Fixed Point Theorem in Controlled Metric Spaces

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ABSTRACT

In this paper, we present a new fixed point theorem that generalizes the result of Souayah and Hidri [5] within the context of the generalized controlled metric space with three control functions.

Keywords: Caristi contraction, Controlled metric space, Double controlled metric space.

INTRODUCTION

Fixed point theory has numerous applications in science and engineering. The proof of existence and uniqueness theorems for solutions of ordinary and boundary value problems often rely on the application of various fixed point theorems. Among them, the Banach contraction principle is the most widely used, and it has been extensively generalized by modifying the contractive condition or by extending the underlying metric space.Kamran et al. [1] originate the theory of an extended *b*-metric space. Mlaiki et al. [2] originate the theory of a controlled metric space. Abdeljawad et al. [3] further extended this idea by addressing the double controlled metric space, while Tasneem et al. [4] advanced the theory by defining a triple controlled metric space. Recently, Souayah and Hidri [5] established a new fixed point theorem inspired by Caristi's contraction principle within the scheme of controlled metric spaces. Furthermore, Singh et al. [6] originate a generalization of the controlled metric space by integrating three control functions. In this paper, we present a new fixed point theorem that generalizes the result of Souayah and Hidri [5] within the context of the generalized controlled metric space with three control functions.

Preliminaries

Definition 2.1 [1] Let $\mathcal{Z} \neq \Phi$, $d : \mathcal{Z} \times \mathcal{Z} \rightarrow [0,\infty)$ be a non – negative function and $\alpha : \mathcal{Z} \times \mathcal{Z} \rightarrow [1,\infty)$ be control functions satisfying the following conditions:

I. $d(f,g) = 0$ if and only if $f = g$,

II. $d(f,g) = d(g,f)$

III. $d(f,g) \leq \alpha(f,g) [d(f,h) + d(h,g)]\ \forall\ f,g,h \in \mathcal{Z}$. Then
d is called a extended b- metric and $(\mathcal{Z}, d)$ is a extended b-metric space.

Definition 2.2 [2] Let $\mathcal{Z} \neq \Phi$, $d : \mathcal{Z} \times \mathcal{Z} \rightarrow [0,\infty)$ be a non – negative function and $\alpha : \mathcal{Z} \times \mathcal{Z} \rightarrow [1,\infty)$ be control functions satisfying the following conditions:

I. $d(f,g) = 0$ if and only if $f = g$,

II $d(f,g) = d(g,f)$

III. $d(f,g) \leq \alpha(f,h)\, d(f,h) + \alpha(h,g)\, d(h,g)\ \forall\ f,g,h \in \mathcal{Z}$. Then
d is called a controlled metric and $(\mathcal{Z},d)$ is a controlled metric space.

Example 2.3[2] Let Ζ = [0,∞) and a function define by d : Ζ x Ζ → [0,∞) such that d(f,g) = 0 if f = g,

$= \frac{1}{f}$, if f is even and g is odd ,

$= \frac{1}{g}$, if f is odd and g is even,

= 1 otherwise,

and the controlled function α : Ζ x Ζ → [1,∞)

α(f,g) = f, if f is even and g is odd

= g, if f is odd and g is odd

= 1, otherwise.

Then d is called a controlled metric and (Ζ, d) is a controlled metric space.

Theorem 2.4 [5] Let (Ζ,d,α) be complete controlled metric space and Ḫ: Ζ→Ζ a function such that for all f,g ε Ζ such that

d(Ḫg,Ḫg,) ≤ (£(f)- £(Ḫf)) d(f, g) Where £: Ζ→R is bounded below function.

Suppose f_0 ε Ζ, the sequence $\{f_n\}$ define by $f_n = Ḫ^n f_0$ satisfies

$\lim_{n\to\infty} \alpha(f,f_n,)$ $\lim_{n\to\infty} \alpha(f_n,f)$ exists and are finite for every f ε Ζ and the controlled function condition holds

$\sup_{m\geq 1}\lim_{i\to\infty} \alpha(f_{i+1},f_{i+2})\, \alpha(f_{i+1},f_m)/ \alpha(f_i,f_{i+1}) < 1/k$ where k ε (0,1). Then Ḫ has a unique fixed point in Ζ.

