An Analytical Approach to Mixed-Constrained Quadratic Optimal Control Problems

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ABSTRACT

This study investigates the analytical solution of quadratic optimal control problems (OCPs) constrained by ordinary differential equations (ODEs) with real and coefficients. The formulation is based on the application of first-order optimality conditions to the Hamiltonian function, which yield a coupled system of first-order differential equations representing the necessary conditions for optimality. The resulting system is solved analytically using the method of eigenvalue decomposition and state transformation to determine the optimal state, control, and adjoint variables. The analytical procedure is illustrated through two examples of quadratic OCPs, confirming the effectiveness and accuracy of the developed method in deriving exact optimal solutions.

INTRODUCTION

Optimization is the process of determining the best possible outcome under given conditions, either by minimizing effort or maximizing desired benefits. It provides the mathematical foundation for decision-making in engineering, science, and economics by expressing objectives as functions of decision variables. The development of optimization theory has profoundly influenced control theory, operations research, and computational mathematics. In particular, optimal control theory extends classical optimization to dynamical systems, seeking control and state trajectories that minimize an objective function subject to system dynamics and constraints. The growing demand for efficient computational strategies has led to the emergence of numerical methods such as the penalty function, Lagrangian, and conjugate gradient approaches for solving constrained optimization problems [13, 14].

Over the years, extensive studies have been carried out to develop efficient algorithms for constrained and unconstrained optimization problems [5-9, 11, 15, 16]. Naidu [10] provided a rigorous foundation for optimal control systems and discussed analytical and numerical techniques for solving such problems. [1] presented a method for solving optimal control problems with mixed constraints by applying the first-order optimality conditions derived from the Hamiltonian function. The resulting system of non-homogeneous first-order ODEs was then solved using the fundamental matrix approach. Similarly, in [2], the analytical solutions of optimal control problems governed by ODEs were investigated. The study employed the first-order optimality conditions of the Hamiltonian function to derive and solve the associated system of first-order ODEs, leading to the determination of the optimal state, control, and adjoint variables, as well as the optimal objective value.

Quadratic optimal control problems constrained by ODEs with real and vector-matrix coefficients were considered. The analytical formulation is derived by applying first-order optimality conditions to the Hamiltonian function, leading to a system of first-order ODEs solved using a state transformation approach. For numerical implementation, the objective functional is discretized via Simpson's one-third rule, while the system dynamics are approximated using a fifth-order implicit integration scheme. The resulting discretized problem is transformed into an unconstrained optimization model using the Augmented Lagrangian Method and solved through the Conjugate Gradient Method (CGM) and FICO Xpress Mosel. Comparative analysis demonstrates that FICO Xpress Mosel achieves faster convergence and higher numerical stability, particularly for large-scale problems, highlighting its efficiency in solving complex quadratic OCPs [3, 12].

The study focuses on a semi-analytical approach for solving a generalized quadratic optimal control problem governed by ODEs. The analytical formulation is derived by applying the first-order optimality conditions to the Hamiltonian function, which provide the necessary conditions for optimality. To solve the resulting general Riccati differential equation, the Adomian Decomposition Method (ADM) is employed, representing the nonlinear system as an infinite series that converges toward the exact solution. This procedure yields the optimal state, control, and adjoint variables, from which the optimal value of the objective functional is determined. The effectiveness of the proposed method is demonstrated through two illustrative examples of optimal control problems constrained by ODEs [4].

METHODOLOGY

In this section, optimal control problems constrained by ODEs with mixed constraints and real co-efficients are considered. The necessary conditions for this class of optimal control problems considered are derived. This leads to the analytical solutions of the optimal control problems constrained by ODEs with mixed constraints and real co-efficients.

In the framework of optimal control theory, we consider a control system aimed at finding an admissible control $u(t)$ and its corresponding state trajectory $x(t)$ that minimize a given cost functional. When only equality restrictions are present, the problem can be formulated as

$$\begin{aligned} \text{Minimize} \quad & f(x,u,t), \\ \text{subjectto} \quad & h(x,u,t)=0, \end{aligned} \qquad (2.1)$$

where f denotes the performance index and h represents the set of equality constraints.

If inequality conditions are imposed, the optimal control problem is expressed as

$$\begin{aligned} \text{Minimize} \quad & f(x,u,t), \\ \text{subjectto} \quad & g(x,u,t)\leq 0, \end{aligned} \qquad (2.2)$$

where g is a vector function specifying the inequality restrictions that define the feasible region.

