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Establishing the A(A)-Stability of Extended Stable Parameter Dependent Nested Linear Multistep Methods with Off-Step Points for Stiff Ordinary Differential Equations

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ABSTRACT

This study established the $A(\alpha)$-stability of extended stable parameter dependent nested linear multistep method with off-step points of order seven for step number $k \leq 5$ for the numerical integration of stiff and non-stiff initial value problems (IVPs) in ordinary differential equations (ODEs). This method forms a super class of the linear multistep methods (LMMs), with properties that embedded the characteristics of LMM and hybrid methods. The method is hybrid in nature as a result of the incorporation of two or moreoff-step points and this makes this method a unique one. The primary aim of this method is to bypassing the Dahlquist order and stability barrierwhich serve as a set back to many works in linear multistep method. The stability properties of the methods were investigated and the intervals of absolute stability of the methods with step number $k \leq 5$ are presented using the boundary locus techniques. The proposed method was found to attained a better stability property. The method is *A- stable* for $k =1$ and *A(α) –stable* for $2 \leq k \leq 5$. The instability of the new methods set in when k >5.which makes the methods more suitable for stiff initial value problems.

Keywords: Nested multi-step methods, stiffly stable**,** boundary locus, A-stability, A(α)-stability, instability.

INTRODUCTION

The class of methods for finding numerical solution to stiff initial value problems in ordinary differential equation is of the type

$$y'(x) = f(x, y(| x)), \quad y(x_0) = y_0 \qquad (1)$$

where $f(x, y(x))$ and $y(x)$ may be vectors, can be based on extended stable parameter dependent nested linear multi-step methods (PDNLMMs) with off-step points in the interval (x_n, x_{n+k}). The methods in this regard are those of (Olatunji 2019) and (Ekoro et al 2018). A method in this class produces approximate solution y_{n+k} to y (x_{n+k}).

In this paper, we establish the *A(α)-stability* of extended stable parameter dependent nested linear multistep methods with off-step points for stiff ordinary differential equations. The author in (Ekoro et al 2018) introduced the work of nested. Further studies of nested method can be found also in the works of (Olatunji 2021), (Kulikov 2020). Nested method is an extension of the General Linear method (GLM) in (Olatunji 2021). The purpose for the Nested method is to by-pass the order and stability barrier in multistage and multivalue algorithm. An example of nested multistage method can be found in the work of (Okuonghae and Ikhilke 2013) and (Olatunji 2021). Also, an example of nested multivalue method can be found in the work (Kulikov 2015) and (Olatunji 2019).

According to (Akhanolu and Agbeboh 2023), the general form of linear multistep method (LMMs) is

$$Y_0 = \sum_{j=0}^{k} \phi_j y_{n+j} + h\lambda_k f_{n+k}; \qquad Y_0 = y_{n+c_0}. \tag{2}$$

and its hybrid counterpart

$$\begin{aligned} Y_{i+1} &= \sum_{j=0}^{k} \varphi_j y_{n+j} + h\rho_i f(Y_i); \qquad i = 0(1)\, s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} &= \sum_{j=0}^{k-1} \delta_j y_{n+j} + h\theta_{s-1} f(Y_{s-1}) + h\phi_k (f_{n+k} + a\, f_{n+k-1}); \quad c_i = k - \frac{1}{2^{k-i}}, \quad 0 \le c \le k. \end{aligned} \tag{3}$$

where Y_{i+1} and y_{n+k} are the hybrid inputs and output (Daramola et al 2024).

Hence, the general form of extended stable parameter dependent nested hybrid multistep method with off-step points is

$$\begin{cases} Y_0 = \sum_{j=0}^{k} \phi_j y_{n+j} + h\lambda_k f_{n+k}; \qquad Y_0 = y_{n+c_0} \\ Y_{i+1} = y_{n+k-1} + \sum_{j=0}^{k} \varphi_j f_{n+j} + h\rho_i f(Y_i); \qquad i = 0(1)\, s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} = y_{n+k-1} + \sum_{j=0}^{k-2} \delta_j y_{n+j} + h\theta_{s-1} f(Y_{s-1}) + h\gamma_k (f_{n+k} + a\, f_{n+k-1}); \quad 0 \le c \le k. \end{cases} \tag{4}$$

