Some Applications of Ramanujan's Master Theorem

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ABSTRACT

Within this article we shall try to find whether it is possible to have a formulae, and prove a formula for solutions of the equations of the form $\mathrm{a}x^{\alpha} + \mathrm{b}x^{\beta} + \mathrm{c} = 0$ .Where $a, b, c, \alpha,\ \beta$ are real numbers and $\alpha > \beta$. And we will get a formula with the help of Ramanujan's Master theorem.

INTRODUCTION

The Egyptian Mathematician Berlin Papyrus, dating back to the middle kingdom (2050BC to 1650 BC) and Greek Mathematician Euclid (circa 300BC) used geometric methods to solve quadratic equations. The Greek Mathematician Diophantus, in his work Arithmatica, (circa 250 AD) solved quadratic equations with a method more recognizably algebraic than the geometric algebra of Euclid, his solutions gives only one root, even when both roots exists. Indian Mathematician Brahmgupta described the quadratic formula in his treatise Brãhmãsphutãsiddhãnta published in 628 AD, although he described it in words, according to this solution of the quadratic equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{(\sqrt{(4ac + b^2)} - b}{2a}.$$

Another Indian Mathematician Sridhrãcãryya (870-930 AD) came up with similar formula having no consideration for both the roots. The quadratic formula covering all cases was first obtained by Simon Stevin in 1594 AD. In 1637 Rene Descartes published La Geometriecontaining special cases of quadratic formula in the form we know today.

Equations of the form $\mathrm{a}x^{\alpha} + \mathrm{b}x^{\beta} + \mathrm{c} = 0$ Where a, b, c, α, β are real numbers and $\alpha > \beta$, are common in theoretical physics especially quantum dynamical equations, but till date we have only numerical solutions for such equations which gives no insight of the further evolution of the dynamical systems.

This article will find and prove a formula for solutions of the equations of the form

$$\mathrm{a}x^{\alpha} + \mathrm{b}x^{\beta} + \mathrm{c} = 0. \qquad (1.1)$$

Ramanujan's Master theorem

If a complex-valued function f(x) has an expansion of the form

$$f(x) = \sum_{k=0}^{\infty} \frac{\phi(k)(-x)^k}{k!}.$$

then the Mellin transform of $f(x)$ is given by

$$\int_0^{\infty} x^{s-1} f(x)dx = \Gamma(s)\phi(s).$$

where Γ(s) is the gamma function.

Main Results

Proposition-1-Suppose in equation (1.1) α, β are rational numbers, then equation (1.1) is a polynomial equation.

Proof. Put $\alpha = \frac{p_1}{q_1}$ and $\beta = \frac{p_2}{q_2}$ where p_1, q_1, p_2, q_2 are natural numbers, also put $x^{\frac{1}{q_1 q_2}} = y$, get

$$x^{\alpha} = x^{\frac{p_1}{q_1}} = y^{q_2 p_1};\ x^{\beta} = x^{\frac{p_2}{q_2}} = y^{q_1 p_2} \qquad (3.1)$$

from (1.1) and (3.1) get

$$ay^{q_2 p_1} + by^{q_1 p_2} + c = 0. \qquad (3.2)$$

equation (3.1) is a polynomial equation. Since $\alpha > \beta$, therefore $q_2 p_1 > q_1 p_2$, hence equation (3.2) has degree $q_2 p_1$.

Remark-1. If equation (3.2) has degree $q_2 p_1 \geq 5$,then it could not be solved in radical as follows from the Galois theory.

Remark-2. If $\alpha > \beta$ in (1.1) are not rational then equation (1.1) turns to be a transcendental equation as it could not be reduced to a polynomial equation by fundamental algebraic operations of addition, subtraction, division and multiplication.

Theorem-1. For $0 < \alpha < 1$, if z_0 is root of equation $y^{\alpha} + y + c = 0$, then $z_0 b^{\frac{\ln a - \ln b}{(1-\alpha)\ln b}}$ is root of $ax^{\alpha} + bx + D = 0$, where $D = cb^{1 - \frac{\ln a - \ln b}{(1-\alpha)\ln b}}$.