By combining (2.1) and (2.2), we obtain the general mixed-constrained optimal control problem:

$$\begin{aligned} \text{Minimize} \quad & f(x,u,t), \\ \text{subjectto} \quad & h(x,u,t)=0, \\ & g(x,u,t)\leq 0, \end{aligned} \qquad (2.3)$$

where

$$f:\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}\to\mathbf{R},\quad h:\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}\to\mathbf{R}^p,\quad g:\mathbf{R}^n\times\mathbf{R}^m\times\mathbf{R}\to\mathbf{R}^r,$$

and $m\leq n,\ p\leq n$. The components of h and g are given by $h_1,h_2,\dots,h_p$ and $g_1,g_2,\dots,g_r$, respectively.

To handle the inequality constraints more conveniently, the mixed-constrained problem in (2.3) can be transformed into an equality-constrained formulation by introducing a vector of nonnegative auxiliary variables $z=(z_1,z_2,\dots,z_r)$. The equivalent problem becomes

$$\begin{aligned} \text{Minimize} \quad & f(x,u,t), \\ \text{subjectto} \quad & h_i(x,u,t) = 0, \quad i = 1,2,\dots,p, \\ & g_j(x,u,t) + z_j^2(x,u,t) = 0, \quad j = 1,2,\dots,r, \end{aligned} \tag{2.4}$$

where the squared slack variables z_j^2 convert inequalities into differentiable equalities while preserving feasibility.

Necessary Conditions for a General Optimal Control Problems with Mixed Constraints

The general formulation of an optimal control problem involving both equality and inequality constraints can be expressed as

$$\begin{aligned} \text{Minimize} \quad I(x,u) = & \int_0^T f(x,u,t)\,dt, \\ \text{subjectto} \quad \dot{x}(t) = & \; h(x,u,t), \quad x(0) = x_0, \\ & g(x,u,t) \le 0, \quad t \in [0,T], \end{aligned} \tag{2.5}$$

where $x(t) \in \mathbf{R}^n$ and $u(t) \in \mathbf{R}^m$. The functions $f: \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}$, $h: \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^p$, and $g: \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R} \to \mathbf{R}^r$ are assumed to be continuously differentiable, and T represents the terminal time of the process.

The Hamiltonian function associated with (2.5) is defined as

$$H(x,u,\lambda,t) = f(x,u,t) + \lambda\, h(x,u,t), \tag{2.6}$$

where $H: \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$, and $\lambda(t) \in \mathbf{R}^n$ denotes the adjoint (or costate) variable.

To incorporate the inequality constraints into the formulation, the Lagrangian function is extended as

$$L(x,u,\lambda,\mu,t) = H(x,u,\lambda,t) + \mu\, g(x,u,t), \tag{2.7}$$

where $\mu(t) \in \mathbf{R}^q$ is the Lagrange multiplier corresponding to the inequality constraint and satisfies the complementary slackness conditions

$$\mu \ge 0, \qquad \mu\, g(x,u,t) = 0. \tag{2.8}$$

Applying the Euler–Lagrange principle to the Lagrangian in (2.7) gives

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0, \tag{2.9}$$

$$\frac{\partial L}{\partial u} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = 0, \tag{2.10}$$

$$\frac{\partial L}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = 0, \tag{2.11}$$

$$\frac{\partial L}{\partial \mu} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mu}}\right) = 0. \tag{2.12}$$

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Substituting (2.7) into (2.9)–(2.12) yields

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial h}{\partial x} + \mu \frac{\partial g}{\partial x} + \dot{\lambda} = 0, \quad (2.13)$$

$$\frac{\partial f}{\partial u} + \lambda \frac{\partial h}{\partial u} + \mu \frac{\partial g}{\partial u} = 0, \quad (2.14)$$

$$\frac{\partial f}{\partial \lambda} + h \frac{\partial \lambda}{\partial \lambda} + \mu \frac{\partial g}{\partial \lambda} = 0, \quad (2.15)$$

$$\frac{\partial f}{\partial \mu} + \lambda \frac{\partial h}{\partial \mu} + g \frac{\partial \mu}{\partial \mu} = 0. \quad (2.16)$$