the parameters $\phi_j, \gamma_k, \delta_j, \rho_i, \theta_{s-1}, \lambda_k$ with i= 1, · · · , s-1, j = 0, 1, · · · , k are all real coefficients. The $f_{n+j} = f(x_{n+j}, y_{n+j})$ is the first derivative function, $h = x_{n+1} - x_n$ denotes the mesh size, k and s represents the step number, and the stage respectively, while $Y_i = y(x_n + c_i h) + Oh^{q_{i+1}}$, i= 0, · · ·, s-1, and y_{n+k} are the stage and the output point of the methods in (3). $f(Y_i) = \{f(x_n + c_i h, Y_i)\}_{i=0}^{s-1}$ denotes the derivative of the stages. The stability region of the formulae in (3) depends on the choice of the parameter "a". $c = [c_0, c_1, \cdots, c_s]^T$ is the abscissa vector of the input methods. The elements in c are computed from the abscissa generator: $c_i = \left\{k - \frac{1}{2^{k-i}}\right\}_{i=0}^{s-1}$, $k = 1, 2 \ldots$ s-1.

Interestingly, many cases of these nested methods are known to be stiffly stable for the numerical solution of stiff initial value problems in ODEs (Kulikov 2020) with

$$y' = f(x, y),\ x \in [x_0, X],\ y(x_0) = y_0,\ f : \Re \times \Re^m \to \Re^m. \tag{5}$$

In ordinary differential equations where $y(x)$ and $f(x, y)$ may be vectors, we seek the numerical solution of this problem using extended stable parameter dependent nested multistep methods with off-step points (PDNLMMs) and its hybrid counterpart obtained by the transformation of the continuous linear multistep method of (Ekoro et al 2018). Such formulae are the adaptive nested Runge - Kutta (ANRK) methods (Ekoro et al 2018), (Okuonghae 2014), hybrid multistep formulae with nested hybrid predictors (Olatunji and Ikhile 2020), second derivative linear multistep with nested predictors [9] which was introduced to overcome the limitation of order reduction that is often encountered during the implementation process in some diagonally and fully implicit Runge-Kutta methods and also to overcome the limitation encountered by nested method in the area of long execution time when applied to large scale initial value problems. These formulae developed did very well in extensive numerical computations of stiff ODEs. Authors(Ebhohimen and Oluwayemi 2023) and (Oguniran et al 2025) also exploited interesting schemes in solving certain numerical problems. The interest of this paper however is to establish the A(α)-stability of extended Stable parameter dependent nested linear multistep methods with off-step points for the solution of the IVPs (1).

Order conditions

By Taylor's series technique, the local truncation error of the stage method and the output method in (4) at x_n gives the following order conditions:

$$h^{q_0}:\begin{cases}\sum_{j=0}^{k}\phi_j = 1, & q_0 = 0,\\ \sum_{j=1}^{k} j\phi_j + \lambda_k = c_0, & q_0 = 1,\\ \frac{1}{q_0!}\sum_{j=1}^{k} j^{q_0}\phi_j + \frac{1}{(q_0-1)!}k^{q_0-1}\lambda_k = \frac{c_0^{q_0}}{q_0!}; & q_0 = 2,3,\ldots,\end{cases} \tag{6a}$$

The error constant of the method in (6a) is

$$C_{q_0+1} = \frac{1}{(q_0+1)!}\sum_{j=0}^{k} j^{q_0+1}\phi_j + \frac{1}{q_0!}k^{q_0}\lambda_k = \frac{c_i^{q_0+1}}{(q_0+1)!} + \mathrm{O}(\mathrm{h}^{q_0+1}). \tag{6b}$$

Equations (6a) and (6b) are the order conditions and error constants of the first input method in (4) while the order conditions and error constants of the second input method in (4) are

$$h^{q_i}:\begin{cases}\sum_{j=0}^{k}\varphi_j + \rho_0 = 4 + c_1, & i = 1,2,\ldots,s,\quad i = 0(1)s-1,\\ \sum_{j=1}^{k} j\varphi_j + c_0\rho_i = 8 - \frac{c_i^2}{2!}, & q_i = 1,\\ \frac{1}{q_i!}\sum_{j=1}^{k} j^{q_i}\varphi_j + \rho_0 c_0^{q_i-1} = -\frac{c_0^{q_i}}{q_i!} + \frac{64}{q_i!}; & q_i = 2,3,\ldots,\end{cases} \tag{7a}$$

The error constant of the method in (7a) is

$$C_{q_i+1} = \frac{1}{(q_i+1)!}\sum_{j=0}^{k-1} j^{q_i+1}\varphi_j + \rho_0 c_0^{q_i} = \frac{c_i^{q_i+1}}{(q_i+1)!} + \frac{64}{(q_i+1)!}\mathrm{O}(\mathrm{h}^{q_i+1}). \tag{7b}$$

