} dt$$

$$= \int_0^\pi (|\ k_n(t)|^p)^{\frac{1}{p}}\left\{\frac{1}{2\pi}\int_0^{2\pi} |[\phi(x+y,t) + \phi(x-y,t) - 2\phi(x,t)]|^p\,dx\right\}^{\frac{1}{p}} dt$$

$$= \int_0^\pi \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\,dt$$

$$= \int_0^{\frac{1}{n+1}} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\,dt$$

$$+ \int_{\frac{1}{n+1}}^\pi \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p |\ k_n(t)|\,dt$$

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$$= I_1 + I_2(say). \quad \text{.............(5.5)}$$

Using lemma 4(a) and 4(c) and the monotonically of $\frac{\omega(t)}{v(t)}$ with respect to t we have

$$I_1 = \int_0^{\frac{1}{n+1}} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p| \; k_n(t)| \, dt$$

$$= \int_0^{\frac{1}{n+1}} o\left(v(y)\frac{\omega(t)}{v(t)}\right) o(n)\, dt$$

$$= o\left(nv(y)\int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)}\, dt\right).$$

Using second mean value theorem of integral we have

$$I_1 \leq o\left(nv(y)\int_0^{\frac{1}{n+1}} \frac{\omega(t)}{v(t)}\, dt\right)$$

$$= o\left(\frac{n}{n+1}\, v(y)\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)})\right)$$

$$= o\left(\, v(y)\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)})\right).$$

.............(5.6)

For I_2 using lemma 4(b) and 4(c) we have

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} \|\phi(.+y,t) + \phi(.-y,t) - 2\phi(.,t)\|_p| \; k_n(t)| \, dt$$

$$= o\left(\int_{\frac{1}{n+1}}^{\pi}\left(v(y)\frac{\omega(t)}{v(t)}\right)\frac{1}{t}\, dt\right)$$

$$= o\left(v(y)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\, v(t)}\right)\; dt\right)$$

.............(5.7)

from (5.5) (5.6) and (5.7) we get

$$\|l_n(.+y) + l_n(.-y) - 2l_n(.)\|_p = o\left(\, v(y)\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)})\right) + o\left(v(y)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\, v(t)}\right)\; dt\right)$$

$$\sup_{y\neq 0}\frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)\|_p}{v(y)} = o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\, v(t)}\right)\right). \quad \text{..........(5.8)}$$

Again using Lemma we have

$$\|l_n(.)\|_p \leq \left(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi} .\right)\|\phi(.,t)\|\,|K_n(t)|\, dt$$

$$= o\left(n\int_0^{\frac{1}{n+1}} \omega(t)dt\right) + o\left(\int_{\frac{1}{n+1}}^{\pi} \frac{\omega(t)}{t}\, dt\right)$$

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$$= o\left(\frac{n}{n+1}\,\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}\,dt\right)$$

$$= o\left(\,\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}\,dt\right) \qquad \text{..............(5.9)}$$

from (5.8) and (5.9) we have

$$\|l_n(.)\|_p^v = \|l_n(.)\|_p + \sup_{y\neq 0}\frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)\|_p}{v(y)}$$

$$= o\left(\,\omega\left(\frac{1}{n+1}\right)\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}\,dt\right) + o\left(\frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + o\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\,v(t)}\right)\right)$$

$$= \sum_{i=1}^{4} J_i.$$

Now we write J_1 in terms of J_3 *and* J_2 , J_3 *in* term of J_4,

in view of the monotonicity of v(t) we have

$$\omega(t) = \left(\frac{\omega(t)}{v(t)}\right),\quad v(t) \leq v(\pi)\left(\frac{\omega(t)}{v(t)}\right) = o\left(\frac{\omega(t)}{v(t)}\right) \quad for\ 0 < t \leq \pi$$

therefore we can write

$$J_1 = o(\,J_3).$$

Again using monotonicity of v(t)

$$J_2 = \int_{\frac{1}{n+1}}^{\pi}\frac{\omega(t)}{t}\,dt = \int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\,v(t)}\right)dt \;\leq\; v(\pi)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\,v(t)}\right)dt \;= o(J_4)$$

