Mathematical Properties and Inference of Multivariate Multi-Sigmoidal Gompertz and Weibull Diffusion Processes for Epidemiological Modeling

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ABSTRACT

This paper develops the mathematical theory of multivariate multi-sigmoidal Gompertz and Weibull diffusion processes for modeling complex growth phenomena with multiple inflection points. We extend univariate formulations to a multivariate framework by incorporating time-dependent and fixed covariates through a flexible polynomial structure $Q(t, z)$ of degree four, enabling the representation of up to three distinct growth phases. The deterministic ordinary differential equations are derived, closed-form solutions are obtained, and stochastic extensions via Itô calculus are rigorously established. We derive the complete maximum likelihood estimation framework, including the score function, observed Hessian, and Fisher information matrix. The theoretical results demonstrate that a fourth-degree polynomial yields up to three inflection points, consistent with multi-sigmoidal behavior and enabling the representation of three distinct growth phases. The models are applied to US COVID-19 data as a demonstration of the mathematical framework, with parameter estimates confirming the theoretical properties and revealing the practical limitations of the lognormal assumption. This work provides a comprehensive mathematical foundation for multivariate multi-sigmoidal diffusion processes with applications in epidemiology, biology, and economics.

Keywords: diffusion processes; multi-sigmoidal growth; Gompertz model; Weibull model; maximum likelihood estimation; stochastic differential equations; inflection points

INTRODUCTION

Growth models describe the forces that govern population, disease, and resource-use dynamics, typically through first-order ordinary differential equations (ODEs) whose parameters are estimated from data (Pérez-Maldonado et al., 2022; Brauer et al., 2019). Diffusion processes — continuous-time Markov processes with almost-surely continuous sample paths — extend these deterministic models by embedding stochastic fluctuations, offering a tractable alternative to the discrete-jump Markov processes traditionally used for SIR-type epidemic models (Guy et al., 2015).

Classical sigmoidal growth curves each carry distinct strengths. The logistic model (Verhulst, 1838; Bacaër, 2011) introduced the carrying-capacity concept but is symmetric about its inflection point. The Gompertz model (Gompertz, 1825) is asymmetric, growing slowly at first before accelerating and then decelerating — well suited to epidemic waves with a sharp rise and a long tail (Manca, 2022). The Weibull model's shape parameter lets it represent increasing, decreasing, or constant hazards, and it uniquely accommodates both proportional-hazards and accelerated-failure-time interpretations. The Richards model (Richards, 1959) added a shape parameter to the logistic curve for asymmetric single-wave dynamics, while the Hyperbolastic family (Tabatabai et al., 2005) built further flexibility into the logistic form. None of these classical curves, however, can reproduce more than one epidemic wave.

Multi-sigmoidal extensions address this limitation by summing several sigmoidal components. Di Crescenzo et al. (2021) proposed deterministic and stochastic multi-sigmoidal logistic models to study inflection points and first-passage times, and Barrera et al. (2021) extended this to multi-sigmoidal Gompertz and Weibull diffusion

processes, embedding a polynomial predictor inside the classical ODEs and estimating parameters by maximum likelihood and Bayesian methods on Spanish COVID-19 data. Di Crescenzo et al. (2025) further proposed a multi-sigmoidal lognormal diffusion process for epidemiological data. These stochastic, multi-sigmoidal formulations capture both the deterministic shape and the random fluctuations of real epidemic trajectories more realistically than single-wave deterministic curves.

This paper builds directly on Barrera et al. (2021) by extending their univariate multi-sigmoidal Gompertz and Weibull diffusion framework to a multivariate setting that jointly incorporates time-varying and fixed covariates. The objectives are to (i) embed the classical Gompertz and Weibull growth ODEs in a stochastic differential equation (SDE) framework that captures random fluctuations in real epidemic data, and (ii) compare the two models' dynamic behavior, including inflection-point structure and sensitivity to covariates. We derive the deterministic ODEs and their closed-form solutions, extend them to SDEs via Itô calculus, and develop the complete maximum-likelihood inference machinery (score, observed Hessian, and Fisher information). The models are then applied to US COVID-19 data as a demonstration.

By modeling epidemic peaks and covariate effects jointly and stochastically, this framework offers more realistic and interpretable tools for infectious-disease modeling, intervention evaluation, and public-health decision-making than traditional single-wave deterministic models. As with Barrera et al. (2021), the lognormal-diffusion assumption may not fully capture sharp regime shifts, and the models treat each population as a single unit without cross-population interaction — limitations we return to in the Conclusion.

2. Gompertz and Weibull Multi-Sigmoidal Curves

In this case we present two Multivariate multi-sigmoidal Gompertz and Weibull curves. Both are derived by incorporating the inclusion of the use of the polynomials function in the ordinary differential equations whose solutions are curves.

We consider the multivariate multi-sigmoidal Gompertz function to be.

$$f(t,z) = k\left[\exp\left(-\alpha \exp\left(-Q(t,z)\right)\right)\right] \qquad (1)$$

and the multivariate multi-sigmoidal Weibull is as a multiplier.

$$f(t,z) = k(1 - \exp(-Q(t,z)^b)) \qquad (2)$$

Where $\alpha > 0$ is a scaling parameter, $k > 0$ are the asymptotic values of every curve, i.e., the carrying capacity of each model and $Q(t,z)$ is a polynomial of degree $p > 0$ define as follows

$$Q(t,z) = \sum_{i=1}^{d} \beta_i t^i + \sum_{j=1}^{p} \delta_j z_j(t) + \sum_{j=1}^{p} \gamma_j \cdot t \cdot z_j(t) + \sum_{1 \le j < k \le p} \varepsilon_{jk}\, z_j(t) z_k(t) \qquad (3)$$

Table1: Description of the Polynomial Nature

Term	Description
$\sum_{i=1}^{d} \beta_i t^i$	Polynomial time trend (e.g., captures epidemic acceleration, saturation, etc.)
$\sum_{j=1}^{p} \delta_j z_j(t)$	Linear covariate effects (e.g., interventions, mobility, temperature, etc.)

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$\sum_{j=1}^{p} \gamma_j \cdot t \cdot z_j(t)$	Time varying effect of each covariate (interaction with time, but still linear and interpretable)
$\sum_{J<K} \mathcal{E}_{JK} z_J(t) z_K(t)$	Covariate interactions (pairwise effects, e.g., vaccination × mobility)

We differentiate equation (3) as follows;

Differentiate the Polynomial term:

$$\sum_{i=1}^{d} \beta_i t^i = \frac{d}{dt}\left(\sum_{i=1}^{d} \beta_i t^i\right) = \sum_{i=1}^{d} \beta_i i t^{i-1}$$

Differentiate the Linear covariate effects

$$\sum_{j=1}^{p} \delta_j z_j(t) = \frac{d}{dt}\left(\sum_{j=1}^{p} \delta_j z_j(t)\right) = \sum_{j=1}^{p} \delta_j \frac{dz_j(t)}{dt}$$

Differentiate the Interaction between time and covariates;

$$\sum_{j}^{p} \Upsilon_j [t. z_j(t)] = \frac{d}{dt}\left(\sum_{j=1}^{p} \gamma_j \cdot t \cdot z_j(t)\right) = \sum_{j=1}^{p} \gamma_j \cdot t \cdot z_j(t) \frac{dz_j(t)}{dt})$$

Differentiate the Pairwise interaction terms

$$\sum_{1\le j<k\le p} \varepsilon_{jk}\, z_j(t) z_k(t) = \frac{d}{dt}\left(\sum_{1\le j<k\le p} \varepsilon_{jk} z_J(t) z_k(t)\right) = \sum_{1\le j<k\le p} \varepsilon_{jk}(\frac{dz_j(t)}{dt} z_k(t) + z_j(t)\ \frac{dz_k(t)}{dt})$$

$$\frac{dQ(t,z)}{dt} = \sum_{i=1}^{d} \beta_i\, i\, t^{i-1} + \sum_{j=1}^{p} \delta_j \frac{dz_j(t)}{dt} + \sum_{j=1}^{p} \gamma_j [z_j(t) + t \frac{dz_j(t)}{dt}] + \sum_{1\le j<k\le p} \varepsilon_{jk} [\frac{dz_j(t)}{dt} z_k(t) + z_j(t)\ \frac{dz_k(t)}{dt}] \qquad (4)$$

2.1The ODE for the Gompertz

Let $\frac{\mathrm{dQ(t,z)}}{\mathrm{dt}} = \mathrm{p(t,z)}$

We differentiate Gompertz type model from equation (1) using chain rule

$$\mathrm{f(t,z) = k\left[\exp\left(-\alpha \exp\left(-Q(t,z)\right)\right)\right]}$$

Let $\mathrm{u(t) = \exp\left(-Q(t,z)\right)}$

Then

$$\mathrm{f(t,z) = k \exp(-\alpha u(t))}$$

$$\frac{\mathrm{df(t,z)}}{\mathrm{dt}} = -\alpha \mathrm{k} \exp(-\alpha \mathrm{u(t)}) \frac{\mathrm{du}}{\mathrm{dt}}$$

But $\mathrm{u(t) = \exp\left(-Q(t,z)\right)}$ then $\frac{\mathrm{du}}{\mathrm{dt}} = -\mathrm{p(t,z)}\exp(-\mathrm{Q(t,z)}$

$$\frac{\mathrm{df(t,z)}}{\mathrm{dt}} = \left[-\alpha \mathrm{k} \exp\left(-\alpha \mathrm{u(t)}\right)\right][-\mathrm{p(t,z)} \exp(-\mathrm{Q(t,z)})] = \alpha \mathrm{k} \exp\left(-\alpha \exp\left(-\mathrm{Q(t,z)}\right)\right)\mathrm{p(t,z)} \exp(-\mathrm{Q(t,z)})$$

But $\mathrm{f(t,z) = k\left[\exp\left(-\alpha \exp\left(-Q(t,z)\right)\right)\right]}$

$$\frac{df(t,z)}{dt} = \alpha\, p(t,z)\exp\big(-Q(t,z)\big)\, f(t,z) \qquad (5)$$

From equation (5),

let $h(t,z) = \alpha\, p(t,z)\exp\big(-Q(t,z)\big)$ (6)

Decelerating growth is contained in equation (6). With higher value of the polynomial $Q(t,z)$, the exponent $\exp\big(-Q(t,z)\big)$ decreases thus slowing the increase or slowing down the decrease of the covariates over time, $p(t,z)$ are regulated by their effects on the acceleration or deceleration of the covariates over time.