For the output method in (4), the order conditions and error constants are

$$h^{p}:\begin{cases}\sum_{j=0}^{k-1}\delta_j + \theta_k = 1, & p = 0,\\ \sum_{j=1}^{k-1-} j\,\delta_j + \theta_k(5+a) = c_4, & p = 1,\\ \frac{1}{p!}\sum_{j=1}^{k-1} j^{p}\delta_j + \frac{1}{(p-1)!}\theta_{s-1}c_{s-1}^{p-1} + \frac{1}{(p-1)!}\left(a(-1+k)^{p-1} + k^{p-1}\right)\gamma_k = \frac{k^p}{p!}; & p = 2,3,\ldots,\end{cases} \tag{8a}$$

and

$$C_{p+1} = \frac{1}{(p+1)!}\sum_{j=1}^{k-1} j^{p+1}\delta_j + \frac{1}{p!}\theta_{s-1}c_{s-1}^{p} + \frac{1}{p!}\left(a(-1+k)^{p} + k^{p}\right)\gamma_k = \frac{k^{p+1}}{(p+1)!} + \mathrm{O}(\mathrm{h}^{p+1}) \tag{8b}$$

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Definition 1: Stiffness (Okuonghae 2014)

The method (4) is said to be stiff over the interval [a, b] if for every $x\epsilon[a,b]$, the eigenvalues {λ,(x), s = 1,…, m} of the Jacobian matrix $J = \frac{\partial f}{\partial y}$ satisfy the following conditions:

a. Re $\lambda_s(x) < 0, s = 1, \ldots, m$
where Re$\lambda_s(x)$ represents the real component of the complex number $\lambda_s(x)$

b. Stiffness Ratio = $\frac{\lambda_r(x)}{\lambda_s(x)} \geq 1,\ \forall r, s = 1, \ldots, m$ for the linear initial value problem

$$y' = Ay + g, \qquad y(x_0) = y_0$$
$$\frac{\partial f}{\partial y} = \frac{\partial f'}{\partial y} = A$$

is m x m matrix and $g\epsilon R^m$.

Definition 2 (Agbeboh and Esekhaigbe 2016): A numerical method is said to be A-stable if its region of absolute stability contains the whole of the left-hand half-plane $Re\ h\lambda < 0$.

However, A-stable is a necessary requirement for a numerical method for stiffness to be satisfied.

Definition 3:(Akhanolu and Agbeboh 2023): A numerical method is said to be stiffly-stable if

(i) the region of absolute stability contains $\Re_1$ *and* $\Re_2$ and
(ii) if it is accurate for all $h \in \Re_2$ when applied to the test equation $y' = \lambda y, \lambda$ a complex constant with $Re\ h\lambda < 0$, where

$$\Re_1 = \{[h\lambda] Re\ h\lambda < -a\},$$

$\Re_2 = \{[h\lambda] - a \leq Re\ h\lambda \leq b, -c \leq Imh\lambda \leq c\}$, and a, b *and* c are positive constants.

Definition 4: A(α) – Stability:(Akhanolu and Agbeboh 2023). A numerical method is said to be **A(α) – Stable** if for some αϵ(0, $\frac{\pi}{2}$) if $R_A = \{h \left| Arg(-\bar{h}) \right| < \alpha, \bar{h} \neq 0\}$.

Definition 5: A method is said to be **Parameter Dependent** if the method is relying on the chosen value of a parameter "a" i.e. $0 \leq a \leq 1$ for a better stability property of the method.

Derivation of the Methods

Using the general form of the *k*-step of the proposed method in (4) above, the method is derived for k_i, s by applying Taylor's series expansion to k_i's respectively and by setting $k = 1\ s = 1$, $p = q + 1 = 3$, $c_0 = \frac{1}{2}$ and $a = \frac{1}{2}$.

For *k=1*, the input method *Y₀* is derived using Taylor's series expansion;

$$Y_0 = S_0 y_n + S_1 y_{n+1} + h\varpi_1 f_{n+1} \quad (9)$$

where

$$\begin{aligned} Y_0 &= y(x_n + c_0 h) \\ y_{n+1} &= y(x_n + h) \\ f_{n+1} &= y'(x_n + h) \end{aligned} \quad (10)$$

Substituting (10) into (9) and equating to zero, yields. (11)

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Expanding (11) using Taylor's series expansion about x_n, gives

$$-(-1+S_0+S_1)y_{x_n}+(c_0-S_1-\varpi_1)y'_{x_n}h+\frac{1}{2}(c_0^2-S_1-2\varpi_1)y''_{x_n}h^2 + \frac{1}{6}(c_0^3-S_1-3\varpi_1)y'''_{x_n}h^3+\ldots+=0 \tag{12}$$