Proof. Rewrite equation $ax^{\alpha} + bx + D = 0$ as $ax^{\alpha} + b^{1-k}.b^k x + D = 0$,where k is a non zero real number. Put $b^k x = y$, then $x = yb^{-k}$, this gives $ax^{\alpha} = ab^{-\alpha k} y^{\alpha}$.Thus equation $ax^{\alpha} + bx + D = 0$, becomes

$$ab^{-\alpha k} y^{\alpha} + b^{1-k} y + D = 0. \qquad (3.3)$$

Choose k so that $ab^{-\alpha k} = b^{1-k}$, this gives

$$k = \frac{\ln b - \ln a}{(1-\alpha)\ln b}$$

Divide equation (3.3) by b^{1-k}, to get

$$y^{\alpha} + y + c = 0. \qquad (3.4)$$

Where $c = Db^{\frac{\ln b - \ln a}{(1-\alpha)\ln b} - 1}$ Now if z_0 is root of equation (3.4) then $x = b^{-k} y$, gives that $b^{\frac{-(\ln b - \ln a)}{(1-\alpha)\ln b}} z_0$ is root of $ax^{\alpha} + bx + D = 0$.

Theorem2. For $(\alpha/\beta) > 1$,if z_0 is root of equation $y^{\alpha/\beta} + y + c = 0$, then $z_0^{\frac{1}{\beta}} b^{\frac{\beta(\ln a - \ln b)}{(\beta-\alpha)\ln b}}$ is root of $ax^{\alpha} + bx^{\beta} + D = 0$, where $c = D.b^{\frac{\beta(\ln a - \ln b)}{(\beta-\alpha)\ln b} - 1}$

Proof. Rewrite equation $ax^{\alpha} + bx^{\beta} + D = 0$ as $ax^{\alpha} + b^{1-k}.b^k x^{\beta} + D = 0$,where k is a non zero real number. Put $b^k x^{\beta} = y$, then $x = (yb^{-k})^{\frac{1}{\beta}}$, this gives $ax^{\alpha} = ab^{\frac{-k\alpha}{\beta}} y^{\frac{\alpha}{\beta}}$. Thus equation $ax^{\alpha} + bx^{\beta} + D = 0$, becomes

$$ab^{\frac{-k\alpha}{\beta}}y^{\frac{\alpha}{\beta}} + b^{1-k}y + D = 0. \qquad (3.5)$$

Choose k so that $ab^{\frac{-k\alpha}{\beta}} = b^{1-k}$, this gives $k = \frac{\beta(\ln a - \ln b)}{(\beta-\alpha)\ln b}$

Divide equation (3.5) by b^{1-k}, to get

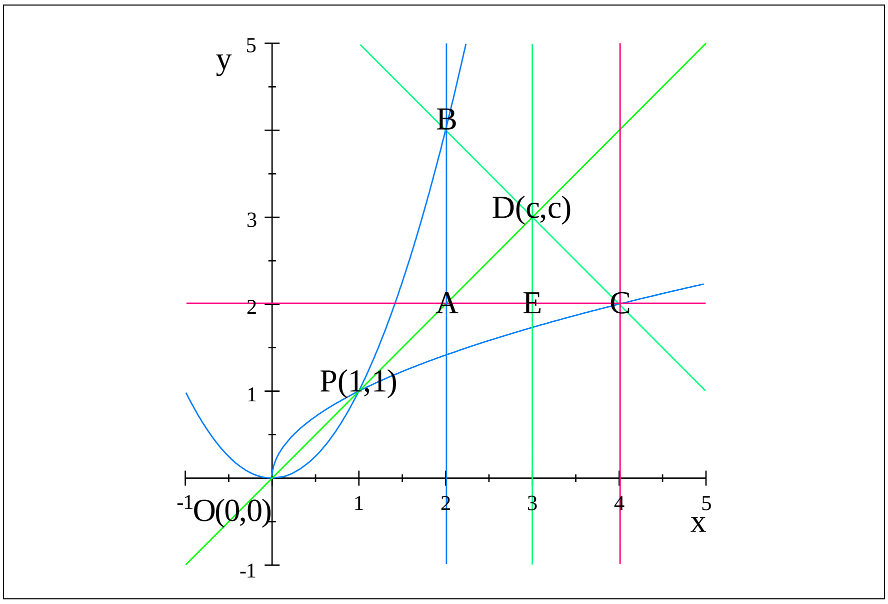
$$y^{\frac{\alpha}{\beta}} + y + c = 0. \qquad (3.6)$$

Now if z_0 is root of equation (3.6) then $x = (yb^{-k})^{\frac{1}{\beta}}$, gives that $z_0^{\frac{1}{\beta}} b^{\frac{(\ln b - \ln a)}{(\beta-\alpha)\ln b}}$ is root of $ax^{\alpha} + bx^{\beta} + D = 0$. This completes the proof.