From these expressions, the necessary optimality conditions are obtained as

$$\dot{\lambda} = -\frac{\partial H}{\partial x} - \mu \frac{\partial g}{\partial x}, \quad \text{(Adjoint Equation)}, \quad (2.17)$$

$$\frac{\partial f}{\partial u} = -\mu \frac{\partial g}{\partial u} - \lambda \frac{\partial h}{\partial u}, \quad \text{(Optimality Condition)}, \quad (2.18)$$

$$\dot{x}(t) = h(x, u, t), \quad \text{(State Equation)}, \quad (2.19)$$

$$\frac{\partial H}{\partial \mu} = -g \frac{\partial \mu}{\partial \mu} - \lambda \frac{\partial h}{\partial \mu}, \quad \text{(Stationary Condition)}. \quad (2.20)$$

Equations (2.17) and (2.19) form a system of first-order ODEs that can be solved simultaneously once suitable boundary conditions are imposed. When only one boundary condition is specified, the remaining one is obtained through the transversality (free-end) condition given by

$$\frac{\partial H}{\partial \dot{x}} = 0 \quad \text{orequivalently} \quad \lambda(T) = 0, \quad \text{(Transversality Condition. (2.21)}$$

For a solution to be optimal, all the necessary conditions defined in (2.17)–(2.21) must hold simultaneously. If any of these are violated, the control and state trajectories are not optimal. The state and adjoint (costate) equations are differential (dynamic) in nature, whereas the control equation is algebraic (static). Together, these equations form a two-point boundary value problem: the state equation evolves forward in time, while the costate equation evolves backward. Such problems often require iterative numerical procedures for their resolution, which typically lead to open-loop optimal control laws.

Analytical Solution of Optimal Control Problems with Mixed Constraints and Real Co-efficients

Consider a quadratic optimal control problem governed by a linear state equation. The objective is to determine the control $u(t)$ and the corresponding state $x(t)$ that minimize a performance index subject to both dynamical and inequality constraints. The problem can be formulated as

$$\text{Minimize} \quad I(x, u) = \int_0^T (px^2(t) + qu^2(t))\, dt, \quad (2.22)$$

$$\text{subjectto} \quad \dot{x}(t) = ax(t) + bu(t), \quad (2.23)$$

$$cx(t) + du(t) \leq 0, \quad x(0) = x_0, \quad (2.24)$$

where a, b, c, d are real parameters and $p, q > 0$ are positive weighting constants.

Theorem 2.1 *Let $u^*(t)$ denote the optimal control that minimizes $I(x, u)$ in the admissible set U, and let $x^*(t)$ be the corresponding optimal state satisfying (2.23). Then there exists an adjoint variable $\mu(t)$ that satisfies*

$$\dot{\mu}(t) = -2p\, x(t) - a\, \mu(t) - c\, \lambda, \quad t \in [0, T], \quad (2.25)$$

together with the transversality and optimality conditions

$$\mu(T) = 0, \quad (2.26)$$

$$u^*(t) = \frac{-b\,\mu(t) - d\,\lambda}{2q}. \quad (2.27)$$

Proof. To derive the necessary conditions, we construct the augmented performance functional corresponding to (2.22)–(2.24). The Hamiltonian is defined as

$$H(x,u,\mu) = p\,x^2(t) + q\,u^2(t) + \mu(a\,x(t) + b\,u(t)), \quad (2.28)$$

while the Lagrangian incorporating the inequality constraint is given by

$$L(x,u,\mu,\lambda) = p\,x^2(t) + q\,u^2(t) + \mu(a\,x(t) + b\,u(t)) + \lambda(c\,x(t) + d\,u(t)). \quad (2.29)$$

Applying the Euler–Lagrange principle to L, treated as a function of x, u, μ, λ, yields the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}, \quad (2.30)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial u}, \quad (2.31)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mu}}\right) = \frac{\partial L}{\partial \mu}, \quad (2.32)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = \frac{\partial L}{\partial \lambda}. \quad (2.33)$$

Solving (2.30)–(2.33) gives

$$\dot{\mu}(t) = -2p\,x(t) - a\,\mu(t) - c\,\lambda, \quad (2.34)$$

$$u^*(t) = \frac{-b\,\mu(t) - d\,\lambda}{2q}, \quad (2.35)$$

$$\dot{x}(t) = a\,x(t) - b\left(\frac{b\,\mu(t) + d\,\lambda}{2q}\right), \quad (2.36)$$

$$c\,x(t) + d\,u(t) = 0. \quad (2.37)$$

Equations (2.34) and (2.36) form a coupled two-point boundary value problem that provides the necessary conditions for optimality. In compact form, this system can be written as