Collecting the co-efficient of the power of *h*, gives the following system of equations;

$$\begin{aligned} -(-1+S_0+S_1)&=0\\ (c_0-S_1-\varpi_1)&=0\\ \frac{1}{2}(c_0^2-S_1-2\varpi_1)&=0 \end{aligned} \tag{13}$$

Solving the arising system of equations in (13) for the unknown values of S_0, S_1, ω_1, gives;

$$\begin{aligned} S_0&=1-2c_0+c_0^2\\ S_1&=2c_0-c_0^2\\ \varpi_1&-c_0+c_0^2 \end{aligned} \tag{14}$$

Substituting the value of $c_0 = 1/2$ yields;

$S_0 = 1/4$, $S_1 = 3/4$ *and* $\varpi_1 = -1/4$

while substituting the values into the input method Y_0 in (9) above gives;

$$Y_0=\frac{y_n}{4}+\frac{3}{4}y_{n+1}-\frac{h}{4}f_{n+1} \tag{15}$$

and expanding the output method y_{n+k} for $k=1$ using Taylor's series expansion;

$$y_{n+1}=I_0y_n+h\theta_0 f(Y_0)+h\lambda_1(f_{n+1}+af_n) \tag{16}$$

where

$$\begin{aligned} y_{n+1}&=y(x_n+h)\\ f(Y_0)&=y'(x_n+c_0h)\\ f_{n+1}&=y'(x_n+h)\\ f_n&=y'_{x_n} \end{aligned} \tag{17}$$

Substituting (17) into (16) and equating to zero, yields;

$$y(x_n+h)-I_0y_n-h\theta_0y'(x_n+c_0h)-h\lambda_1[y'(x_n+h)+ay'_{xn}]=0 \tag{18}$$

Expanding (18) using Taylor's series expansion about x_n, gives;

$$-(-1+I_0)y_{xn}-(-1+\theta_0+(1+a)\lambda_1)y'_{xn}h-\frac{1}{2}(-1+2c_0\theta_0+2\lambda_1)y''_{xn}h^2$$
$$-\frac{1}{6}(-1+3c_0^2I_0+3\lambda_1)y'''_{xn}h^3+...+=0 \qquad (19)$$

Collecting the co-efficient of the power of *h*, gives the following system of equations;

$$\begin{aligned}&-(-1+I_0)=0\\&-(-1+\theta_0+(1+a)\lambda_1)=0\\&=\frac{1}{2}(-1+2c_0\theta_0+2\lambda_1)=0\end{aligned} \qquad (20)$$

Solving the arising system of equations in (20) for the unknown values gives;

$$I_0=1, \quad \theta_0=\frac{1-a}{2(-1+c_0+ac_0)} \; and \quad \lambda_1=\frac{1-2c_0}{2(-1+c_0+ac_0)}$$

Substituting the value of $c_0=1/2$ and $a=1/2$, yields;

$$I_0=1,\ \theta_0=1\ and\ \lambda_1=0$$

Inserting the values above into the output method in (16), gives;

$$y_{n+1}=y_n+hf(Y_0) \qquad (21)$$

Bringing the input method (15) and the output method (21) together for $k = 1$, yeilds

$$\begin{aligned}Y_0&=\frac{y_n}{4}+\frac{3}{4}y_{n+1}-\frac{h}{4}f_{n+1}\\y_{n+1}&=y_n+hf(Y_0)\end{aligned} \qquad (22)$$

Continuing in the same manner, we obtain the methods for $k = 2, 3, 4$ *and* 5

Stability Analysis of the Methods

This work tends to establish the *A(*α*)-stability* properties of the formula in (4) which depends on the choice of values of the parameter "*a*". The *A(*α*)-stability* properties of extended stable parameter dependent nested linear multistep methods with off-step points was analyzed by applying the method (4) on step numbers $k_{i's}\ \forall\ i=1(1)5$ to the standard scalar test equation $y'=\lambda y,\quad \mathrm{Re}(\lambda)<0$ (23)

This gives the general stability polynomial of the algorithms in (4) as

$$\pi_1(w,z)=w^{k-i}-\sum_{j}^{k-2}\partial_j w^j-z\theta_{s-1}f(R_1(Y_{s-1})-z\gamma_1(w^k+aw^{k-1})=0,\quad 0\le c\le k \qquad (24)$$

where

$$Y_0=\sum_{j=0}^{k}\phi_j w^j+z\beta_k w^k \qquad (24a)$$

$$Y_{i+1} = w^{k-1} + \sum_{j=0}^{k} \varphi_j w^j + z\rho_i f(Y_i) \ . \tag{24b}$$