Example 2.5 [5] Let Ζ= [0,1] , d(f,g) = $|f-g|^2$ and α(f,g) = f + g +2. It is easy to prove that (Ζ,d,α) is complete controlled metric space but not traditional metric space or b- metric space. Let the function Ḫ: Ζ→Ζ and £ : Ζ→R defined by Ḫ(f) = f/5+1/2 and £(f) = $2f^2 + 3f$.

(£(f)- £(Ḫf))d(f,g) = $2f^2 + 3f - \{ 2(f/5+1/2)^2 + 3(f/5+1/2)\} |f-g|^2$

= $(48f^2/25 +2f-2)\ |f-g|^2$

However, the original problem restates the whole expression as

(£(f)- £(Ḫf))d(f, g) = $(52f^2/25 +4f+2)\ |f - g|^2$

d(Ḫf,Ḫg) = $|f - g|^2/25$. Hence,

d(Ḫf,Ḫg) ≤ (£(f)- £(Ḫf))d(f, g)

Finally Ḫ has a unique fixed point is Ḫ(5/8) = 5/8 ε [0,1].

Definition 2.6 [3] Let Ζ ≠ Φ, d : Ζ x Ζ → [0,∞)be a non- negative function and α, β : Ζ x Ζ → [1,∞) be two control functions satisfying the following conditions:

I. d(f,g) = 0 if and only if f = g,

II. d(f,g) = d(g,f)

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III. $d(f,g) \leq \alpha(f,h)d(f,h) + \beta(h,g)d(h,g)\ \forall\ f,g,h\ \varepsilon\ Ȥ$. Then d is called a double controlled metric and (Ȥ,d) is a double controlled metric space.

Example2.7 [3]. Let Ȥ = [0,∞) and a function define d : Ȥ x Ȥ→ [0,∞) such that

$d(f,g) = 0$, if x = y,

$= \frac{1}{f}$, if f ≥ 1 and g ε [0,1)

$= \frac{1}{g}$, if g ≥ 1 and f ε [0,1)

= 1, otherwise

and the controlled functions α, β : Ȥ x Ȥ → [1,∞) by

$\alpha(f,g) = f$, if f, g ≥ 1 and $\beta(f,g) = 1$, if f,g < 1

= 1, otherwise = max{f,g}, otherwise.

Then d is a double controlled metric and (Ȥ,d) is a double controlled metric space.

Singh et al. introduce a generalization of the controlled metric space by introducing three control functions.

Definition 2.8 [6] Let Ȥ ≠ Φ, d : Ȥ x Ȥ x Ȥ → [0,∞)be a non- negative function and α,β,γ : Ȥ x Ȥ x Ȥ → [1,∞) be three control functions satisfying the following conditions:

I. d(f,g,h) = 0 if at least two of the three points are the same.

II. For f,g ε Ȥ such that f ≠ g there exist a point h ε Ȥ such that d(f,g,h) ≠ 0.

III. For f,g,h ε Ȥ, d(f,g,h) = d(f,h,g) = d(g,h,f) = d(h,f,g) = d(g,f,h) = d(h,g,f)

IV. For f,g,h,a ε Ȥ, $d(f,g,h) \leq \alpha(f,g,a)d(f,g,a) + \beta(g,h,a)d(g,h,a) + \gamma(h,f,a)d(h,f,a)$.

Then d is called controlled metric and (Ȥ,d) is a controlled metric space.

Example 2.9 Let Ȥ = (0,1) and d : Ȥ x Ȥ x Ȥ → [0,∞)is defined by

d(f,g,h) = 0 , if at least two of the three points are the same.

$= \mu(f,g,h)\, e^{1/2|f-g| + 3/8|g-h| + 1/8|h-f|}$ otherwise. For continuous α(f,g,h) ≥ 0 symmetric in f,g,h. If suffices to only verify property (IV) of definition 2.8.Which typically in such generalized setting takes a form like.