Theorem3. For $0 < \alpha < 1$, equation $x^{\alpha} + x - 2c = 0$, has a root at, $c + cf$, and for $\alpha > 1$, it has a root at $c - cf$, where $f = \frac{\Gamma(\alpha)\phi(\alpha)}{c^{1-\alpha}} - 1$,

$$\phi(k) = (k! + \frac{k!\,\alpha}{1!} + \ldots + \frac{k!\,\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}.$$

Proof. Plot $y = x^{\alpha}$ and $y = x^{\frac{1}{\alpha}}$, $0 < \alpha < 1$, $y = x$ and on Cartesian plane (for illustration take $\alpha = \sqrt{2}$



As plotted above in the figure, curves $y = x^{\alpha}$ and $y = x^{\frac{1}{\alpha}}$, are symmetric about line $y = x$, points B and C respectively lies at the intersection of the curves $y = x^{\frac{1}{\alpha}}$ and $y = x^{\alpha}$, and the line perpendicular to line $y = x$ and passing through point D(c, c). Thus in ΔABC, $AE = EC = d$ (say). Therefore x-coordinate of point C is $c + d$, and x-coordinate of point B is $c - d$. Since $y = x^{\alpha}$, and line passing through points B, C and D(c,c) i.e. $y = -x + 2c$ intersects each other at point C, therfore at C we must have $x^{\alpha} = -x + 2c$.

$$x^{\alpha} + x - 2c = 0. \qquad (3.7)$$

Since x-coordinate of point C is $c + d$,therefore at C we must have

$$(c+d)^{\alpha} + (c+d) - 2c = 0$$

$$(c+d)^{\alpha} = c - d. \qquad (3.8)$$

Similarly at point B, x-coordinate of point B is $c - d$,therefore at B we must have

$(c-d)^{\frac{1}{\alpha}} = c+d.$ (3.9)

Thus to find root of equation (3.7) we need to find d in terms of c and α, for this rewrite equation (3.8) as

$$(1+f)^{\alpha} = c^{1-\alpha}(1-f).$$

Where $f=(d/c)$. Note from geometric representation as in figure-1 $AC=2d, OD=c$ gives $d<c$, without geometric consideration inequality $d<c$ could not be established, now we can use binomial expansion of $(1+f)^{\alpha}$ *wherever needed*, To find f, rewrite above equation as

$$\frac{(1+f)^{\alpha}}{(1-f)} = c^{1-\alpha}.$$

, For $|\ xf\ |<1$ we have $\frac{(1+xf)^{\alpha}}{(1-f)} = \left(1+\alpha xf+\frac{\alpha(\alpha-1)}{1!}(xf)^2+\frac{\alpha(\alpha-1)(\alpha-2)}{2!}(xf)^3+\ldots\right)(1+f+f^2\ldots)$

, now collect coefficients of $(xf)^k$, multiply numerator and $denominator\ by\ k!$, and use the fact that

$\frac{1}{2\pi}\int_0^{2\pi} e^{im\theta}.e^{-in\theta}\mathrm{d}\theta = \begin{cases} 1\ if\ m\neq n \\ 0\ otherwise \end{cases}.$ (3.9A)

$$\frac{(1+xf)^{\alpha}}{(1-f)}$$

$=\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_0^{\infty}\frac{\left((k!+\frac{k!\alpha}{1!}+\ldots+\frac{k!\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}\right)(-xe^{i\theta+i\pi})^k}{k!}\right)\frac{1}{1-fe^{-i\theta}}d\theta.$ (3.9B)