Applying the method for $k = 1$, $s = 1$, $p = q + 1 = 3$, $c_0 = \frac{1}{2}$ and $a = \frac{1}{2}$ to the test equation in (23) gives the stability polynomial

$$\pi_1(w,z) = w - 1 - z\theta_0(R_1(w,z)) - z\gamma_1(w + a) = 0, \quad z = \lambda h, \tag{25}$$

where

$$R_1(w,z) = \sum_{j=0}^{k} \hat{\alpha}_j w^j + z\beta_1 w \tag{25a}$$

setting $z = 0$ in (25) gives the stability polynomial for $k = 1$

$$\pi_1(w) = w - 1 \tag{26}$$

Again, applying the method for $k = 2$, $s = 2$, $p = q_1 = q_0 + 1 = 4$, $c_0 = \frac{7}{4}$ and $c_1 = \frac{3}{2}$, $a = \frac{1}{4}$ to the test equation in (23) gives the stability polynomial

$$\pi_2(w,z) = w^2 - w - z\sum_{j=0}^{0} \delta_j w^j - z\theta_1(R_2(w,z)) - z\gamma_2(w^2 + aw) = 0, \quad z = \lambda h \tag{27}$$

where

$$R_1(w,z) = \sum_{j=0}^{2} \phi_j w^j + z\lambda_2 w^2 \tag{27a}$$

$$R_2(w,z) = (w + z\sum_{j=0}^{2} \varphi_j w^j + z\rho_0(R_1(w,z)) \tag{27b}$$

setting $z = 0$ in (27) gives the stability polynomial for $k = 2$.

$$\pi_2(w) = \frac{1}{23} - \frac{24w}{23} + w^2 \ . \tag{28}$$

For $k = 3$, $s = 3$ and $p = q_2 = q_1 = q_0 + 1 = 5$, $c_0 = \frac{23}{8}$, $c_1 = \frac{11}{4}$, and $c_2 = \frac{5}{2}$, $a = \frac{1}{2}$, we have a method of order five and applying it to the test equation in (23) gives the stability polynomial

$$\pi_3(w,z) = w^3 - w^2 - z\sum_{j=0}^{1} \delta_j w^j - z\theta_2(R_3(w,z)) - z\gamma_3(w^3 + aw^2) = 0, \quad z = \lambda h \tag{29}$$

where

$$R_1(w,z) = \sum_{j=o}^{3} \phi_j w^j + z\lambda_3 w^3 \tag{29a}$$

$$R_2(w,z) = w^2 + z\sum_{j=0}^{3} \varphi_j w^j + z\rho_0(\sum_{j=o}^{3} \phi_j w^j + z\lambda_3 w^3) \tag{29b}$$

$$R_3(w,z) = (w^2 + z\sum_{j=0}^{3}\varphi_j w^j + z\rho_1(R_2(w,z)(R_1(w,z)))) \tag{29c}$$

setting $z=0$ in (29), gives the stability polynomial for $k = 3$.

$$\pi_3(w) = -\frac{1}{163} + \frac{9w}{163} - \frac{171w^2}{163} + w^3. \quad (30)$$

For $k=4, s=4$, $p = q_3 = q_2 = q_1 = q_0 + 1 = 6$, $c_0 = \frac{63}{16}$, $c_1 = \frac{23}{8}$, $c_2 = \frac{15}{4}$, and $c_3 = \frac{7}{2}$, $a = \frac{1}{2}$, gives the method of order six and applying it to the test equation in (23) gives the stability polynomial

$$\pi_4(w,z) = w^4 - w^3 - z\sum_{j=0}^{2}\delta_j w^j - z\theta_3(R_4(w,z)) + z\gamma_4(w^4 + aw^3) = 0, \quad z = \lambda h \tag{31}$$

where

$$R_1(w,z) = \sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4 \tag{31a}$$

$$R_2(w,z) = w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_0(\sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4) \tag{31b}$$

$$R_3(w,z) = w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_1(w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_0(\sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4)) \tag{31c}$$

$$R_4(w,z) = (w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_2(R_3(w,z)(R_2(w,z)(R_1(w,z)))) \tag{31d}$$

setting $z=0$ in (31) gives the stability polynomial for $k = 4$.

$$\pi_4(w) = \frac{19}{11155} - \frac{166w}{11155} + \frac{162w^2}{2231} - \frac{11818w^3}{11155} + w^4 \tag{32}$$