For all f,g,h,t ε Ȥ .

$d(f,g,h) = \alpha(f,g,t)\, d(f,g,t) + \beta(g,h,t)\, d(g,h,t) + \gamma(h,f,t)\, d(h,f,t)$

for some control function α,β,γ : Ȥ x Ȥ x Ȥ → [1,∞).

Using Jensen's inequality applied of the convex function $f(x) = e^x$ along with modulus power │ a│.

$d(f,g,h) = \mu(f,g,h)\, e^{1/2|f-g| + 3/8|g-h| + 1/8|h-f|}$

$\leq \mu(f,g,h)\, [1/2\, e^{|f-g|} + 3/8\, e^{|g-h|} + 1/8\, e^{|h-f|}]$

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$= \mu(f,g,h)\ [1/2\ e^{1/2|f-g|+1/2|f-g|} + 3/8\ e^{1/2|g-h|+1/2|g-h|} + 1/8\ e^{1/2|h-f|+1/2|h-f|}]$

$\leq [\ 1/2e^{1/2|f-g|}+1]\ \mu(f,g,h)\ \ e^{1/2|f-g|+3/8|g-t|+1/8|t-f|}$

$+ [\ 3/8e^{1/2|h-g|}+1]\ \mu(f,g,h)\ \ e^{1/2|h-g|+3/8|g-t|+1/8|t-h|}$

$+ [\ 1/8e^{1/2|h-f|}+1]\ \mu(f,g,h)\ \ e^{1/2|h-f|+3/8|f-t|+1/8|t-h|}$

$= \alpha(f,g,t)\ d(f,g,t) + \beta(h,g,t)\ d(h,g,t) + \gamma(h\ f,t)\ d(h,f,t)$

Where $\alpha(f,.g,t) = 1/2\ e^{1\backslash 2|f-g|} + 1 \geq 1$

$\beta(g,h,t) = 3/8\ e^{1\backslash 2|g-h|} + 1 \geq 1$

$\gamma(h,f,t) = 1/8\ e^{1\backslash 2|h-f|} + 1 \geq 1.$

Definition 2.10 [6] Let (Ⱬ,d) be a controlled metric space. Let Ⱨ : Ⱬ →Ⱬ we say Ⱨ is continuous at the pair (a,b) ε Ⱬ x Ⱬ if for every ε > 0, there exist r > 0 such that

Ⱨ ($B_r(f,g)$) c $B_c(f,g)$.

Definition 2.11 [6] Let (Ⱬ,d) be a controlled metric-type space.

I. The sequence $\{f_n\}_{n\ \varepsilon\ N}$ converges to f ε Ⱬ. If for every ε > 0, there exists an integer n ε N such that for all n ≥ N. $d(f_n, f, t) < \varepsilon$ for some fixed t ε Ⱬ.

II. The sequence $\{f_n\}_{n\ \varepsilon\ N}$ is a Cauchy sequence,. If for every ε > 0, there exist an integer n ε N such that for all n,m ≥ N $d(f_n, f_m, t) < \varepsilon$ for some fixed t ε Ⱬ.

III. The space (Ⱬ,d) is complete, if every Cauchy sequence in Ⱬ is convergent.

In order to simplify results, we define the following products as follows:

Define

$U_0(t, f_n, f_{n+1}) = \alpha$

$U_1(f_m, t , f_{n+1}, f_{n+2}) = \beta\ \gamma$

$U_2(f_m, t , f_{n+1}, f_{n+2}, f_{n+3}) = \beta\ \alpha\ \gamma$

$U_3(f_m, t , f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}) = \beta\ \alpha\ \alpha\ \gamma$

In general

$U_{m-n-1}(f_m, t , f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}\ \ldots, f_{m-1}) = \beta\ (\alpha\ldots\ (m-n-2))$ times) γ

Define

$V_0(t, f_n, f_m) = \gamma$

$V_1(f_m, t , f_{n+1}, f_{n+2}) = \beta\ \beta$

$V_2(f_m, t , f_{n+1}, f_{n+2}, f_{n+3}) = \beta\ \alpha\ \beta$

$V_3(f_m, t , f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}) = \beta\ \alpha\ \alpha\ \beta$

In general

$V_{m-n-1}(f_m, t, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4} \ldots, f_{m-1}) = \beta\ (\alpha\ldots\ (m-n-2))$ times) β

.