Then equation (5) becomes

$$\frac{df(t,z)}{dt} = h(t,z)f(t,z) \qquad (7)$$

From the equation above, $f(t,z)$ is the cumulative response variable (e.g. cumulative infections), depending on time t and covariates z, $h(t,z)$ is the instantaneous growth rate function**,** which varies with both time and the covariates.

equation (7) also implies that the rate of change of the process have a direct proportionality to the current state $f(t,z)$ with a growth rate defined by $h(t,z)$. i.e. the rate of change increases as $h(t,z)$ increases

Equation (7) is a separable ODE which lead to $\frac{1}{f(t,z)}\frac{df(t,z)}{dt} = h(t,z)$

$$\int \frac{1}{f(t,z)}\frac{df(t,z)}{dt}dt = \int h(t,z)dt$$

$$\ln f(t,z) = \int h(t,z)dt + C$$

Taking the exponential of both side $e^{\ln f(t,z)} = e^{\int h(t,z)dt + C}$

$$f(t,z) = e^{\int h(t,z)dt + C}$$

$$f(t,z) = e^{\int h(t,z)dt} \text{ x } e^{C}$$

Let $e^{C} = C'$

$$f(t,z) = C'e^{\int h(t,z)dt} \qquad (8)$$

Recall that $h(t,z) = \alpha\, p(t,z)\exp\big(-Q(t,z)\big)$

$$f(t,z) = C'e^{\int \alpha\, p(t,z)\exp\left(-Q(t,z)\right)dt}$$

Also recall that $u(t) = \exp\big(-Q(t,z)\big)$

$$\frac{du}{dt} = -p(t,z)\exp(-Q(t,z)$$

$$\int p(t,z)e^{(-Q(t,z))}dt = e^{(-Q(t,z))} + C$$

Then $\int h(t,z)dt = -\alpha\exp\big(-Q(t,z)\big) + C$

Then equation (8) becomes $f(t,z) = C' \exp\left(-\alpha \exp\left(-Q(t,z)\right)\right)$ (9)

We obtain C' using the initial condition $f(t_0, z(t_0)) = f_0$

Equation (9) becomes $f(t_0, z_0) = C' \exp\left(-\alpha \exp\left(-Q(t_0, z(t_0))\right)\right) = f_0$

$$C' = \frac{f_0}{\exp\left(-\alpha \exp\left(-Q(t_0, z(t_0))\right)\right)}$$

$$C' = f_0 \exp\left(\alpha \exp\left(-Q(t_0, z(t_0))\right)\right) \quad (10)$$

Substitute equation (10) into equation (9)

$$f(t,z) = f_0 \exp\left(\alpha \exp\left(-Q(t_0, z(t_0))\right)\right) \exp\left(-\alpha \exp\left(-Q(t,z)\right)\right)$$

$$= f_0 \exp\left[-\alpha\left(\exp\left(-Q(t,z)\right) - \exp\left(-Q(t_0, z(t_0))\right)\right)\right] \quad (11)$$

From (11) above f_0 is the Initial value at baseline time t_0 with covariates z_0 . it also shows how the outcome changes over time based on sigmoidal growth modulated by time and covariates, Moreso, the change $\exp\left(-Q(t,z)\right) - \exp\left(-Q(t_0, z(t_0))\right)$ tracks how the process has progressed from the initial state. Generally, we can say that the rate of change diminishes as t increases or covariates increase $Q(t,z)$.

2.2The ODE for the Weibull type Model

From equation (2) we differentiate the Weibull type model

$$f(t,z) = k\left(1 - \exp(-Q(t,z)^b)\right)$$

$$\frac{df(t,z)}{dt} = k.\frac{d}{dt}\left(1 - \exp\left(-Q(t,z)^b\right)\right) = -k\frac{d}{dt}\exp\left(-Q(t,z)^b\right) \quad (12)$$

We differentiate the exponential function using chain rule

Let $u(t) = \left(Q(t,z)^b\right)$, then

$$\frac{d}{dt}\left(\exp\left(-u(t)\right)\right) = -\exp(-u(t))\frac{du}{dt}$$

Equation (12) now becomes

$$\frac{df(t,z)}{dt} = -k\left[-\exp\left(-u(t)\right)\frac{du}{dt}\right] = k\exp\left(-u(t)\right)\frac{du}{dt} \quad (13)$$

From $u(t) = \left(Q(t,z)^b\right)$

$$\frac{du}{dt} = \frac{d}{dt}\left(Q(t,z)^b\right) = bQ(t,z)^{b-1}\frac{dQ(t,z)}{dt} \quad (14)$$

Substitute equation (14) into equation (13)

$$\frac{df(t,z)}{dt} = k\,bQ(t,z)^{b-1}\exp\left(-u(t)\right)\frac{dQ(t,z)}{dt}$$

Recall that $\frac{dQ(t,z)}{dt} = p(t,z)$

$$\frac{df(t,z)}{dt} = k\, bQ(t,z)^{b-1}\exp(-Q(t,z)^b)p(t,z) \quad (15)$$

From equation (2)

$$f(t,z) = k\left(1 - \exp(-Q(t,z)^b)\right)$$

$$k\exp\left(-Q(t,z)^b\right) = k - f(t,z)$$

$$\exp\left(-Q(t,z)^b\right) = 1 - \frac{f(t,z)}{k} \quad (16)$$

Substitute equation (16) into Equation (15)

$$\frac{df(t,z)}{dt} = k\, bQ(t,z)^{b-1}\left(1 - \frac{f(t,z)}{k}\right)p(t,z)$$

$$\frac{df(t,z)}{dt} = bQ(t,z)^{b-1}p(t,z)\left(k - f(t,z)\right) \quad (17)$$

In general equation (17) is applied at;

- The Early stages When $f(t,z) \ll k$, and growth is approximately exponential.
- The Mid stages where Growth slows down as $f(t,z)$ increases.
- As $f(t,z) \to k$, $k - f(t,z) \to 0$ and growth decelerates, this occurs at the later stages

Let $$h(t,z) = bQ(t,z)^{b-1}p(t,z) \quad (18)$$

The expression in (18) above allows for flexible growth patterns, including S-shapes and other behaviors that are nonlinear and the growth rate increases with $Q(t,z)$ when the shape parameter $b > 1$ and decreases when it is $0 < b < 1$. $p(t,z)$ as stated earlier adjusts the growth rate best on the time sensitivity of the covariate effects.

Then equation (17) now becomes

$$\frac{df(t,z)}{dt} = h(t,z)\left(k - f(t,z)\right)$$

$$\frac{df(t,z)}{dt} = -h(t,z)(f(t,z) - k) \quad (19)$$

Equation (19) is a logistic type formulation that is commonly used in Weibull type growth models to describe saturating processes like the spread of infections where growth slows as it moves close to a carrying capacity k. $\left(k - f(t,z)\right)$ is the Remaining growth potential, i.e. the gap between current level and saturation. (19) also implies rate of change of $f(t,z)$ is proportional to how far it is from the maximum value or carrying capacity k scaled by a varying growth rate $h(t,z)$.

Equation (19) is a separable ODE

$$\frac{df(\,)}{(f(t,z) - k)} = -h(t,z)dt$$

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We integrate both side

$$\int \frac{df(t,z)}{(f(t,z)-k)} df = \int -h(t,z)dt$$

$$\ln|f(t,z)-k| = -\int h(t,z)dt + C$$

We evaluate the right-hand side of the integral

$$\int h(t,z)dt = \int bQ(t,z)^{b-1}\frac{dQ(t,z)}{dt}dt$$

$$\int bQ(t,z)^{b-1}dQ(t,z) = Q(t,z)^{b} + C'$$

Therefore, $\ln|f(t,z)-k| = -Q(t,z)^{b} + C - C'$

Let $C - C' = C''$

$$\ln|f(t,z)-k| = -Q(t,z)^{b} + C''$$

Take exponential of both side

$$|f(t,z)-k| = e^{-Q(t,z)^{b}+C''}$$

$$|f(t,z)-k| = e^{-Q(t,z)^{b}}.e^{C''}$$

Let $e^{C''} = C_1$,

Then $|f(t,z)-k| = C_1 e^{-Q(t,z)^{b}}$

$f(t,z)-k = C_1 e^{-Q(t,z)^{b}}$

$f(t,z) = k + C_1 e^{-Q(t,z)^{b}}$ (20)

Making used of the initial condition $f(t_0, z_0) = f_0$

Then $f_0 = k + C_1 e^{-Q(t_0,z_0)^{b}}$, solve for C_1

$$C_1 = (f_0 - k).e^{Q(t_0,z_0)^{b}}$$

Plug back C_1 into equation (20)

$$f(t,z) = k + (f_0 - k).e^{Q(t_0,z_0)^{b}}.e^{-Q(t,z)^{b}}$$

$$f(t,z) = k + (f_0 - k).\exp\left(Q(t_0,z_0)^{b} - Q(t,z)^{b}\right) \quad (21)$$

Model growth is seen to be a saturating growth toward k from (21) above. Furthermore, when

$t = t_0$, the exponential attains the value of 1and the curve at $f(t_0, z_0)$ becomes f_0 and the shape of the curve is dictated by the exponent b.

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Each of Gompertz and Weibull functions in (1) and (2) can be viewed as the solution of the Malthusian-type linear differential equation as shown by (7) and (19).

Together, the growth-rate function $h(t,z)$, the resulting ODE, and the closed-form solution derived above for the Gompertz case (equations 5, 7, 11) and the Weibull case (equations 17, 19, 21) constitute the complete parallel formulation for both models.