For $k=5$, $s=5$, $p = q_4 = q_3 = q_2 = q_1 = q_0 + 1 = 7$, $c_2 = \frac{39}{8}$, $c_3 = \frac{19}{4}$, and $c_4 = \frac{9}{2}$, $a = \frac{1}{2}$, gives the method of order seven and applying it to the test equation in (23) gives the stability polynomial

$$\pi_5(w,z) = w^5 - w^4 - z\sum_{j=0}^{3}\delta_j w^j - z\theta_4(R_5(w,z)) + z\gamma_5(w^5 + aw^4) = 0, \quad z = \lambda h \tag{33}$$

where

$$R_1(w,z) = \sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5 \tag{33a}$$

$$R_2(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5) \tag{33b}$$

$$R_3(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_1(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5)) \tag{33c}$$

$$R_4(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_2(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_1(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5))) \quad (33d)$$

$$R_5(w,z) = (w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_3(R_4(w,z)(R_3(w,z)(R_2(w,z)(R_1(w,z)))))) \quad (33f)$$

setting $z = 0$ in (33) gives the stability polynomial for $k = 5$.

$$\pi_5(w) = -\frac{12}{24067} + \frac{115w}{24067} - \frac{550w^2}{24067} + \frac{2010w^3}{24067} - \frac{25630w^4}{24067} + w^5. \quad (34)$$

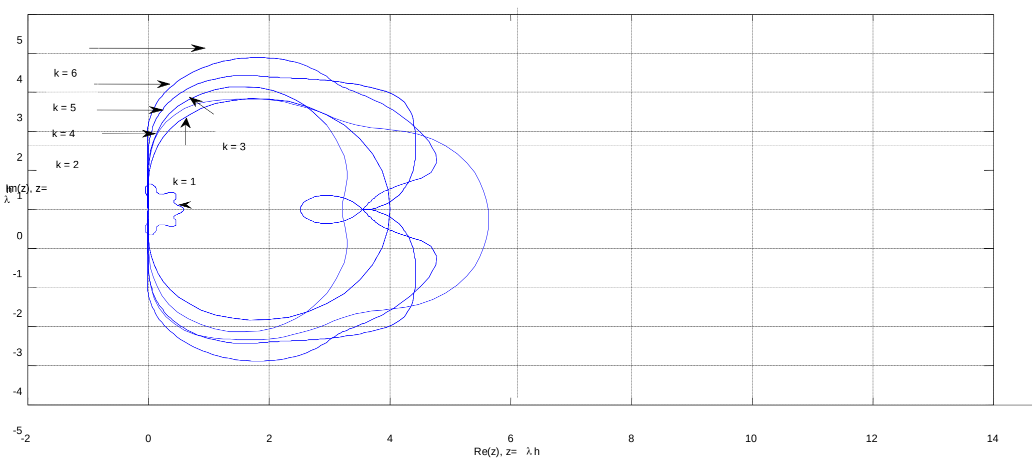

Fig 1: shows the boundary locus plot of the stability polynomial for $k = 1,2,3,4,5$.

k	a	Stability properties
1	½	A-Stable (-∞ , 0]
2	¼	A(α)-Stable (α = 89.9^0)
3	½	A(α)-Stable (α = 47.1^0)
4	½	A(α)-Stable (α = 82.5^0)
5	½	A(α)-Stable (α = 61.9^0)

Table 1: show the value of "a" for $k_{i's}, \forall\ i = 1(1)5$ and the stability properties of the method (4).

Numerical Experiments

In this section, we consider the application of the derived method Y_0 the LMMs and y_{n+k} its hybrid counterpart for *k=1*.

$$Y_0 = \frac{1}{4}(y_n + 3y_{n+1}) - \frac{h}{4}f_{n+1}; \qquad q_0 = 2, \quad C_3 = \frac{1}{48},$$
$$y_{n+1} = y_n + hf(Y_0);\ \ p = 2, \quad C_3 = \frac{1}{24}. \quad (35)$$

This method for *k=1* is of order 3, and also independently derived using Taylor's series expansion on the two problems below;

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Problem 1: A 2x2matrixnonlinear chemical problem (Olajunji and Ikhile 2018)

$$y' = \begin{bmatrix} -0.1 & -199.9 \\ -200 & 0 \end{bmatrix} y,\ y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ h = 0.0001,\ x = [0,\ 1],\quad y = [y_1,\ y_2]^T$$

$$Exact\ solution: y_1(x) = \exp^{-0.1x} + \exp^{-200x}\ \ and\ \ y_2(x) = \exp^{-200x}$$

Problem 2: [13] Root Mean Square Runge-Kutta (RMS-RK-4)

$y' = y^2 - y,\ \ y(0) = 2,\ x \in [0,1]$ and $h = 0.1$

In solving the IVPs above, the implicitness in the methods has to be resolved using the Newton Raphson iterative scheme below as suggested by (Kulikov 2015) and (Ebhohimen and Oluwayemi 2023).