Main Result

Theorem 3.1 Let Ɀ ≠ Φ and (Ɀ,d) be a complete controlled metric space. Let

Ƕ : Ɀ →Ɀ be a functions satisfying

(3.1) $d(Ƕf,Ƕg,t) \leq (£(f) - £(Ƕf))d(f,g.t)\ \forall\ f,g,t\ \varepsilon$ Ɀ.

Where £: Ɀ→R is bounded function.

Assume that

(3.2) $\sup_{m\geq1} \lim_{j\to\infty} I\ \ U_{j+1}(f_m, t, f_{n+1},\ldots f_{j+2})/ U_j(f_m, t, f_{n+1},\ldots f_{j+1})\ I < 1/k$

(3.3) $\sup_{m\geq1} \lim_{j\to\infty} I\ \ V_{j+1}(f_m, t, f_{n+1},\ldots f_{j+2})/ V_j(f_m, t, f_{n+1},\ldots f_{j+1})\ I < 1/k$, where $k\varepsilon\ (0,1)$

Also assume that for each f ε Ɀ, the limits

(3.4) $\lim_{n\to\infty} \alpha(f,f_n,t)$, $\lim_{n\to\infty} \beta(f,f_n,t)$ and $\lim_{n\to\infty} \gamma(f,f_n,t)$ exists. Then Ƕ has a unique fixed point in Ɀ.

Proof Let $f_0\ \varepsilon$ Ɀ be arbitrary. Define sequence $\{f_n\}_{n\ \varepsilon\ N}$ by $f_{n+1} = Ƕf_n$ for n ε N. Then from (3.1), we get

$d(f_n,f_{n+1},t) = d(Ƕf_{n-1},Ƕf_n,t) \leq (£(f_{n-1}) - £(Ƕf_{n-1}))\ d(f_{n-1},f_n.t)$

$= (£(f_{n-1}) - £(f_n))\ d(f_{n-1},f_n.t)$

Let, $b_n = d(f_{n-1},f_n.t)$.Then

(3.5) $b_{n+1}/ b_n \leq £(f_{n-1}) - £(f_n)\ \forall\ n\ \varepsilon\ N$.

This implies that the sequence $\{£(f_n)\}$ is non- increasing and bounded below hence convergent. Thus,

$\lim_{n\to\infty} £(f_n) = r > 0$. Summing the inequality (3.5), we get

$\Sigma_{i=1}^{n}\ b_{i+1}/b_i \leq \Sigma_{i=1}^{n}\ (£(f_{i-1}) - £(f_i))$

$= £(f_0) - £(f_n)$,

which is finite,so

(3.6) $\lim_{i\to\infty} b_{i+1}/b_i = 0$.

Hence there exists $i_0\ \varepsilon$ N such that for all $i \geq i_0$

(3.7) $b_{i+1}/b_i \leq k$ for k ε (0,1), so

(3.8) $d(f_i,f_{i+1},t) \leq k\ d(f_{i-1},f_i,t)$ for all $i \geq i_0$

By recursion

(3.9) $d(f_i,f_{i+1},a) \leq k^i\ d(f_0,f_1,t)$ for all $i \geq i_0$

Showing Cauchy Property

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Let m,n εN with m> n , we obtain

$d(f_n,f_m,t) \leq \alpha(f_n,f_m,f_{n+1})\, d(f_n,f_m,f_{n+1}) + \beta(f_m,t,f_{n+2})\, d(f_m,t,f_{n+2}) + \gamma(t, f_n,f_{n+1})\, d(t, f_n,f_{n+1})$