2.3 The Points of Inflection

In order to locate the inflection point, we differentiate

$$\frac{df(t,z)}{dt}\alpha k\ exp\big(-\alpha\exp\big(-Q(t,z)\big)\big)p(t,z)\exp(-Q(t,z)) \qquad Gompertz$$

$$\frac{df(t,z)}{dt} = k\ bQ(t,z)^{b-1}\exp(-Q(t,z)^b)p(t,z) \qquad Weibull$$

Start with

$$\frac{df(t,z)}{dt}\alpha k\ exp\big(-\alpha\exp\big(-Q(t,z)\big)\big)p(t,z)\exp(-Q(t,z)) \qquad Gompertz$$

Let define $Q = Q(t,z),\ p = \frac{dQ}{dt} = p(t,z), f = f(t,z), and\ f' = \frac{df}{dt}$

The we rewrite $\frac{df}{dt} = \alpha k p e^{-Q}.e^{(-\alpha e^{-Q})}$

Let denote $A = \alpha k,\ B = p,\ C = e^{-Q}\ and\ D = e^{(-\alpha e^{-Q})} = e^{(-\alpha C)}$

Then $f' = A.B.C.D$, using Product rule

$$f'' = A[B'CD + BC'D + BCD'] \qquad (22)$$

We compute each derivative

$$B' = \frac{dp}{dt} = p'(t,z),\ \ C' = \frac{d}{dt}(e^{-Q}) = -pe^{-Q} = -pC, D' = \frac{d}{dt}\big(e^{(-\alpha e^{-Q})}\big) = \alpha p e^{-Q}.e^{(-\alpha e^{-Q})} = \alpha pCD$$

Plug in the above back into (22)

$$f'' = A[p'CD - p^2CD + \alpha p^2\ C^2D]$$

We eliminate common terms $f'' = \alpha k.CD[p'(t,z) - p^2 + \alpha p^2 C$

$$\frac{d^2f}{dt^2} = \alpha k.e^{-Q}.e^{(-\alpha e^{-Q})}[p' - p^2 + \alpha p^2 e^{-Q}]$$

$$\frac{d^2f}{dt^2} = \alpha k.e^{-Q(t,z)}.e^{(-\alpha e^{-Q(t,z)})}[p'(t,z) - [p(t,z)]^2 + \alpha\big[p(t,z)\big]^2 e^{-Q(t,z)}\big] = 0 \qquad (23)$$

$$\frac{df(t,z)}{dt} = k\ bQ(t,z)^{b-1}\exp(-Q(t,z)^b)p(t,z) \qquad Weibull$$

From above let $Q = Q(t,z),\ p(t,z) = \frac{dQ(t,z)}{dt}, and\ \frac{dt(t,z)}{dt} = \frac{d^2Q(t,z)}{dt^2}$

Let denote $A = k.b$, then $f'(t,z) = A.Q^{b-1}.e^{-Q^b}.p$

We apply product rule $f'' = A.\frac{d}{dt}(Q^{b-1}.e^{-Q^b}.p)$

$$\frac{d}{dt}(UVW) = U'VW + UV'W + UVW'$$

Where $U = Q^{b-1}, V = e^{-Q^b}, W = p$

We compute the derivatives

$$U' = (b-1)Q^{b-2}.p,\ V' = \frac{d}{dt}\left(e^{-Q^b}\right) = -bQ^{b-1}.p.e^{-Q^b}, W' = \frac{dp}{dt}$$

Plug back into the product rule

$$f'' = A[(b-1)Q^{b-2}.p.e^{-Q^b}.p + Q^{b-1}.\left(-bQ^{b-1}.p.e^{-Q^b}\right).p + Q^{b-1}.e^{-Q^b}\frac{dp}{dt}$$

$$f'' = (b-1)Q^{b-2}p^2.e^{-Q^b} + -bQ^{2b-2}.p^2.e^{-Q^b} + Q^{b-1}\frac{dp(t,z)}{dt}e^{-Q^b}$$

$$\frac{d^2f(t,z)}{dt^2} = kb.e^{-Q(t,z)^b}[(b-1)Q(t,z)^{b-2}.\left(p(t,z)\right)^2 - bQ(t,z)^{2b-2}.\left(p(t,z)\right)^2 + Q(t,z)^{b-1}.\frac{d^2Q(t,z)}{dt^2} = 0 \quad (24)$$

equations (23) and (24) above are the inflection points of the curves. Many growth phenomena usually present at least one inflection, representing the moment when the growth rate changes. Multivariate multi-sigmoidal models may have multiple inflection points, and are therefore useful in modeling such types of growth phenomena. (Barrera *et al,* 2021)

3 Gompertz and Weibull Multi-Sigmoidal Diffusion Processes

3.1 Gompertz-Type Stochastic Diffusion Model

We define the Stochastic Differential Equation (SDE) from the deterministic form in equation (5) and introduce multiplicative noise to it via a Brownian motion

$$dX(t) = \mu(t)X(t)dt + \sigma X(t)dW(t) \quad (25)$$

Where $\mu(t) = \alpha p(t,z)e^{-Q(t,z)}$

$\sigma > 0$ *is a constant volatiliy parameter*

$W(t)$ *is the standard Wiener process* (*model of random shock*

Divide bothside of (25) by $X(t)$

$$\frac{dX(t)}{X(t)} = \mu(t)d + \sigma dW(t) \quad (26)$$

Equation (25) Is a geometric type SDE, meaning the noise scales with the process (multiplicative noise). Solving this directly is very hard unless we transform it

$$Y(t) = lnX(t) \quad (27)$$

Where

$X(t)\ is\ a\ stochastic\ process\ i.e.,\ the\ solution\ of\ the\ SDE\ in\ (25)$

$Y(t)\ is\ the\ logarithm\ of\ the\ entire\ process$

Equation (26) permit us to convert the multiplicative SDE into an additive one, which is easier to solve. by definition Itô's Lemma says if a process $X(t)$ follows $dX(t) = a\big(t, X(t)\big)dt + b\big(t, X(t)\big)dW(t)\ and$

$$Y(t) = g\big(X(t)\big)\ where\ a\big(t, X(t)\big)\ is\ the\ drift\ term\ and\ b\big(t, X(t)\big)\ is\ the\ diffusion.$$

Then;

$$\mathrm{d}Y(t) = g'\big(X(t)\big)dX(t) + \frac{1}{2}g''\big(X(t)\big)\big(dX(t)\big)^2 \tag{28}$$

We used the Itô's Lemma to derive the SDE from (27).

$$\text{Let } g\big(X(t)\big) = lnX(t), then$$

$$g'\big(X(t)\big) = \frac{1}{X(t)}\ and\ g''\big(X(t)\big) = -\frac{1}{X(t)^2}$$

We plug back into (28)

$$\mathrm{d}Y(t) = d\big(lnX(t)\big) = \frac{1}{X(t)}dX(t) - \frac{1}{2X(t)^2}\big(dX(t)\big)^2 \tag{29}$$

$$from\ (25), it\ implies\ that\ \big(dX(t)\big)^2 = (\mu(t)X(t)dt + \sigma X(t)dW(t))^2$$

Expanding the above using binomial expansion led to

$$\big(dX(t)\big)^2 = (\mu X(t)dt)^2 + 2(\mu X(t)dt)\big(\sigma X(t)dW(t)\big) + (\sigma X(t)dW(t))^2$$

Using the Itô rule

$$(dt)^2 = 0, dt.dW(t) = 0\ and\ \big(dW(t)\big)^2 = dt$$

$$\text{This lead to } (\sigma X(t)dW(t))^2 = \sigma^2X(t)^2\big(dW(t)\big)^2 = \sigma^2X(t)^2dt$$

$$which\ imply\ that\ \big(dX(t)\big)^2 = \sigma^2X(t)^2dt$$

$$then\ \frac{1}{2X(t)^2}\big(dX(t)\big)^2 = \frac{1}{2X(t)^2}.\sigma^2X(t)^2dt = \frac{1}{2}\sigma^2dt \tag{30}$$

We substitute (26) and (30) into (29)

$$\mathrm{d}Y(t) = d\big(lnX(t)\big) = \mu(t)d + \sigma dW(t) - \frac{1}{2}\sigma^2dt \tag{31}$$

equation (31) is the Stochastic Differential Equation (SDE) where $\mathrm{d}Y(t)$ is the infinitesimal change in the process in equation (27). We integrate both side of the SDE (31) over time interval $[t_0, t]$ and change the t inside the integral to s and then integrate with respect to s.

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$$\int_{t_0}^{t} dY(s) = \int_{t_0}^{t} \left[\mu(s) - \frac{1}{2}\sigma^2\right] ds + \int_{t_0}^{t} \sigma dW(s)$$

$$\text{Where } \mu(s) = \alpha p(s,z) e^{-Q(s,z)}$$

Integrating the left-hand side

$$\int_{t_0}^{t} dY(s) = Y(t) - Y(t_0)$$

Integrating the right-hand side

$\int_{t_0}^{t} \left[\mu(s) - \frac{1}{2}\sigma^2\right] ds$ *is the drift integral*

The second term is the itô stochastic integral

$$\int_{t_0}^{t} \sigma dW(s) = \sigma\big(W(t) - W(t_0)\big)$$

because the integral of $dW(s)$ *is just the change in the brownian motion*

$$Y(t) - Y(t_0) = \int_{t_0}^{t} \left[\mu(s) - \frac{1}{2}\sigma^2\right] ds + \sigma\big(W(t) - W(t_0)\big)$$

$$Y(t) - Y(t_0) = \int_{t_0}^{t} \left[\alpha p(s,z) e^{-Q(s,z)} - \frac{1}{2}\sigma^2\right] ds + \sigma\big(W(t) - W(t_0)\big) \qquad (32)$$

$$from\ (30), \qquad Y(t_0) = \ln X_0$$

$$Y(t) = \ln X_0 + \left(\int_{t_0}^{t} \left[\alpha p(s,z) e^{-Q(s,z)} - \frac{1}{2}\sigma^2\right] ds + \sigma\big(W(t) - W(t_0)\big) \right)$$

Taking the exponential of both side

$$X(t) = X_0 . exp\left(\int_{t_0}^{t} \left[\alpha p(s,z) e^{-Q(s,z)} - \frac{1}{2}\sigma^2\right] ds + \sigma\big(W(t) - W(t_0)\big) \right) \qquad (33)$$

Equation (33) is the close form solution of the inhomogeneous lognormal diffusion process corresponding to the stochastic Gompertz model.

3.2 Weibull-Type Stochastic Diffusion Model

$$\frac{\mathrm{df(t,z)}}{\mathrm{dt}} = \mathrm{bQ(t,z)^{b-1} p(t,z)\big(k - f(t,z)\big)}, \qquad \text{from (17)}$$

Let the Stochastic process be X(t) as we define earlier.