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\mu}(t) \end{pmatrix} = \begin{pmatrix} -2p & -a \\ a & -\frac{b^2}{2q} \end{pmatrix} \begin{pmatrix} x(t) \\ \mu(t) \end{pmatrix} + \begin{pmatrix} -c\,\lambda \\ -\frac{b\,d\,\lambda}{2q} \end{pmatrix}. \quad (2.38)$$

Let

$$M = \begin{pmatrix} -2p & -a \\ a & -\frac{b^2}{2q} \end{pmatrix}.$$

The eigenvalues of M are obtained as

$$\lambda_1 = -\frac{4pq - \sqrt{-(b^2 + 4aq - 4pq)(-b^2 + 4aq + 4pq)} + b^2}{4q}, \quad (2.39)$$

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$$\lambda_2 = -\frac{4pq+\sqrt{-(b^2+4aq-4pq)(-b^2+4aq+4pq)}+b^2}{4q}. \quad (2.40)$$

The corresponding eigenvectors are

$$U_1 = \begin{pmatrix} \frac{4b^2-(4pq-\sqrt{-(b^2+4aq-4pq)(-b^2+4aq+4pq)}+b^2)}{4aq} \\ 1 \end{pmatrix}, \quad (2.41)$$

$$U_2 = \begin{pmatrix} \frac{4b^2-(4pq+\sqrt{-(b^2+4aq-4pq)(-b^2+4aq+4pq)}+b^2)}{4aq} \\ 1 \end{pmatrix}. \quad (2.42)$$

Hence, the complementary solution of (2.38) is

$$V(t) = c_1 \overleftarrow{U_1} e^{\lambda_1 t} + c_2 \overleftarrow{U_2} e^{\lambda_2 t}. \quad (2.43)$$

Using the method of undetermined coefficients, we assume a particular solution of the form

$$\begin{pmatrix} x_p(t) \\ \mu_p(t) \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \overleftarrow{\gamma}. \quad (2.44)$$

Substituting (2.44) into (2.38) gives

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2p & -a \\ a & -\frac{b^2}{2q} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} -c\,\lambda \\ -\frac{b\,d\,\lambda}{2q} \end{pmatrix}. \quad (2.45)$$

Solving this linear system yields

$$\alpha = \frac{\lambda\,(b^2c-a\,b\,d)}{2(a^2q-b^2p)}, \quad (2.46)$$

$$\beta = -\frac{\lambda\,(a\,c\,q-b\,d\,p)}{2(a^2q-b^2p)}. \quad (2.47)$$

Therefore, the complete solution of (2.38) can be expressed as

$$V(t) = c_1 \overleftarrow{U_1} e^{\lambda_1 t} + c_2 \overleftarrow{U_2} e^{\lambda_2 t} + \overleftarrow{\gamma}, \quad (2.48)$$

where c_1 and c_2 are constants of integration. Knowing the initial condition $x(0)$, one can determine $\mu(T)$ to satisfy the transversality condition. The constants c_1 and c_2 are then obtained by substituting these boundary conditions into (2.48).

RESULTS AND DISCUSSIONS

Example 1

$$Minimize\ \ I(u) = \int_0^T \left(x^2(t) + u^2(t)\right) \quad (3.1)$$

$$Subjectto\ \ \dot{x}(t) = \alpha u(t),\ x(0) = x_0 \quad (3.2)$$

$$u(t) \geq 0,\ 0 \leq t \leq T. \quad (3.3)$$

Take $T = 1$ and $x_0 = 1$.

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Solution 1 *The Hamiltonian is given as*

$$H = (x^2(t) + u^2(t)) + \mu(\dot{x}(t) - u(t)) \tag{3.4}$$

Applying the necessary conditions for optimality

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{x}}] = \frac{\partial H}{\partial x} \tag{3.5}$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{u}}] = \frac{\partial H}{\partial u} \tag{3.6}$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{\mu}}] = \frac{\partial H}{\partial \mu} \tag{3.7}$$

Applying equations (3.5), (3.6) and (3.6) on equation (3.4), we have

$$\dot{\mu} = 2x \tag{3.8}$$

$$u^* = 2u - \mu \tag{3.9}$$

$$\dot{x} = u \tag{3.10}$$

From Equation (3.9)

$$u^* = \frac{\mu}{2} \tag{3.11}$$

In view of equation (3.11), equation (3.10) becomes

$$\dot{x}(t) = \frac{\mu}{2} \tag{3.12}$$

Next, we solve for x and μ using matrix method

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \mu^* \end{bmatrix} \tag{3.13}$$

Let $A = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}$, the eigenvalues A using the characteristics equation $|A - \lambda I| = 0$ are $\lambda = \pm 1$.