$\leq \alpha(f_n,f_m,f_{n+1})\, d(f_n,f_m,f_{n+1}) + \beta(f_m,t,f_{n+1})\, \alpha(f_n,f_m,f_{n+1})\, d(f_m,t,f_{n+1})$

$+ \beta(f_m,t,f_{n+1})\, \beta(t,f_{n+1},f_{n+2})\, d(t,f_{n+1},f_{n+2}) + \beta(f_m,t,f_{n+1})\, \gamma(f_{n+1},f_m,f_{n+2})\, d(f_{n+1},f_m,f_{n+2})$

$+ \gamma(t, f_n,f_{n+1})\, d(t, f_n,f_{n+1})$

$\leq \alpha(f_n,f_m,f_{n+1})\, d(f_n,f_m,f_{n+1}) + \beta(f_m,t,f_{n+1})\, \alpha(f_m,t,f_{n+2})\, \alpha(f_m,t ,f_{n+3})\, d(f_m,t ,f_{n+3})$

$+\beta(f_m,t,f_{n+1})\, \alpha(f_m,t,f_{n+2})\beta(t,f_{n+2},f_{n+3})\, d(t,f_{n+2},f_{n+3})$

$+\beta(f_m,t,f_{n+1})\, \alpha(f_m,t,f_{n+2})\, \gamma(f_{n+2},f_m,f_{n+3})\, d(f_{n+2},f_m,f_{n+3})$

$+\beta(f_m,t,f_{n+1})\beta(t,f_{n+1},f_{n+2})d(t,f_{n+1},f_{n+2}) + \beta(f_m,t,f_{n+1})\, \gamma(f_{n+1},f_m,f_{n+2})\, d(f_{n+1},f_m,f_{n+2})$

$+ \gamma(t, f_n,f_{n+1})\, d(t, f_n,f_{n+1})$

$d(f_n,f_m,t) \leq \alpha(f_n,f_m,f_{n+1})\, d(f_n,f_m,f_{n+1}) + \beta (f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \alpha (f_m,t,f_{n+3})\, \alpha (f_m,t,f_{n+4})\, d(f_m,t, f_{n+4})$

$+ \beta (f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \alpha (f_m,t,f_{n+3})\, \beta (t, f_{n+3},f_{n+4})\, d(t,f_{n+3}, f_{n+4})$

$+ \beta (f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \alpha (f_m,t,f_{n+3})\, \gamma(f_{n+3},f_m, f_{n+4})\, d(f_{n+3},f_m, f_{n+4})$

$+ \beta (f_m,t,f_{n+1})\, \beta (f_m,t,f_{n+2})\, \beta (t, f_{n+2}, f_{n+3})\, d(t,f_m, f_{n+3})$ (3.10)

$+ \beta (f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \gamma(f_{n+2},f_m, f_{n+3})\, d(f_{n+2},f_m, f_{n+3})$

$+ \beta (f_m,t,f_{n+1})\, \beta (t, f_{n+1},f_{n+2})\, d(t,f_{n+1}, f_{n+2})$

$+ \beta(f_m,t,f_{n+1})\, \gamma(f_{n+1}, f_m,f_{n+2})\, d(f_{n+1}, f_m,f_{n+2})$

$+ \gamma(t, f_n,f_{n+1})\, d(t, f_n,f_{n+1})$

Using 3.9, we get

$d(f_n,f_m,t) \leq \alpha (f_n,f_m,f_{n+1})k^n d(f_0,f_m,f_1)+\beta(f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \alpha (f_m,t,f_{n+3})\, \alpha (f_m,t,f_{n+4})\, d(f_m,t, f_{n+4})$

$+ \beta (f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \alpha (f_m,t,f_{n+3})\, \beta (t, f_{n+3},f_{n+4})k^{n+3}\, d(t,f_0, f_1)$

$+ \beta (f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \alpha (f_m,t,f_{n+3})\, \gamma(f_{n+3},f_m, f_{n+4})k^{n+3}\, d(f_0,f_m, f_1)$