We then define the drift term (mean rate of change)

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$\mu(t) = bQ(t,z)^{b-1}p(t,z)\,(k - X(t))$ and introduce a multiplicative noise (diffusion term) through a Wiener process $W(t)$ then the Stochastic Differential Equation become

$$dX(t) = h(t,z)\big(k - X(t)\big)dt + \sigma X(t)dW(t)$$

$$dX(t) = a(t)dt + b(t)dW(t) \qquad (34)$$

$$\text{where } a(t) = h(t,z)\big(k - X(t)\big) \text{ and } b(t) = \sigma X(t)$$

$$\text{Recall that } Y(t) = \ln X(t) \text{ from (27)}$$

We then fine the differential of the log of the process $dY(t)$ using Itô's Lemma.

$$dY(t) = d\big(\ln X(t)\big) = \frac{1}{X(t)}dX(t) - \frac{1}{2X(t)^2}\big(dX(t)\big)^2$$

$$\text{square both side of (34)}, \qquad \big(dX(t)\big)^2 = \big(a(t)dt + b(t)dW(t)\big)^2$$

Applying binomial expansion to solve lead to

$$\big(dX(t)\big)^2 = a(t)^2 + dt^2 + 2a(t)b(t)dtdW(t) + b(t)^2\big(dX(t)\big)^2$$

Applying the itô rule of differentiation stated earlier lead us to

$$\big(dX(t)\big)^2 = b(t)^2.dt$$

$$b(t) = \sigma X(t)$$

$$\text{which imply that } b(t)^2 = \sigma^2 X(t)^2$$

Substituting into the Itô's Lemma

$$dY(t) = \frac{1}{X(t)}\big[h(t,z)\big(k - X(t)\big)dt + \sigma X(t)dW(t)\big] - \frac{1}{2X(t)^2}.\sigma^2 X(t)^2$$

$$dY(t) = \left[\frac{h(t,z)\big(k - X(t)\big)}{X(t)} - \frac{1}{2}\sigma^2\right]dt + \sigma dW(t)$$

$$\text{simplifying further} \quad \frac{h(t,z)\big(k - X(t)\big)}{X(t)} = h(t,z).\left(\frac{k}{X(t)} - 1\right)$$

$$dY(t) = \left[h(t,z).\left(\frac{k}{X(t)} - 1\right) - \frac{1}{2}\sigma^2\right]dt + \sigma dW(t) \qquad (35)$$

equation (35) is the Stochastic Differential Equation (SDE) for the Weibull model.

$$\text{Recall that } dX(t) = h(t,z)\big(k - X(t)\big)dt + \sigma X(t)dW(t)$$

$$\text{to solve this SDE, let } Y(t) = k - X(t) \text{ which imply that } X(t) = k - Y(t)$$

$$\text{then } dY(t) = -dX(t)$$

Plug in $dX(t)$

$$dY(t) = -\left[h(t,z)\left(k - X(t)\right)dt + \sigma X(t)dW(t)\right] \quad (36)$$

We using $Y(t) = k - X(t)$ and $X(t) = k - Y(t)$ because the SDE is not in standard lognormal form this is because the drift term involves both $X(t)$ and $k - X(t)$ so that we have

$$dY(t) = -h(t,z)Y(t)dt - \sigma\left(k - Y(t)\right)dW(t) \quad (37)$$

But (37) is still non-linear due to multiplicative noise with Y(t) embedded in both terms. Therefore, we transform again by defining

$$Z(t) = \frac{Y(t)}{k}, \text{then } Y(t) = kZ(t) \text{ subtitute this into (37)}$$

$$kdZ(t) = -h(t,z)kZ(t)dt - \sigma\left(k - kZ(t)\right)dW(t) \quad \text{divide both side by k}$$

$$dZ(t) = -h(t,z)Z(t)dt - \sigma\left(1 - Z(t)\right)dW(t) \quad (38)$$

This is a nonlinear SDE because of the $\left(1 - Z(t)\right)$ term. By definition this type of SDEs are generally not solvable in closed form unless they match a known structure. To do this we apply the Itô's Lemma in a reverse form. what we have been doing is to using the lemma to find the differential of a function of a stochastic process. I.e., we start with a process $X(t)$ and ask what is $dY(t)$, where $Y(t) = g\left(X(t)\right)$. But what we want to do now is the reverse i.e., we already know $Y(t)$ (log of $Z(t)$) and want to get differential of $Z(t)$.

but $Z(t) = e^{Y(t)}$, we used Itô's Lemma to compute $dZ(t)$ in terms of $dY(t)$

$$\text{Let } Z(t) = g\left(Y(t)\right) = e^{Y(t)} \text{ (exponential of the stochastic process)}$$

Apply Itô's Lemma to this transformation

$$g'\left(Y(t)\right) = \frac{d}{dY}e^{Y(t)} = e^{Y(t)}$$

$$g''\left(Y(t)\right) = \frac{d^2}{dY^2}e^{Y(t)} = e^{Y(t)}$$

$$dZ(t) = g'\left(Y(t)\right)dY(t) + \frac{1}{2}g''\left(Y(t)\right)(dY(t))^2$$

$$dZ(t) = e^{Y(t)}dY(t) + \frac{1}{2}e^{Y(t)}(dY(t))^2$$

$$\text{since } Z(t) = e^{Y(t)}$$

$$dZ(t) = Z(t)dY(t) + \frac{1}{2}Z(t)\left(dY(t)\right)^2 \quad (39)$$

$$\text{We have shown that } dY(t) = \mu(t)dt + \sigma dW(t) \text{ and}$$

$$\left(dY(t)\right)^2 = \sigma^2 dt, \quad \text{then (39) become}$$

$$dZ(t) = Z(t)[\mu(t)dt + \sigma dW(t)] + \frac{1}{2}Z(t).\sigma^2 dt \quad (40)$$

But in our Gompertz or Weibull derivation we might have $\mu(t) = -h(t,z) - \frac{1}{2}\sigma^2$ in this case the term simplifies to $dZ(t) = Z(t)[-h(t,z)]dt + \sigma Z(t)dW(t)$ which is the correct SDE for $Z(t)$ we want to verify.

$$\text{Recall from the transformation } X(t) = k - kZ(t)$$

Substituting the close form solution of $Z(t)$ into the above

$$X(t) = k - kZ_0 \exp\left(\int_{t_0}^{t} h(s,z)ds - \frac{1}{2}\sigma^2(t-t_0) + \sigma\big(W(t) - W(t_0)\big)\right)$$

set $Z_0 = 1 - \frac{X_0}{k}$ so that $X(t_0) = X_0$ then

$$X(t) = k - (k - X_0)\exp\left(\int_{t_0}^{t} h(s,z)ds - \frac{1}{2}\sigma^2(t-t_0) + \sigma\big(W(t) - W(t_0)\big)\right) \qquad (41)$$

Equation (41) is the lognormal Weibull process of the model.

4. Inference

4.1 Maximum Likelihood Estimate (MLE)

In this section, the maximum likelihood estimation of the parameters of the two multi-sigmoidal processes is discussed, following the general methodology suggested by (Román-Román *et al*., 2018).

Let d be a discrete sampling paths and each path i is observed at times t_{ij} for $i = 1, \dots, d$ and $j = 2, \dots, n_i$ where all the paths are observed at the same time instants. Furthermore, let ${X_i}^T = \left(X(t_{i1}), \dots, X\left(t_{in_i}\right)\right)$ be the vector corresponding to the i^{th} sample path $(i = 1, \dots, d)$. Then the full data vector will become $X = ({X_1}^T \left| {X_2}^T \right| \dots \left| {X_d}^T \right)^T$. Let assume the process is lognormal so $lnX(t) \sim N\big(\mu_Y(t), \sigma^2(t_1 - t_0)\big)$ i.e., the increments of $lnX(t)$ are normally distributed.

To derive the log likelihood let define the following notation and parameters

$$\tilde{q}: \text{vector of model parameters (Gompertz or Wiebull)}$$

$$z = \left(m_1, {S_1}^2\right): \text{parameters for the initial distribution}$$

$$S^2: \text{diffusion coefficient squared}$$

$$\Delta t_{ij} = t_{ij} - t_{i,j-1}: \text{Time increments}$$

$$X_0 \sim LN\left(m_1, {S_1}^2\right) \qquad (42)$$

$$Let\ Y_{ij} = log\left(\frac{X(t_{ij})}{X(t_{i,j-1})}\right) \qquad (43)$$

From the solution of $X(t)$ take the form of Gompertz general lognormal form in (33) therefore the transition from one time step to another also follow a lognormal law

$$\frac{X(t_{ij})}{X(t_{i,j-1})} \sim \text{Lognormal}\left(H_q(t_{i,j-1}, t_{ij}), \sigma^2(t_{ij} - t_{i,j-1})\right) \qquad (44)$$

$$\text{If } X(t_{ij}) = X(t_{i,j-1}) \exp\left(H_q(t_{i,j-1}, t_{ij}) + \sigma\left(W(t_{ij}) - W(t_{i,j-1})\right)\right) \quad (45)$$

Taking the logarithm and dividing both side of (52) by $X(t_{i,j-1})$

$$\log\left(\frac{X(t_{ij})}{X(t_{i,j-1})}\right) = H_q(t_{i,j-1}, t_{ij}) + \sigma\left(W(t_{ij}) - W(t_{i,j-1})\right) \quad (46)$$

$$\text{Then} \quad Y_{ij} \sim N\left(H_q(t_{i,j-1}, t_{ij}) + \sigma\left(W(t_{ij}) - W(t_{i,j-1})\right)\right)$$

We denote

$S_1{}^2$: Variance of the initial lognormal

S^2: Varianvce of the Brownian Motion

$$\widetilde{H_q}(t_{i,j-1}, t_{ij}) = H_q(t_{i,j-1}, t_{ij}) - \frac{1}{2}S^2(t_{ij} - t_{i,j-1})$$

$$\text{Total number of transitions } n\sum_{i=1}^{d}(n_i - 1)$$

Initial point (observation) log likelihood , for each path i, the first observation X_{i1} is lognormal

$$\log X_{i1} \sim N\left(m_1, S_1{}^2\right) \quad (47)$$

Then the log likelihood condition is

$$L_{(init)} = \prod_{i=1}^{d} \frac{1}{X_{i1}\sqrt{2\pi S_1{}^2}} \exp\left(-\frac{(\log X_{i1} - m_1)^2}{2S_1{}^2}\right)$$

$$\log L_{(init)} = -\frac{d}{2}\log(2\pi) - \frac{d}{2}\log S_1{}^2 - \sum_{i=1}^{d}\log X_{i1} - \frac{1}{2S_1{}^2}\sum_{i=1}^{d}(\log X_{i1} - m_1)^2 1) \quad (48)$$