When $\lambda = 1$, we obtain the eigenvectors

$$\begin{bmatrix} -1 & \frac{1}{2} \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.14}$$

$$-x + \frac{1}{2}\mu = 0 \tag{3.15}$$

$$2x - \mu = 0 \tag{3.16}$$

This implies that when $x = 1, \mu = 2$.

When $\lambda = -1$,

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.17}$$

$$x + \frac{1}{2}\mu = 0 \tag{3.18}$$

$$2x + \mu = 0 \tag{3.19}$$

When $x = 1, \mu = -2$.

Therefore, the general solution of 3.13 is given as

$$V(t) = c_1 \overleftarrow{U_1} e^{\lambda_1 t} + c_2 \overleftarrow{U_2} e^{\lambda_2 t} + \overleftarrow{\gamma} \tag{3.20}$$

where c_1 and c_2 are constants of integration and $\overleftarrow{U_1}$ and $\overleftarrow{U_2}$ represent the eigenvectors of A. Since we know $x(0)$, the task is to choose $\mu(T)$ so that the transversality condition is satisfied. Hence,

$$x^*(t) = c_1 e^t + c_2 e^{-t} \tag{3.21}$$

$$\mu^*(t) = 2c_1 e^t - 2c_2 e^{-t} \tag{3.22}$$

Applying the initial conditions $x(0) = 1$ and $\mu(0) = 1$, we have

$$x^*(0) = c_1 + c_2 = 1 \tag{3.23}$$

$$\mu^*(1) = 2c_1 e^1 - 2c_2 e^{-1} = 0 \tag{3.24}$$

Solving (3.23) and (3.24) simultaneously, we obtain the values for $= c_2 e^{-2}$ and $c_2 = \frac{1}{e^{-2}+1}$ Hence,

$$x^*(t) = \frac{e^{t-2} + e^{-t}}{e^{-2}+1} \tag{3.25}$$

and

$$\mu^*(t) = \frac{2(e^{t-2} - e^{-t})}{e^{-2}+1} \tag{3.26}$$

Recall from equation (3.11) that $u = \frac{\mu}{2}$. Hence,

$$u^*(t) = \frac{e^{t-2} - e^{-t}}{e^{-2}+1} \tag{3.27}$$

Substituting $x^*(t)$ and $u^*(t)$ into $J(x, u) = \int_0^1 (x^2(t) + u^2(t))dt$, and solving the resulting integral, we obtain an optimal value for $J(x, u)$

$$J^*(x, u) = \int_0^1 (\frac{e^{t-2} + e^{-t}}{e^{-2}+1})^2 + (\frac{e^{t-2} - e^{-t}}{e^{-2}+1})^2 dt \tag{3.28}$$

$$J^*(x,u) = \int_0^1 \frac{2(e^{2t-4}+e^{-2t})}{(e^{-2}+1)^2} dt \quad (3.29)$$

$$J^*(x,u) = \frac{2}{(e^{-2}+1)^2} \int_0^1 (e^{2t-4} + e^{-2t}) dt \quad (3.30)$$

$$J^*(x,u) = \frac{2}{(e^{-2}+1)^2} [\frac{e^{2t-4}}{2} - \frac{e^{-2t}}{2})], (0 \leq t \leq 1) \quad (3.31)$$

Applying the lower and upper limits, we have,

$$J^*(x,u) = \frac{2}{(e^{-2}+1)^2} [(\frac{e^{-2}}{2} - \frac{e^{-2}}{2}) - (\frac{e^{-4}}{2} - \frac{e^{0}}{2})] \quad (3.32)$$

$$J^*(x,u) = \frac{2}{(e^{-2}+1)^2} [0.4908421806] \quad (3.33)$$

$$J^*(x,u) = \frac{0.9816843611}{1.288986205} = 0.761594156 \quad (3.34)$$

Example 2

$$Minimize\ \ I(u) = x(T) + \int_0^T (u(t))^2 dt \quad (3.35)$$

$$Subjectto\ \ \dot{x} = ax(t) - u(t),\ x(0) = x_0 \quad (3.36)$$

$$u(t) \leq 0,\ \ 0 < t < T. \quad (3.37)$$

Solution 2 *The Hamiltonian is given as*

$$H = u^2(t) + \mu(\dot{x}(t) - ax(t) + u(t) \quad (3.38)$$

Thus, the E-L system can be written as

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{x}}] = \frac{\partial H}{\partial x} \quad (3.39)$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{u}}] = \frac{\partial H}{\partial u} \quad (3.40)$$

$$\frac{d}{dt}[\frac{\partial H}{\partial \dot{\mu}}] = \frac{\partial H}{\partial \mu} \quad (3.41)$$