Where $\left(\sum_0^{\infty}\frac{\left((k!+\frac{k!\alpha}{1!}+\ldots+\frac{k!\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}\right)}{k!}(-xe^{i\theta+i\pi})^k\right)$ is the coefficient of $(xf)^k e^{ik\theta}$ without f^k , f^k

Could be recovered upon integration and using orthogonality property as mentioned in equation (3.9A) Recognise $f(x)$ of the Ramanujan's master theorem as

$\Psi(x,\theta)=\sum_0^{\infty}\frac{\left((k!+\frac{k!\alpha}{1!}+\ldots+\frac{k!\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}\right)}{k!}(-xe^{i\theta+i\pi})^k.$ (3.10)

Since infinite series in (3.10) has infinite radius of convergence in variable x we can apply Mellin's transform on $\Psi(x,\theta)$, Then by Ramanujan's master theorem and equation (3.9B) get,

$$\int_0^{\infty}\frac{x^{\alpha-1}(1+xf)^{\alpha}}{(1-f)}dx = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}x^{\alpha-1}\Psi(x,\theta)\frac{1}{1-fe^{-i\theta}}dxd\theta,$$

$\Gamma(\alpha)\phi(\alpha)\sum_{k=0}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{ik\theta}e^{ik\pi}}{1-fe^{-i\theta}}d\theta = c^{1-\alpha}$, gives

$\sum_{k=0}^{\infty}e^{ik\pi}f^k = \frac{1}{1+f} = \frac{c^{1-\alpha}}{\Gamma(\alpha)\phi(\alpha)},$

$f=\frac{\Gamma(\alpha)\phi(\alpha)}{c^{1-\alpha}}-1$ (3.12)

where $\phi(\alpha)=\left(k!+\frac{k!\alpha}{1!}+\ldots+\frac{k!\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}\right)$. Equation (3.12) gives root of the equation $x^{\alpha}+x-2c=0$ as

$$c+\boldsymbol{d}=c+cf=c+c\left(\frac{\Gamma(\alpha)\phi(\alpha)}{c^{1-\alpha}}-1\right)=\Gamma(\alpha)\phi(\alpha)c^{\alpha} \quad (3.13)$$

Therefore roots are given by This completes the proof of the theorem.

Remark-1: Since α need not be an integer, as $k!=\Gamma(k+1)$, we can write generalised approximate for $\alpha!=\Gamma(\alpha+1)$, for this generalization by Katugampola, Niels Henrik Abel, Podlubny see (**6., 7., 8.**). So root given in equation (3.13) is approximate.

Corollary-1: From Theorem-2, Theorem-3 and equation (3.13) root of equation $ax^{\alpha}+bx^{\beta}+D=0$, is given by $\Gamma(\alpha)\phi(\alpha)c^{\alpha}z_0^{\frac{1}{\beta}}b^{\frac{\beta(\ln a-\ln b)}{(\beta-\alpha)\ln b}}$

Proof. From Theorem-2 $c=D.b^{\frac{\beta(\ln a-\ln b)}{(\beta-\alpha)\ln b}-1}$, from Theorem-3 we have $-2c=D.b^{\frac{\beta(\ln a-\ln b)}{(\beta-\alpha)\ln b}-1}$, therefore from (3.13) get solution of $y^{\alpha/\beta}+y+c=0$

$$z_0=\Gamma(\alpha)\phi(\alpha)\left(\frac{e^{i\pi}D.b^{\frac{\beta(\ln a-\ln b)}{(\beta-\alpha)\ln b}-1}}{2}\right)^{\alpha}. \quad (3.14)$$

Therefore From Theorem-2, Theorem-3 and equation (3.14) root of the equation $ax^{\alpha}+bx^{\beta}+D=0$, is given by

$$x=\left(\Gamma(\alpha)\phi(\alpha)\left(\frac{e^{i\pi}D.b^{\frac{\beta(\ln a-\ln b)}{(\beta-\alpha)\ln b}-1}}{2}\right)^{\alpha}\right)^{\frac{1}{\beta}}b^{\frac{\beta(\ln a-\ln b)}{(\beta-\alpha)\ln b}}.$$

Revision: All the revisions indicated by reviewer has been incorporated in this version of manuscript.

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Declaration

This paper is in original form and no version of it will be submitted for publication elsewhere.

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