Then the transition (observation) likelihood.

$$Y_{ij} = \log\left(\frac{X(t_{ij})}{X(t_{i,j-1})}\right) \sim N\left(\widetilde{H_q}(t_{i,j-1}, t_{ij}), S^2(t_{ij} - t_{i,j-1})\right) \quad (49)$$

The expression above follows from a lognormal distribution theory. Then the transition likelihood for each pair is

$$L_{(trans)} = \prod_{i=1}^{d}\prod_{j=2}^{n_i} \frac{1}{X_{i1}\sqrt{2\pi S^2 \Delta t_{ij}}} \exp\left(-\frac{1}{2S^2\Delta t_{ij}}\left[Y_{ij} - \widetilde{H_q}(t_{i,j-1}, t_{ij})\right]^2\right)$$

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$$\log L_{(\text{trans})} = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log S^2 - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\log\Delta t_{ij} - \frac{1}{2S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{\left(Y_{ij} - \widetilde{H_q}(t_{i,j-1}, t_{ij})\right)^2}{\Delta t_{ij}}$$

$$\text{Let} \quad Z_1 = \sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{(Y_{ij})^2}{\Delta t_{ij}}$$

$$F_{\tilde{q}} = \sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{\left(\widetilde{H_q}(t_{i,j-1}, t_{ij})\right)^2}{\Delta t_{ij}}$$

$$G_{\tilde{q}} = \sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{Y_{ij}\widetilde{H_q}(t_{i,j-1}, t_{ij})}{\Delta t_{ij}}$$

$$\log L_{(\text{trans})} = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log S^2\Delta t_{ij} - \frac{1}{2S^2}\left(Z_1 + F_{\tilde{q}} - 2G_{\tilde{q}}\right) \qquad (50)$$

Adding equation (48) and (50) we have the full likelihood for the process over all paths.

$$\log L(\text{ж}) = -\frac{n+d}{2}\log(2\pi) - \frac{d}{2}\log {S_1}^2 - \frac{n}{2}\log S^2\Delta t_{ij} - \sum_{i=1}^{d}\log X_{i1} - \frac{1}{2{S_1}^2}\sum_{i=1}^{d}(\log X_{i1} - m_1)^2$$
$$- \frac{1}{2S^2}\left(Z_1 + F_{\tilde{q}} - 2G_{\tilde{q}}\right) \qquad (51)$$

We assume that the initial state of the process (i.e., the first observation of each path at time t_0) follows a lognormal distribution as stated in (42) and (47). So, to estimate these parameters from the data we have the following.

$$\widehat{m_1} = \frac{1}{d}\sum_{i=1}^{d}\log X_{i1} \qquad (52)$$

$$\widehat{{S_1}^2} = \frac{1}{d}\sum_{i=1}^{d}(\log X_{i1} - m_1)^2 \qquad (53)$$

Equation (52) and (53) are the standard MLE estimates for normal data applied to the logarithm of initial observation X_{i1}. The rest of the process value X_{i1} for $j = 2, \dots, n_i$ are model via a lognormal transition i.e.,

$$Y_{ij} = \log\left(\frac{X(t_{ij})}{X(t_{i,j-1})}\right) \sim N\left(\widetilde{H_q}(t_{i,j-1}, t_{ij}), \sigma^2\Delta t_{ij}\right), \qquad j \geq 2$$

is the deterministic mean of the transition depending on the parameter vector q.

And the model implied transition mean

$H_q(t_{i,j-1}, t_{ij})$ which also depends on the parameter q.

Now in likelihood estimation, we always compare the observed increment Y_{ij} with the predicted mean increment H_q. So, we define

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$$r_{ij}(q) = Y_{ij} - H_q(t_{i,j-1}, t_{ij}) \quad (54)$$

(54) is the residual term i.e., the error term between what we observed (Y_{ij}) and what the deterministic part of the model predicts (H_q). More so, another motivation for defining $r_{ij}(q)$ is because later we will need to expand the quadratic forms like $\sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{(Y_{ij}-\widetilde{H_q}(t_{i,j-1},t_{ij}))^2}{\Delta t_{ij}}$ since $\widetilde{H_q}(t_{i,j-1}, t_{ij}) = H_q(t_{i,j-1}, t_{ij}) - \frac{1}{2}S^2\Delta t_{ij}$

And the squared error decomposition is

$$Z_1 + F_{\tilde{q}} - 2G_{\tilde{q}}$$

$$= \sum_{i,j}\frac{{Y_{ij}}^2}{\Delta t_{ij}} + \left(\frac{{H_q}^2(t_{i,j-1}, t_{ij})}{\Delta t_{ij}} - S\sum_{i,j}H_q(t_{i,j-1}, t_{ij}) + \frac{i}{4}S^2\sum_{i,j}\Delta t_{ij}\right)$$

$$-2\left(\sum_{i,j}\frac{Y_{ij}H_q(t_{i,j-1}, t_{ij})}{\Delta t_{ij}} - \frac{i}{2}S\sum_{i,j}Y_{ij}\right)$$

$$= \sum_{i,j}\frac{{Y_{ij}}^2}{\Delta t_{ij}} + \sum_{i,j}\frac{{H_q}^2(t_{i,j-1}, t_{ij})}{\Delta t_{ij}} - 2\sum_{i,j}\frac{Y_{ij}H_q(t_{i,j-1}, t_{ij})}{\Delta t_{ij}} + \left(-S\sum_{i,j}H_q(t_{i,j-1}, t_{ij}) + S^2\sum_{i,j}Y_{ij}\right)$$

$$+\frac{1}{4}S^2\sum_{i,j}\Delta t_{ij}$$

$$\text{Where } \sum_{i,j}\frac{{Y_{ij}}^2}{\Delta t_{ij}} + \sum_{i,j}\frac{{H_q}^2(t_{i,j-1}, t_{ij})}{\Delta t_{ij}} - 2\sum_{i,j}\frac{Y_{ij}H_q(t_{i,j-1}, t_{ij})}{\Delta t_{ij}} = A_0$$

$$\sum_{i,j}Y_{ij} - \sum_{i,j}H_q(t_{i,j-1}, t_{ij}) = A_1$$

$$\sum_{i,j}\Delta t_{ij} = A_2$$

$$\text{Therefore } Z_1 + F_{\tilde{q}} - 2G_{\tilde{q}} = A_0 + SA_1 + \frac{1}{4}S^2A_2$$

Then equation (50) now becomes

$$\log L_{(\text{trans})} = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log S\Delta t_{ij} - \frac{1}{2S}\left(A_0 + SA_1 + \frac{1}{4}S^2A_2\right) \quad (55)$$

$$\text{where } S = S^2$$

Differentiating with respect to S set to zero

$$\frac{d}{dS}\log L_{(\text{trans})} = -\frac{n}{2S} - \frac{1}{2}\left(\frac{A_0{}' + A_1 + \frac{1}{2}SA_2}{S} - A_0 + SA_1 + \frac{1}{4}S^2A_2\right)$$

But A_0, A_1, A_2 do not depend on S so $A_0{}' = 0$. we then simplify to

$$-\frac{n}{2S}+\frac{A_0+SA_1+\frac{1}{4}S^2A_2}{2S^2}-\frac{A_1+\frac{1}{2}S^2A_2}{2S^2}=0$$

$$\text{Multiply by } 2S^2 \text{ and simplify lead us to } A_0-\frac{1}{4}S^2A_2-nS$$

$$\text{Rearrange } A_2S^2+4nS-4A_0=0$$

We solve the quadratic and pick positive root using quadratic formula

$$a=A_2, b=4n \text{ and } c=-4A_0$$

$$\hat{S}(q)=\frac{-4n\pm\sqrt{(4n)^2+16A_2A_0}}{2A_2}$$

$$=\frac{-4n\pm\sqrt{16n^2+16A_2A_0}}{2A_2}=\frac{-4n\pm4\sqrt{n^2+A_2A_0}}{2A_2}=\frac{-2n\pm2\sqrt{n^2+A_2A_0}}{A_2}$$

We must choose the positive root since $S>0$

$$\hat{S}(q)=\frac{2\left(\sqrt{n^2+A_2A_0}-n\right)}{A_2}$$

Let $R=\sqrt{n^2+A_2A_0}$, then $\hat{S}(q)=\frac{(R-n)}{A_2}$. We multiply both the numerator and the denominator by $R+n$.

$$\hat{S}(q)=\frac{(R-n)(R+n)}{A_2(R+n)}=\frac{2(R^2-n^2)}{A_2(R+n)}.$$

$$\text{But } R^2=n^2+A_2A_0, \quad \text{so } R^2-n^2=A_2A_0. \text{ hence}$$

$$\hat{S}(q)=\frac{2A_2A_0}{A_2(R+n)}=\frac{2A_0}{(R+n)}. \text{ Replacing R back let to}$$

$$\widehat{S^2}(q)=\frac{2A_0}{n+\sqrt{n^2+A_2A_0}} \qquad (56)$$

Equation (56) is the numerically stable equivalent. We need to obtain this estimate because the one in (53) is valid only when $\widetilde{H_q}$ does not depend on S (i.e., H_q is treated as fixed). Because H_q contains $-\frac{1}{2}S\Delta t_{ij}$, hence the formula in (53) is incorrect thus the proper closed form is the one in (56).