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - (F'(y_{n+k}^{[s]}))^{-1} F(y_{n+k}^{[s]}), \quad s = 0,1,\ldots \tag{36}$$

where $F'(y_{n+k}^{[s]})$ is the Jacobean matrix? While an explicit one-step method

$$y_{n+1}^{[0]} = y_n + \frac{h}{2}(f_{n-1} + f_n). \tag{37}$$

is used to generate the starting value for the schemes.

x	EPDNLMM y_n	SDLMM y_n	Exact Solution $y_{(x)}$	EPDNLMM Error = $\lvert y_{(x)} - y_n \rvert$	SDLMM Error = $\lvert y_{(x)} - y_n \rvert$
0.2	$4.2772760718e \times 10^{-18}$	4.3131677052	$4.3341767041e \times 10^{-18}$	$5.6900632264e \times 10^{-20}$	$1.4858821195e \times 10^{-7}$
0.4	$1.8295146962e \times 10^{-35}$	1.8391145693	$1.8413118044e \times 10^{-35}$	$1.1797108180e \times 10^{-37}$	$1.6070167507e \times 10^{-7}$
0.6	$7.8253635434e \times 10^{-53}$	7.8226341188	$7.8225448396e \times 10^{-53}$	$2.8187037014e \times 10^{-56}$	$1.5004672493e \times 10^{-7}$
0.8	$3.3471343363e \times 10^{-70}$	3.3331973634	$3.3232941657e \times 10^{-70}$	$2.3840170623e \times 10^{-72}$	$1.3658646103e \times 10^{-7}$
1.0	$2.0681544656e \times 10^{-9}$	2.0931756254	$2.1027916876e \times 10^{-10}$	$3.4637222005e \times 10^{-11}$	$1.2373236302e \times 10^{-7}$

Table 2: The Numerical Solution of stiffly stable **EPDNLMM** Vs **SDLMM** (Esuabana and Ekoro 2017)

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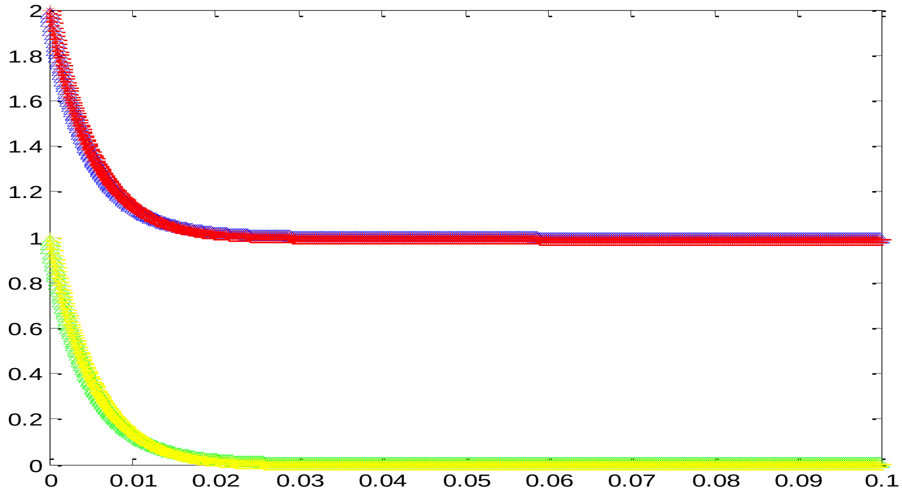

Fig. 2: :Graph of the stiffly stable EPDNLMM for problem 1 using MATHEMATICA.

N	TSOL	$y(x_n)$	Error in **RMS-RK4** **Error =** $\lvert y_{(x)} - y_n \rvert$	**EPDNLMM** **Error =** $\lvert y_{(x)} - y_n \rvert$
0.1	2.2351E+00	2.2351E+00	1.4016E-05	3.40352e-05
0.2	2.5687E+00	2.5688E+00	5.6286E-05	3.08843e-05
0.3	3.0763E+00	3.0765E+00	1.9805E-04	3.76084e-05
0.4	3.9356E+00	3.9364E+00	7.8005E-04	2.49631e-05
0.5	5.6935E+00	5.6976E+00	4.0994E-03	2.49509e-04
0.6	1.1243E+01	1.1268E+01	2.4043E-02	2.91428e-04
0.7	-1.4543E+02	1.4498E+02	2.9041E+02	2.5608e-02
0.8	-8.8676E+00	4.8510E+15	4.8510E+15	2.2939e-01
0.9	-4.3516E+00	0.4406E+232	0.4406E+232	3.3641e+10
1.0	-2.7844E+00	NAN	NAN	NAN