............(5.10)

using the fact $\frac{\omega(t)}{v(t)}$ is positive and non decreasing , we have

$$J_4 = \int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\,v(t)}\right)dt \;= \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{t}\right)dt \;\geq\; \frac{\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}$$

therefore we can write

$$J_3 = o(J_4).$$

So we have

$$\|l_n(.)\|_p^v = o(J_4) = o\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\,v(t)}\right)dt\right).$$

Hence

$$E_n(g) = \inf\|l_n(.)\|_p^v = o\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\,v(t)}\right)dt\right)$$

This completes the proof Theorem 1.

Proof of Theorem 2 – We have from theorem 1

$$E_n(g) = \inf\|l_n(.)\|_p^v = o\left(\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\omega(t)}{t\,v(t)}\right)\,dt\right)$$

we assume that $\frac{w(t)}{tv(t)}$ is positive and decreasing in t then

$$E_n(g) = \inf\|l_n(.)\|_p^v = o\left(\left(\frac{\omega\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}v\left(\frac{1}{n+1}\right)}\right)\int_{\frac{1}{n+1}}^{\pi}\,dt\right)$$

$$E_n(g) = \inf\|l_n(.)\|_p^v = o\left(\left(\frac{\omega\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}v\left(\frac{1}{n+1}\right)}\right)(t)_{\frac{1}{n+1}}^{\pi}\right)$$

$$E_n(g) = \inf\|l_n(.)\|_p^v = o\left(\left(\frac{\omega\left(\frac{1}{n+1}\right)}{\frac{1}{n+1}v\left(\frac{1}{n+1}\right)}\right)\left(\pi - \frac{1}{n+1}\right)\right)$$

$$E_n(g) = \inf\|l_n(.)\|_p^v = o\left(\left(\frac{(n+1)\omega\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)\left(\pi - \frac{1}{n+1}\right)\right).$$

This completes the proof Theorem 2.

REFERENCES

1. Kim, J. (2021). Degree of Approximation of Function of Class $Zyg^{w}(\alpha,\gamma)$ by Cesaro Means of Fourier series. *East Asian Math. J.*, Vol. 37. No. 3 pp. 289 – 293.
2. Leinder, L. (1981). Strong approximation and generalized Zygmund class. *Acta Sci. Math. 43, no.* 3-4, 301 – 309.
3. Moricz, F. (2010) . Enlarged Lipschitz and Zygmund classes of function and Fourier transformation. *East J. Approx. 16*. No. 3, 259 – 271.
4. Moricz, F. & Nemeth(2007) . Generalized Zygmund classes of function and Approximation by Fourier series . *Acta Sci. Math.* , no. 3 – 4, 637 – 647.
5. Mishra, A., Padhy, B. P., & Mishra, U. (2020). On approximation of signal in the Generalized Zygmund class using (E, r) (N, qn) mean of conjugate derived Fourier series. *EJPAM* , Vol 13 , No. 5, 1325 – 1336.
6. Pradhan, T., Paikray, S. K., Das, A. A. &dutta, H. (2019). On approximation of Signal in the generalised Zygmund class via (E,1) $(\bar{N}, P_n)$ summability mean of Conjugate Fourier series. *Proyecciones (Antofagasta. Online*) , Vol 38, n. 5, 981- - 998.
7. Singh, M. V., Mittal, M. L. , & Rhoades, B. E. (2017).Approximation of functions in the generalized Zygmund class using Hausdorff means. *Journal of Inequalities and Applications*. 2017:101 DOI 10.1186/s13660-017-1361-8 . pp. 1- 11.
8. Sinha, S .K. & Shrivastava, U. K. (2022). Degree of Approximation of Function of Class $\mathbf{Z^{w}(\alpha,\gamma)}$ by $(\mathbf{N},\mathbf{p},\mathbf{q})\mathbf{C_1}$ Mean of Fourier Series. *South East Asian J of Mathematics and Mathematical Science,* (18)2, 117 -124.