4.2 The Score and the Hessian

Recall that $\widetilde{H_q}(t_{i,j-1},t_{ij})=H_q(t_{i,j-1},t_{ij})-\frac{1}{2}S^2\Delta t_{ij}$ Then the derivative of $\widetilde{H_q}$ *with respect to q is*

$$\frac{\partial\widetilde{H_q}}{\partial q}=\frac{\partial H_q}{\partial q} \quad since\ \frac{1}{2}S^2\Delta t_{ij}\ is\ a\ constant\ i.e.,\ it\ does\ not\ depend\ on\ q\ .$$

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Recall that

$$logL_{(trans)} = -\frac{n}{2}\log(2\pi) - \frac{n}{2}logS^2 - \frac{1}{2}\sum_{i=1}^{d}\sum_{j=2}^{n_i} log\Delta t_{ij} - \frac{1}{2S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{\left(Y_{ij}-\widetilde{H_q}(t_{i,j-1},t_{ij})\right)^2}{\Delta t_{ij}}$$

Differentiating with respect to q

$$\frac{\partial logL_{(trans)}}{\partial q} = -\frac{1}{2S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{\left(Y_{ij} - \widetilde{H_q}(t_{i,j-1},t_{ij})\right)^2}{\Delta t_{ij}}$$

$$= -\frac{1}{2S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{2\left(Y_{ij} - \widetilde{H_q}\right)(-)\frac{\partial\widetilde{H_q}}{\partial q}}{\Delta t_{ij}} = \frac{1}{S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{Y_{ij} - \widetilde{H_q}(t_{i,j-1},t_{ij}).\frac{\partial H_q(t_{i,j-1},t_{ij})}{\partial q}}{\Delta t_{ij}}$$

We define the vector X_q as follows

$$X_q = \sum_{i=1}^{d}\sum_{j=2}^{n_i}\frac{1}{\Delta t_{ij}}\left(Y_{ij} - \widetilde{H_q}(t_{i,j-1},t_{ij})\right).\frac{\partial\widetilde{H_q}(t_{i,j-1},t_{ij})}{\partial q}$$

Were Y_{ij}: $is\ the$ log $increment\ between\ two\ time\ points$

Then

$$\frac{\partial logL_{(trans)}}{\partial q} = \bigcup_q (q,s) = \frac{1}{S^2}X_q \qquad (57)$$

Equation (57) is the Score function.

To obtain the Hessian, we find the second derivative of $logL_{(trans)}\ with\ respect\ to\ q$ using product and chain rule.

$$From\ the\ score\ \widetilde{H_q}(t_{i,j-1},t_{ij}) = H_q(t_{i,j-1},t_{ij}) - \frac{1}{2}S^2\Delta t_{ij}.$$

$$here\ we\ treat\ S^2\ as\ fixed\ when\ differentiating\ with\ respect\ to\ q.This\ is\ standard$$

$$when\ computing\ the\ Hessian\ for\ a\ given\ \ S^2.e.g.,in\ profiled\ iterations.$$

$$for\ each\ given\ increment\ (i,j)\ define;$$

$$Scalar \qquad a_{ij}(q) = \frac{Y_{ij} - \widetilde{H_q}(t_{i,j-1},t_{ij})}{\Delta t_{ij}} = \frac{Y_{ij} - H_q(t_{i,j-1},t_{ij})}{\Delta t_{ij}} + \frac{S^2}{2}$$

$$Column\ vector\ (of\ dimension\ p+1\ if\ q \in \mathbb{R}^{p+1})$$

$$b_{ij}(q) = \frac{\partial H_q(t_{i,j-1},t_{ij})}{\partial q}$$

Then the score can be written compactly as

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$$\bigcup_q(q,s) = \frac{1}{S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i} a_{ij}(q)\ b_{ij}(q) \tag{58}$$

We now differentiate the score using product rule to obtain the observed Hessian

$$H_{obs}(q,s) = \frac{\partial^2 logL_{(trans)}}{\partial q \partial q^T} = \frac{\partial \bigcup_q(q,s)}{\partial q^T}$$

$$\frac{\partial}{\partial q^T} a_{ij}b_{ij} = \underbrace{\left(\frac{\partial a_{ij}}{\partial q^T}\right)}_{row\ (1\times p)} \underbrace{b_{ij}}_{column(p\times 1)} + a_{ij}\underbrace{\left(\frac{\partial b_{ij}}{\partial q^T}\right)}_{(p\times p)}$$

So that

$$H_{obs}(q,s) = \frac{1}{S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\left[\left(\frac{\partial a_{ij}}{\partial q^T}\right)b_{ij} + a_{ij}\left(\frac{\partial b_{ij}}{\partial q^T}\right)\right] \tag{59}$$

$$We\ compute\ the\ two\ derivatives\ \frac{\partial a_{ij}}{\partial q^T}\ and\ \frac{\partial b_{ij}}{\partial q^T}.$$

$$\frac{\partial a_{ij}}{\partial q^T} = -\frac{1}{\Delta t_{ij}}.\frac{\partial H_q(t_{i,j-1}, t_{ij})}{\partial q} = -\frac{1}{\Delta t_{ij}} b_{ij}(q) \tag{60}$$

$$Since\ from\ a_{ij}\ only\ H_q(t_{i,j-1}, t_{ij})\ depends\ on\ q, the\ remaining\ terms\ are\ constant$$

$$with\ respect\ to\ q. we\ write\ it\ as\ row\ vector\ to\ match\ the\ multiplication\ in\ (59)\ .$$

$$\frac{\partial a_{ij}}{\partial q^T} = -\frac{1}{\Delta t_{ij}} b_{ij}(q)^T.\ plugging\ into\ \left(\frac{\partial a_{ij}}{\partial q^T}\right)b_{ij}\ gives$$

$$\left(\frac{\partial a_{ij}}{\partial q^T}\right)b_{ij} = \left(= -\frac{1}{\Delta t_{ij}} b_{ij}(q)^T\right)b_{ij} = -\frac{1}{\Delta t_{ij}} b_{ij}(q)b_{ij}(q)^T \tag{61}$$

$$Equation\ (51)\ above\ is\ the\ outer\ product\ and\ the\ result\ is\ a\ p\times p\ matrix.$$

$$We\ differentiate\ \ b_{ij}(q) = \frac{\partial H_q(t_{i,j-1}, t_{ij})}{\partial q}\ with\ respect\ to\ q$$

$$\frac{\partial b_{ij}}{\partial q^T} = \frac{\partial^2 H_q(t_{i,j-1}, t_{ij})}{\partial q \partial q^T} = \partial^2 H_q(t_{i,j-1}, t_{ij}).\ This\ is\ also\ a\ p\times p\ matrix.$$

$$The\ second\ product\ becomes\ \ a_{ij}\left(\frac{\partial b_{ij}}{\partial q^T}\right) = \frac{Y_{ij} - \widetilde{H_q}(t_{i,j-1}, t_{ij})}{\Delta t_{ij}}.\partial^2 H_q(t_{i,j-1}, t_{ij}) \tag{62}$$

Combining (61) and (62) together

$$H_{obs}(q,s) = \frac{1}{S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\left\{-\frac{1}{\Delta t_{ij}} b_{ij}(q)b_{ij}(q)^T + \frac{Y_{ij} - \widetilde{H_q}(t_{i,j-1}, t_{ij})}{\Delta t_{ij}}.\partial^2 H_q(t_{i,j-1}, t_{ij})\right\}$$

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$$Replacing\ b_{ij}(q) = \partial q H_q\left(t_{i,j-1}, t_{ij}\right), \quad then$$

$$H_{obs}(q,s) = \frac{1}{S^2}\sum_{i=1}^{d}\sum_{j=2}^{n_i}\left\{-\frac{1}{\Delta t_{ij}}\partial q H_q\left(t_{i,j-1}, t_{ij}\right)\left(\partial q H_q\left(t_{i,j-1}, t_{ij}\right)\right)^T + \frac{Y_{ij} - \widetilde{H_q}\left(t_{i,j-1}, t_{ij}\right)}{\Delta t_{ij}}.\partial^2 H_q\left(t_{i,j-1}, t_{ij}\right)\right\} \quad (63)$$

Equation (63) is the observed Hessian.

4.3 Fisher Information

$$\text{If we take expectatio under the model } \mathrm{E}\left[\mathrm{Y_{ij}} - \widetilde{\mathrm{H_q}}\left(\mathrm{t_{i,j-1}}, \mathrm{t_{ij}}\right)\right] = 0,$$

So the second term drops out in the expectation. The fisher information is

$$\mathrm{I_{(q,s)}} = -\mathrm{E}[\mathrm{H_{obs}}(\mathrm{q},\mathrm{s})]$$

$$\mathrm{I_{(q,s)}} = \frac{1}{\mathrm{S^2}}\sum_{\mathrm{i=1}}^{\mathrm{d}}\sum_{\mathrm{j=2}}^{\mathrm{n_i}}\frac{1}{\Delta \mathrm{t_{ij}}}\partial \mathrm{q}\mathrm{H_q}\left(\mathrm{t_{i,j-1}}, \mathrm{t_{ij}}\right)\left(\partial \mathrm{q}\mathrm{H_q}\left(\mathrm{t_{i,j-1}}, \mathrm{t_{ij}}\right)\right)^{\mathrm{T}} \quad (64)$$

The matrix above is positive semi-definite and is what we use for fisher scoring.

$$\mathrm{q^{(k+1)}} = \mathrm{q^{(k)}} + \left(\mathrm{I^{(k)}}\right)^{-1}\bigcup \mathrm{q}^{(\mathrm{k})} \quad (65)$$

$$\text{With profiling at iteration k we plug } \mathrm{S^2} = \widehat{\mathrm{S^2}}\left(\mathrm{q^k}\right) \text{ into both } \bigcup \mathrm{q} \text{ and } \mathrm{I_{(q,s)}}$$

The first term in equation (63) always makes the Hessian negative (concave contribution) along the direction where $\partial \mathrm{q}\mathrm{H_q}\left(\mathrm{t_{i,j-1}}, \mathrm{t_{ij}}\right) \neq 0$. The second term can have either sign i.e. it depends on the realized value of the residuals $\mathrm{Y_{ij}} - \widetilde{\mathrm{H_q}}\left(\mathrm{t_{i,j-1}}, \mathrm{t_{ij}}\right)$. That is why Newton's Raphson method can be less stable than fisher scoring.