Table 3: The Numerical Solution of stiffly stable **EPDNLMM** Vs **RMS-RK4** (Agbeboh and Esekhaigbe 2016)

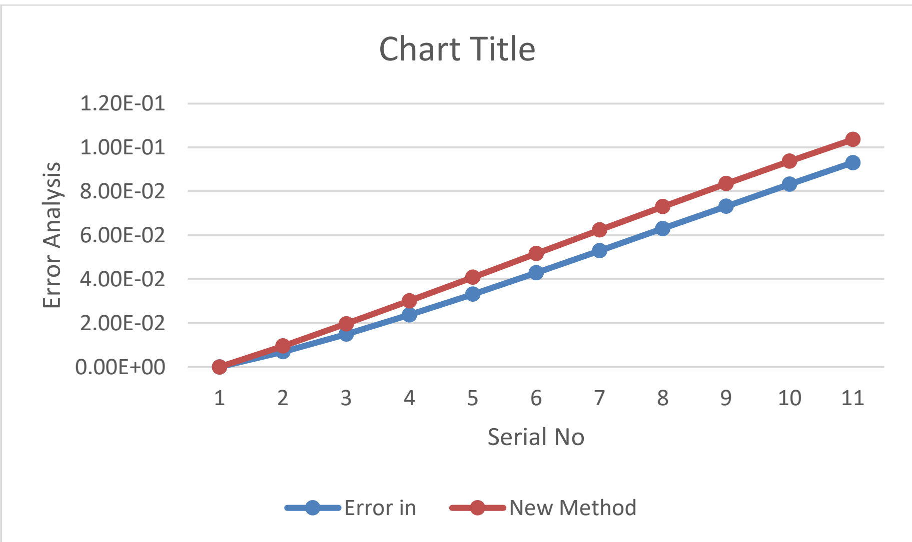

Fig 3: Graph of the stiffly stable EPDNLMM for problem 2 using MATLAB.

DISCUSSION OF RESULTS AND CONCLUSIONS

Table 1: shows the values of the parameters "a" and their corresponding stability properties for $k \leq 5$. This method is relying on the chosen values of the parameter "a" and $a \in [0,1]$ for better stability properties of the method.

Table 2: The table show the results of our stiffly stable extended PDNLMM on problem I solved as compared with the SDLMM with step-size *h= 0.0001* for the problem for $x\varepsilon[0,1]$. It reveals that the new method on the problem solved have a very low error level as compared to the SDLMM. This shows that the new method is accurate, effective and efficient in solving stiff IVPs in ODEs.

Table 3: The table above shows the performance of our stiffly stable extended PDNLMM on problem 2 solved as compared with the **RMS-RK4** with step-size $h = 0.1$ for $x\varepsilon[0,1]$. It shows that the stiffly stable extended PDNLMM compared well with the Implicit 4th Order RKM with good rate of convergence.

Figure 1: Shows the various boundary locus plots of the stability polynomial of the extended stiffly stable extended parameter dependent nested linear multistep methods with off-step points" for step number $1 \leq k \leq 5$ and clearly shows that our methods is *A-stable* and *A(α)- stable*.

Figure 2: shows the Graphical Representation of the new method (Red and Yellow Represent Numerical Solutions while Blue and Green represents Exact Solutions). Also,

Figure 3**:** represented in the graph (Red stands for the Numerical solution while Blue for the Exact solution) shows the consistency and convergence of the stiffly stable extended PDNLMMs. Unlike the Euler and RKM, the consistency and convergence of our method can be ascertained during the implementation process. A linear multistep is consistence if the numerical solution converges to that of the exact solution and also, if the method is able to solve exactly two or more ODE problems with different initial conditions. A linear multistep is convergent if and only if it is stable and consistent. A careful look at the graphs of our methods shows that our methods are consistence and convergence as the numerical solutions converges to that of the exact solutions. This is a proof that our new methods compared favorably well with other existing numerical methods in (Olatunji and Ikhile 20218), (Esuabana and Ekoro 2017) and (Muideen 2024).

Finally, it has been established from Figure 1 above that the stability properties of method (4) is stiffly stable as it is found to be A-stable and A(α)-stable in the region of absolute stability. Hence the "extended stiffly stable parameter dependent nested linear multistep methods with off-step points" is recommended for all ODE solvers as an instrument for determining the stability of real-life problems. The method is therefore suitable for solving stiff initial value problems.

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