$+ \beta (f_m,t,f_{n+1})\, \alpha (f_m,t,f_{n+2})\, \beta (t, f_{n+2}, f_{n+3})k^{n+2}\, d(t,f_0, f_1)$

(3,11) $+ \beta (f_m,t,f_{n+1})\, \alpha(f_m,t,f_{n+2})\, \gamma(f_{n+2},f_m, f_{n+3})k^{n+2}\, d(f_0,f_m, f_1)$

$+ \beta (f_m,t,f_{n+1})\, \beta (t, f_{n+1},f_{n+2})k^{n+1}\, d(t,f_0, f_1)$

$+ \beta (f_m,t,f_{n+1})\, \gamma(f_{n+1}, f_m,f_{n+2})k^{n+1}\, d(f_0, f_m,f_1)$

$+ \gamma(t, f_m,f_{n+1})k^n\, d(t, f_0,f_1).$

It follows that inequality, (3.11), can be written as

$d(f_n,f_m,t) \leq k^n\, d(f_0,f_1,f_m)\, [\, U_0(f_n,f_m,f_{n+1}) + k\, U_1(f_m,t,f_{n+1},f_{n+2}) + k^2\, U_2(f_m,t,f_{n+1},f_{n+2}, f_{n+3}) +\ldots$

$+ k^{m-n-1}\, U_{m-n-1}(f_m,t,f_{n+1},f_{n+2},.. f_{m-1}\ldots)\,]$

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(3.12) $+ k^n d(f_0,f_1,t) [V_0(f_n,f_m,f_{n+1}) + k V_1(f_m,t,f_{n+1},f_{n+2}) + k^2 V_2(f_m,t,f_{n+1},f_{n+2}, f_{n+3}) +\ldots$

$+ k^{m-n-1} V_{m-n-1}(f_m,t,f_{n+1},f_{n+2},.. f_{m-1}\ldots)]$

The ratio test together with 3.2 and3.3, implies that the series converges and thus the sequence $\{f_n\}_{n \varepsilon N}$ is a Cauchy sequence. Since (Ⱬ,d) is complete controlled metric space there exists f* ε Ⱬ such that

$d(f_n,f^*, t) \to 0$ as $n\to\infty$.

Proving f* is a fixed point.Using (3.9) we get

, $d(f^*,Ḥf^*,t) \le \alpha(f^*,Ḥf^*,f_{n+1}) d(f^*,Ḥf^*,f_{n+1}) + \beta(Ḥf^*,t,f_{n+1}) d(Ḥf^*,t,Ḥf^*,f_{n+1}) + \gamma(t,f^*,f_{n+1}) d(t,f^*,f_{n+1})$

$\le \alpha (f^*,Ḥf^*,f_{n+1}) d(f^*,Ḥf^*,f_{n+1}) + \beta (Ḥf^*,t,f_{n+1}) d(Ḥf^*,t,Ḥf_n) + \gamma(t,f^*,f_{n+1}) d(t,f^*,f_{n+1})$

$\le \alpha (f^*,Ḥf^*,f_{n+1}) d(f^*,Ḥf^*,f_{n+1}) + \beta (Ḥf^*,t,f_{n+1}) kd(f^*,t,f_n) + \gamma(t,f^*,f_{n+1}) d(t,f^*,f_{n+1})$.

Taking the limit as $n\to\infty$and using (3.9), we get

$d(f^*,Hf^*,t) \le 0$.

Implies,$Ḥf^* = f^*$.

Uniqueness

Suppose there is another fixed point $f^{**} \neq f^*$. Then

(3.13) $d(f^*,f^{**},t) = d(Ḥf^*,Ḥf^{**},t) \le (£(f^*)- £(f^*)) d(f^*,f^{**},t)$

$d(f^*,f^{**},t) = 0$,

which implies $f^* = f^{**}$. Thus the mapping Ḥ has a unique fixed point f* ε Ⱬ.

Remark: Our result extend the corresponding result of Souayah and Hidri [5] and others.

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