4.4 Inference Procedure

4.4.1 Standardization $[t_0, \mathrm{T}] \rightarrow [0,1]$

Before applying the inference procedure, it is necessary to discuss a modification involving the time interval where the processes are considered. Although such processes have been defined on a generic interval $[t_0, \mathrm{T}]$, two main reasons motivate the standardization of such interval to $[0,1]$. The first one is the analytical complexity of the multivariate multi-sigmoidal lognormal model. When $t_0 = 0$, many exponential functions turn to 1, making calculus more tractable than the general case $t_0 \neq 0$. The second reason is the numerical precision of computational operations. When $\mathrm{T} = 1$, fluctuations of the objective function (derived from the log-likelihood function) allow for a good performance of the numerical algorithms. Such standardization does not, nevertheless, modify the nature of the process. Indeed, the two multivariate multi-sigmoidal lognormal diffusion processes being considered remain the same when the parametric space is transformed from $[t_0, \mathrm{T}]$ to $[0,1]$. In particular, infinitesimal moments fully characterizing the process are equal up to a term coming from the standardization procedure. (Barrera *et al*.,2021). To improve numerical stability, we standardized the observation times $t \in [t_0, \mathrm{T}]$ to a unit time interval

$\tau \in [0,1]$ via $\tau = \frac{(\mathrm{t}-\mathrm{t_0})}{(\mathrm{T}-\mathrm{t_0})}$. Under this change of variables, the SDE

$$dX(t) = \mu(t)X(t)dt + \sigma X(t)dW(t)$$

In equation (25) becomes

$$dY(\tau) = (T - t_0)\mu(\tau)Y(\tau)d\tau + \left(\sqrt{T - t_0}\right)\sigma Y(\tau)dB(\tau) \quad (66)$$

Where $B(\tau)$ is a standard brownain motion with respect to the rescaled time τ.

The polynomial predictors transform by binomial expansion is given as

$$Q(\tau) = \sum_{\ell} \tilde{b}_\ell \, \tau^\ell \quad (67)$$

$$\text{With} \quad \tilde{b}_\ell = \sum_{k \geq \ell} b_k \binom{k}{\ell} t_0{}^{k-\ell}(T - t_0)^\ell.$$

The log-likelihood parameter is also rescaling as

$$\tilde{S}^2 = S^2(T - t_0) \quad (68)$$

With this reparameterization the log-likelihood, score and observed Hessian retain their algebraic form in τ up to an additive constant, while numerical stability is improved.

4.4.2 Initial Solutions

In order to obtain initial solutions, we propose a simple linear regression. Taking $t_0 = 0$ in agreement with the remark made in the previous subsection, from our previous solutions it follows, for the multivariate Gompertz case, that.

$$\log\log\frac{k}{f(t,z)} = \log a - Q(t,z) \quad (69)$$

Where $k = f_0 e^{\alpha}$, and $Q(t,z)$is the multivariate polynomial predictor on

time t and covariate z as stated earlier.

For the multivariate Weibull case, it has

$\log\left(\dfrac{h - \dfrac{f(t,z)}{f_0}}{h - 1}\right) = -Q(t,z)$, which taking into acount the relation $h = \left(1 - \dfrac{f_0}{k}\right)^{-1}$, it becomes

$$\log\left(\frac{k - f(t,z)}{k - f_0}\right) \quad (70)$$

Both equation (69) and (70) can be viewed as a linear regression models over (t, z^g) and (t, z^w), where t is the time vector and z^g, z^w denote the vectors of transformed responses. Specifically

$$\text{For Gompertz:} \quad z_i{}^g = \log\log\left(\frac{\tilde{k}_g}{m_i{}^g}\right), \quad i = 1, \ldots, d$$

For Weibull: $z_i{}^w = \left(\frac{\tilde{k}_w - m_i{}^w}{\tilde{k}_w - m_i{}^w}\right), \qquad i = 1, \dots, d$

Where $\tilde{k}_g, m_i{}^g, \tilde{k}_w, m_i{}^w$ are the final and i^{th} observed values of the sample means

respectively to be modeled by Gompertz and Weibull processes.

The coefficients of the multivariate predictor $Q(t, z)$ can be estimated by regression coefficients of the transformed response vectors (z^g, z^w) of the covariates and time. Initial values of α and h can be found on the relationships between such parameters and limit values k.

The same process can be used to get a starting σ^2 value. It has a regression equation that is due to having a lognormal distribution. In fact, it is well known that with a random sample of a lognormal distribution $LN(\mu, \sigma^2)$, the ratio between the arithmetic and the geometric mean gives an approximation of σ^2. On substituting this in the distribution of $X(t, z)$, we have, at each t_i, the estimate of

$\sigma_0{}^2 + \sigma^2 t_i$; that is, $\sigma_i{}^2 = 2 \log\left(\frac{m_i}{g_i}\right)$, where $m_i \in \{m_i{}^g, m_i{}^w\}$ denotes the arithmetic and $g_i \in \{g_i{}^g, g_i{}^w\}$ denotes the geometric means at time t_i of the Gompertz and Weibull cases. Simple linear regression of the $\sigma_i{}^2$ against t_i is then done to determine the initial value of σ^2. It should be noted that X_0 degenerate implies that $\sigma_0{}^2 = 0$. Otherwise, it is known that $\sigma_0{}^2$ is etimated previously using the values of the sample paths at t_0.

4.4.3 Degree of Polynomial

The degree of the polynomial q in the infinitesimal mean function of the multivariate multi-sigmoidal Gompertz and Weibull models is crucial. Increase degrees are more flexible in capturing nonlinear epidemic dynamics but be prone to overfitting and decreased interpretability, whereas lower degrees may simply be insufficient to capture the complexity induced by covariates. The nonlinear form $Q(t, z)$ determines the nonlinear form of the growth curve and the specification function as a nonlinear form is direct control of the number and position of inflection points and hence the number of distinguishable epidemic waves in the process under analysis.

Theoretical Foundation of Fourth Degree Polynomial

We suppose that: $Q(t, z)$ is a polynomial:

$$Q(t, z) = \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4 + \gamma^t z \qquad (71)$$

z *is a covariates vector and* γ *is the covariate parameter Vector.*

The formulation is a combination of both temporal dynamics and covariate effects in a single formulation.

Theoretically, the degree q of $Q(t, z)$ is the number of possible inflection points of the process $f(t, z)$. In a sigmoidal function $f(t, z)$ the infinitesimal mean of which is a polynomial $Q(t, z)$ of degree q, the maximum number of possible inflection points is $q - 1$. Points where $f(t, z)$ is zero are called the inflection points. In the deterministic model of the multivariate Gompertz process,

$f(t, z) = k\left[\exp\left(-\alpha \exp\left(-Q(t, z)\right)\right)\right]$, to give the second derivative function of $Q'(t, z)$ and $Q''(t, z)$. Through Simplification, we end up with an inflection condition of the form

$$Q'(t, z) - [Q'(t, z)]^2 \alpha \exp\left(-Q(t, z)\right) [Q'(t, z)]^2 = 0.$$

$$Letting\ p(t, z) = Q'(t, z), this\ becomes$$

$$p'(t, z) - p(t, z)^2 \alpha \exp\left(-Q(t, z)\right) p(t, z)^2 = 0.$$

In case $Q(t,z)$ is a q degree polynomial, $p(t,z)$ is a $q-1$ degree polynomial. Therefore, $p(t,z)$ can be at most $q-1$ different real inflection points.

Epidemiological Motivation

The empirical epidemic data have waves or multi-sigmoidal cumulative patterns that are related to several inflection points. $Q(t,z)$ has three inflection points $(q-1=3)$ which can be observed with a fourth-degree polynomial $(q=4)$ that can be used to describe triple-wave epidemic behavior.

Table 2: COVID-19 Pattern Observed

Wave	Epidemiological description	Period	Structural description
1	First outbreak and containment	Spring 2020	First deceleration phase of the acceleration
2	Summer Surge and partial reopening	Mid-2020	Second turning point
3	Vaccination and Winter Wave	Late 2020-2021	Third turning point

A fourth-degree specification offers the minimum complexity needed to reproduce the observed triple-wave epidemic pattern while remaining interpretable.

Comparative Theoretical Framework

it is possible to compare the different degree of polynomials of $Q(t,z)$ their flexibility and parsimony:

Table 3: Parsimony in different degree of $Q(t,z)$

Polynomial degree (q)	Maximum inflection points	Flexibility	Parsimony
2	1	Low	High
3	2	Moderate	Moderate
4	3	Sufficient	Balance
≥5	≥4	excessive	Low

The fourth-degree polynomial gives the best balance of flexibility and parsimony for modeling complex, interpretable multi-wave epidemics. Adding it to the infinitesimal mean functions of the Gompertz and Weibull diffusion processes gives:

$$\mu_g(t,z) = \alpha \exp\big(-for\ Q(t,z)\big)\, Q'(t,z) f_g(t,z) \qquad (72)$$

$$w_g(t,z) = b[[Q(t,z)]^{\{b-1\}Q'(t,z)[k_w - f_{w(t,z)}]} \qquad (73)$$

and Gompertz and Weibull process are respectively (72) *and* (73)

This allows up to three turning points, matching the empirically observed multi-sigmoidal epidemic pattern.

Policy Effect Estimation

The parameter of the policy effects δ_j were estimated by maximum likelihood and parameters in the predictor of policy equation $Q(t,z)$ Equation (3). These δ_j coefficients are the linear impacts of policies $z_j(t)$ on the growth process in the chain rule relation $\frac{df}{dz_j} = \left(\frac{df}{dQ}\right).\left(\frac{dQ}{dz_j}\right)$ *where* $\frac{dQ}{dz_j} = \delta_j + \Upsilon_j.t + \sum_{J<K} \varepsilon_{JK} z_K(t)$

The estimated values of δ directly imply that the policies save (negative δ) or risk-enhancing (positive δ) effects on COVID-19 cases growth.

5. Application to COVID-19 Data

https://github.com/OxCGRT/USA-covid-policyWe apply the models to daily US COVID-19 cumulative case and death counts, with vaccination coverage and school-closure covariates. Data were obtained from Our World in Data (https://ourworldindata.org/covid-data) and the Oxford COVID-19 Government Response Tracker (https://github.com/OxCGRT/USA-covid-policy).

5.1 Exploratory Data Analysis

The cumulative case trajectory shows clear multi-phase growth with three inflection points corresponding to separate pandemic waves, confirming the need for a multi-sigmoidal rather than single-sigmoidal model.

The daily incidence series is poly-modal, with four major peaks and three subordinate peaks, further supporting a multi-sigmoidal framework.

The correlation matrix shows complex relationships between interventions and case trajectories, indicating a need for multivariate modeling that accounts for temporal dynamics and confounding rather than simple correlation analysis.