Applying equations (3.39), (3.40) and (3.41) on equation (3.38), we have

$$\dot{\mu}(t) = -a\mu(t) \quad (3.42)$$

$$2u(t) + \mu(t) = 0 \quad (3.43)$$

$$u^*(t) = -\frac{\mu(t)}{2} \quad (3.44)$$

$$\dot{x}(t) = ax(t) - u(t) \quad (3.45)$$

This implies that

$$\dot{x}(t) = ax(t) + \frac{\mu(t)}{2} \quad (3.46)$$

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$$\dot{\mu}(t) = -a\mu(t) \tag{3.47}$$

The general solution of the first order ordinary differential equation given by equation (3.42) is

$$\mu(t) = ke^{-at} \tag{3.48}$$

From the transversality condition

$$\mu(T) = \frac{\partial x(T)x(T)}{\partial x(T)} = 1 \tag{3.49}$$

$$\mu(T) = ke^{-aT} = 1 \tag{3.50}$$

$$k = \frac{1}{e^{-aT}} = e^{aT} \tag{3.51}$$

Hence

$$\mu(t) = e^{a(T-t)} \tag{3.52}$$

Since $u^*(t) = -\frac{\mu(t)}{2}$, it implies that

$$u^*(t) = -\frac{e^{a(T-t)}}{2} \tag{3.53}$$

In view of equation (3.53), equation (3.45) now implies that

$$\dot{x}(t) = ax(t) + \frac{e^{a(T-t)}}{2} \tag{3.54}$$

$$\dot{x} - ax(t) = \frac{e^{a(T-t)}}{2} \tag{3.55}$$

Using the integrating factor e^{-at} to solve equation (3.55), we have

$$xe^{-at} = \frac{1}{2}\int e^{(aT-2at)}dt \tag{3.56}$$

$$x(t) = -\frac{1}{4a}e^{aT-at} + ce^{at} \tag{3.57}$$

Applying the initial condition $x(0) = x_0$, we obtain the value of the constant of integration c as follows

$$x(t) = -\frac{1}{4a}e^{aT-a(0)} + ce^{a(0)} = x_0 \tag{3.58}$$

$$c = x_0 + \frac{1}{4a}e^{aT} \tag{3.59}$$

This implies that

$$x(t) = -\frac{1}{4a}e^{aT-at} + x_0e^{at} + \frac{1}{4a}e^{aT+at} \tag{3.60}$$

$$x(t) = x_0e^{at} + \frac{1}{4a}e^{aT}(e^{at} - e^{-at}) \tag{3.61}$$

Recall from equation (3.35) that

$$J(x,u) = x(T) + \int_0^T (u(t))^2 dt \tag{3.62}$$

This implies that

$$J(x,u) = x(T) + \int_0^T \left(\frac{-e^{a(T-t)}}{2}\right)^2 dt \tag{3.63}$$

$$= x(T) + \frac{1}{4}\int_0^T e^{2a(T-t)} dt \tag{3.64}$$

$$= x(T) + \left[\frac{-1}{8a} e^{2a(T-t)}\right]_0^T \tag{3.65}$$

$$= x(T) - \frac{1}{8a}\left(e^{2a(T-T)} - (e^{2a(T-0)})\right) \tag{3.66}$$

$$= x(T) + \frac{1}{8a}(e^{2aT} - 1) \tag{3.67}$$

CONCLUSION

This research has presented an analytical framework for solving continuous quadratic optimal control problems governed by ODEs with mixed constraints and real coefficients. The application of first-order optimality conditions to the Hamiltonian function provided the necessary equations for optimality, which were solved analytically to obtain the optimal state, control, and adjoint variables. The analytical approach developed in this study offers a rigorous and computationally effective tool for addressing a broad class of optimal control problems encountered in engineering and applied sciences.

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