5.2 Parameter Estimation

Initial parameter validation gives useful insight before optimization: both models started from a similar carrying-capacity estimate (≈114 million cases) and matched the initial observation of 7 cases, but their predictive performance diverged sharply. The Weibull model was reasonably accurate at this stage (10% overestimation), whereas the Gompertz model collapsed to zero, reflecting high sensitivity to initial parameter values and the need for subsequent maximum-likelihood refinement.

Table 4: Initial Model Framework Validation of Parameter

Parameter	Gompertz Model	Weibull Model	Interpretation
Carrying Capacity (k)	113,780,512	113,780,512	Both models have almost the same theoretical maximum cases
Initial Case Count	7	7	Perfect match to actual initial data point
Final Observed Cases	0	113,780,512	Gompertz fail, Weibull reasonable
Model Prediction	-103,436,836,829	+10,343,683	Gompertz collapsed to zero; Weibull 10% over estimate
Polynomial Q(t,z)	0.1	0.1	Positive structure

Maximum-likelihood estimation converged for both models to nearly identical negative log-likelihood values (Table 5), indicating comparable statistical fit despite their different predictive performance and policy interpretations.

Table 5: Maximum Likelihood Estimation Results

Parameter / Matric	Weibull Model	Gompertz Model	Interpretation
Carrying capacity (k)	126,821,125	149,946,842	Gompertz estimates higher cases
Shape (b)/ Scale (α)	0.5243	2.4876	Weibull: Slow growth; Gompertz: Faster initial growth

Time Polynomial (β_1)	9.4953	-10.8338	Opposite linear time effects
Time polynomial (β_2)	-16.1389	16.7887	Opposite quadratic time effect
Vaccination Effect (δ_1)	12.7060	-23.9502	Weibull positive, Gompertz Negative.
School closure Effect (δ_2)	-0.9194	0.6394	Mixed interaction effect
Time x Vaccination (γ_1)	-0.0184	-0.0631	Both show negative interactions
Time x school (γ_2)	0.7785	13.9101	Gompertz show strong positive interaction
Vacc x School (ε_1)	-19.4989	36.8726	Opposite interaction effects
Noise (σ)	10,168,042	11,849,343	Similar Residuals variability
Initial NLL	26,506	28,712	Weibull started with better initial fit
Final NLL	25,050	25,051	Identical final performance
Iterations	1,001	1,002	Both converge successfully

The final comparison favors Gompertz, which achieved 98.6% predictive accuracy and lower mean absolute error than Weibull's 59.4% accuracy, though Weibull had a marginally smaller RMSE. Overall, Gompertz performs better for multivariate multi-sigmoidal modeling of COVID-19 data.

Table 6: Final Model Fitting Performance Comparison

Matric	Weibull Model	Gompertz Model	Interpretation
Final Prediction	61,429,109	101,983,870	Gompertz perform better
Accuracy	59.4%	98.6%	Gompertz 39% more accurate
RMSE	11,449,904	11,923,009	Similar error magnitude
MAE	9,619,316	8,468,824	Gompertz has lower average error

The effect of the covariates on the growth dynamics of the models demonstrates different impacts on the models with Gompertz model providing epidemiologically coherent policy interpretations with both vaccinations ($\delta_1 = -22.82$) and school closures ($\delta_2 = -24.28$) demonstrating Protective effects i.e., reduced pandemic growth. In contrast, the Weibull model's counterintuitive positive vaccination effect ($\delta_1 = +11.68$) alongside a weak school closure effect ($\delta_2 = -0.56$). This discrepancy in policy effect direction suggests that among many other reasons the Weibull formulation may be misspecifying the underlying epidemic growth mechanism.

Table 7: Policy Effect Interpretation

Policy	Weibull Effect	Gompertz Effect	Epidemiological Interpretation
Vaccinations	+11.68	-22.82	Weibull have wrong sign
School Closures	-0.056	-24.28	Both negative, but Gompertz much stronger
Overall	Mixing/confusing	Consistently Protective	Gompertz provides coherent policy story

5.3 Inflection Points and Multi-Sigmoidal Behavior

The results reveal that COVID-19 pandemic have three inflections points i.e., three waves with Weibull having inflection at day 1, 795 and 1433, Gompertz at day 1, 1304 and 1433. The inflection points at days 795 (Weibull) and 1304 (Gompertz) represent the specific interpretations of model major pandemic transitions potentially corresponding to variant emergence, policy interventions or changes in behaviors. The consensus inflection of both models at day 1433 is probably the transition of pandemic to endemic. This is the inflections points as

determine by the second derivative analysis $(f''(t,z) = 0)$ and gives corroboration of the multi-sigmoidal behavior of both of the models with three distinct inflection points of the same formulation. This observation confirms the fundamental theoretical hypothesis that the dynamics of COVID-19 will naturally go through several phases of growth that will demand complex mult-sigmoidal modeling.

Table 8: Mathematical Test of Multi-Sigmoidal Structure

Mathematical Test	Weibull Model	Gompertz Model	Interpretation
No of Inflection points	3 points	3 points	Both models exhibit mult-sigmoidal behavior
Inflection Days	1, 795, 1433	1, 1304, 1433	Different wave timing prediction

5.4 Lognormal Assumption Validation

Both models violate the core lognormal assumption (P < 2e-16), indicating non-normal, heteroscedastic, and autocorrelated errors. Despite this, both retain clear multi-sigmoidal structure in log-space (Weibull: 468 inflection points; Gompertz: 398), confirming the framework's robustness across multiple waves.

These violations justify the stabilized SDE approach adopted here, since a standard lognormal diffusion process would be misspecified for COVID-19 dynamics.

Table 9: Lognormal Assumption Validation Results

Assumption Test	Weibull	Gompertz	Status
Log-Residual Normality	P<2e-16	P<2e-16	Both Violated
Multiplicative Error Structure	$R^2 = 0.0066$	$R^2 = 0.2417$	Both Violated
Constant Log-Variance	P<2e-16	P<2e-16	Both Violated
No Autocorrelation	P<2e-16	P<2e-16	Both Violated
Multi-sigmoidal Log-Pattern	468 Points	398 Points	Both Valid

5.5 Time-Varying Volatility

Although the SDE framework assumes constant volatility (σ), the data show significant time-varying volatility (heteroscedasticity p < 2e-16), ranging from near-deterministic periods to swings of several million cases. Gompertz shows lower average volatility (more stable predictions) than Weibull, with a minimum of 4.21 marking periods of near certainty — illustrating why a constant-volatility SDE is inadequate.

High- and low-volatility periods correspond respectively to episodes of uncertainty (variant emergence, policy change) and stable transmission, consistent with real COVID-19 dynamics.

Table 10: Time-Varying Volatility Estimation

Volatility Metric	Weibull Model	Gompertz Model	Interpretation
Mean σ(t)	11,096,284	848,217	Weibull is 29% more volatile
Minimum σ(t)	52,455	4.21	Gompertz is near zero low volatility
Maximum σ(t)	6,007,607	5,218,077	Both show extreme high volatility periods
Range	5.9M cases	5.2M	Massive volatility swings

5.6 Uncertainty Quantification

Bootstrap analysis of the residuals shows near-perfect calibration for both models (Gompertz: 94.8% coverage; Weibull: 94.6%), with Gompertz's wider bands (43.8M cases) reflecting a more conservative, assumption-free uncertainty estimate drawn from all 1,436 empirical forecast errors.

Table 11: Bootstrap Uncertainty Quantification

Metric	Weibull Model	Gompertz Model	Interpretation
95% CI Coverage	94.6%	94.8%	Both achieve near perfect coverage
Confidence Band Width	37.1M cases	43.8M cases	Gompertz is wider, more conservative uncertainty
Residuals Used	1,436 points	1,436 points	Comprehensive empirical basis

5.7 SDE Implementation

Direct application of the SDE framework produced numerically unstable paths for both models. We addressed this with a stabilized fallback combining deterministic predictions with empirically calibrated noise. The two models failed for because of different reasons. The Weibull fail because of the singular drift term $[Q(t,z)]^{b-1}$ that causes Euler–Maruyama moments to diverge (Hutzenthaler *et al.,* 2012), and Gompertz fail because of an unnormalized volatility scale that controlled the drift. In all cases the fallback replaces the divergent simulation with a deterministic-plus-noise approximation rather than a convergent SDE solution.

The stabilized Gompertz SDE performed well (98.5% accuracy, 18.3% uncertainty-band width), whereas the stabilized Weibull SDE underestimated cases (61.8% accuracy) with an unrealistically wide interval (5279.1%). The low CI-coverage rates (27.4-35.8%) indicate that these SDEs reflect forward-looking prediction uncertainty rather than retrospective fit.

Table 12: SDE Implementation Performance

Metric	Gompertz SDE	Weibull SDE	Interpretation
Final Accuracy	98.5%	61.8%	Gompertz maintains excellent predictive performance
Path Stability	10/10 stable	Mixed stability	Weibull shows explosive paths
CI Coverage	27.4%	35.8%	Both underestimate uncertainty
Mean CI Width	18.3%	5279.1%	Weibull has unrealistic uncertainty
Path Variability	3.66%	6.56%	Gompertz more stable predictions

CONCLUSIONS

This study compared multivariate Gompertz and Weibull-type growth models within a stochastic diffusion framework that integrates time-dependent and fixed covariates. The Gompertz model proved the more appropriate choice for wave-like infectious diseases such as COVID-19, delivering more accurate predictions, more stable performance, and more epidemiologically coherent policy effects than the Weibull formulation. At the same time, the analysis exposed real challenges: both models breached core lognormal assumptions, and the pure stochastic formulation was numerically unstable, which motivated the stabilized hybrid SDE approach adopted here.

More broadly, the study illustrates the risk of prioritizing statistical fit over biological plausibility in multivariate multi-sigmoidal modeling, and demonstrates a practical workflow — assumption testing, time-varying volatility, bootstrap uncertainty, and stabilized SDEs — for diagnosing and correcting such issues with messy real-world data. Parameter optimization for these models should be bounded to avoid explosive estimates. Future work should compare estimated inflection dates against real-world event timelines to identify their drivers, extend the covariate set (e.g., mask-wearing policy) and its interactions, and address the non-monotonic early-time behavior observed here, potentially by enforcing monotonicity constraints or exploring alternative initial-solution methods.

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