

Anti (T, S) –Fuzzy Ideals of a Hemiring

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ABSTRACT

In this paper, we made an attempt to study the algebraic nature of an anti (T, S) –fuzzy ideals of a hemiring and its properties.

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INTRODUCTION

In universal algebra, hemirings generalize classical rings by relaxing the requirement of additive inverses. A hemiring $(R, +, \cdot)$ consists of a commutative additive semigroup and a multiplicative semigroup satisfying the distributive laws, with a zero element that is absorbing under multiplication. The introduction of fuzzy set theory by Lotfi A. Zadeh provided a framework for handling uncertainty, which was subsequently extended to algebraic structures through the study of fuzzy ideals in hemirings. Further generalizations using triangular norms (T) and triangular conorms (S) led to the development of (T, S) –fuzzy structures. Anti-fuzzy concepts, which emphasize non-membership behaviour, provide another important extension of fuzzy ideal theory. Motivated by these developments, this paper studies anti (T, S) –fuzzy ideals of hemirings and establishes their fundamental properties and related results.

PRELIMINARIES

1.1 Definition: A (T, S) –norm is a binary operations $T: [0,1] \times [0,1] \rightarrow [0,1]$ and $S: [0,1] \times [0,1] \rightarrow [0,1]$ satisfying the following requirements:

- (i) $T(0, x) = 0, T(1, x) = x$ (boundary condition)
- (ii) $T(x, y) = T(y, x)$ (commutativity)
- (iii) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity)
- (iv) if $x \leq y$ and $w \leq z$, then $T(x, w) \leq T(y, z)$ (monotonicity)
- (v) $S(0, x) = x, S(1, x) = 1$ (boundary condition)
- (vi) $S(x, y) = S(y, x)$ (commutativity)
- (vii) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)
- (viii) if $x \leq y$ and $w \leq z$, then $S(x, w) \leq S(y, z)$ (monotonicity).

1.2 Definition: Let $(R, +, \cdot)$ be a hemiring. A fuzzy subset A of R is said to be an anti (T, S) –fuzzy ideal (anti fuzzy ideal with respect to (T, S) –norm) of R if it satisfies the following conditions:

- (i) $\mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$,
- (ii) $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$, for all x and y in R .

1.3 Definition: Let A and B be fuzzy subsets of sets G and H , respectively. The anti-product of A and B , denoted by $A \times B$, is defined as $A \times B = \{(x, y), \mu_{A \times B}(x, y)\} | \text{ for all } x \in G \text{ and } y \in H\}$, where $\mu_{A \times B}(x, y) = \max\{\mu_A(x), \mu_B(y)\}$.

1.4 Definition: Let A be a fuzzy subset in a set S , the anti-strongest fuzzy relation on S , that is a fuzzy relation on A is V given by $\mu_V(x, y) = \max\{\mu_A(x), \mu_A(y)\}$, for all x and y in S .

1.5 Definition: Let R and R' be any two hemirings. Let $f: R \rightarrow R'$ be any function and A be an anti (T, S) –fuzzy ideal in R , V be an anti (T, S) –fuzzy ideal in $f(R) = R'$, defined by $\mu_V(y) = \inf_{x \in f^{-1}(y)} \mu_A(x)$, for all $x \in R, y \in R'$. Then A is called a preimage of V under f and is denoted by $f^1(V)$.

1.6 Definition: Let A be a fuzzy subset of X . For α in $[0, 1]$, the lower level subset of A is the set $A_\alpha = \{x \in X | \mu_A(x) \leq \alpha\}$

PROPERTIES

2.1 Theorem: Let $(R, +, \cdot)$ be a hemiring. If A and B are two anti (T, S) –fuzzy ideals of R . Then $A \cup B$ is an anti (T, S) –fuzzy ideal of R .

Proof: Let $x, y \in R$. Let $A = \{(x, \mu_A(x)) | x \in R\}$ and $B = \{(x, \mu_B(x)) | x \in R\}$ be anti (T, S) –fuzzy ideals of a hemiring R . Let $C = A \cup B$ and $C = \{(x, \mu_C(x)) | x \in R\}$, where $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$. Now, $\mu_C(x + y) = \max\{\mu_A(x + y), \mu_B(x + y)\} \leq \max\{S(\mu_A(x), \mu_A(y)), S(\mu_B(x), \mu_B(y))\} \leq S(\mu_C(x), \mu_C(y))$. Therefore, $\mu_C(x + y) \leq S(\mu_C(x), \mu_C(y))$, for all x and $y \in R$. And, $\mu_C(xy) = \max\{\mu_A(xy), \mu_B(xy)\} \leq \max\{T(\mu_A(x), \mu_A(y)), T(\mu_B(x), \mu_B(y))\} \leq T(\mu_C(x), \mu_C(y))$. Therefore, $\mu_C(xy) \leq T(\mu_C(x), \mu_C(y))$, for all x and $y \in R$. Therefore C is an anti (T, S) –fuzzy ideal of a hemiring R . Hence $A \cup B$ is an anti (T, S) –fuzzy ideal of the hemiring R .

2.2 Theorem: Let $(R, +, \cdot)$ be a hemiring. The union of a family of anti (T, S) –fuzzy ideals of R is an anti (T, S) –fuzzy ideal of R .

Proof: The argument is trivial.

2.3 Theorem: Let A and B be anti (T, S) –fuzzy ideals of the hemirings G and H , respectively. If A and B are anti (T, S) –fuzzy ideals, then $A \times B$ is an anti (T, S) –fuzzy ideal of $G \times H$.

Proof: Let A and B be anti (T, S) –fuzzy ideals of the hemirings G and H respectively. Let x_1, x_2 be in G and y_1, y_2 be in H . Then (x_1, y_1) and (x_2, y_2) are in $G \times H$. Now, $\mu_{A \times B}[(x_1, y_1) + (x_2, y_2)] = \mu_{A \times B}(x_1 + x_2, y_1 + y_2) = \max\{\mu_A(x_1 + x_2), \mu_B(y_1 + y_2)\} \leq \max\{S(\mu_A(x_1), \mu_A(x_2)), S(\mu_B(y_1), \mu_B(y_2))\} \leq S(\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2))$. Therefore, $\mu_{A \times B}[(x_1, y_1) + (x_1, y_2)] \leq S(\mu_{A \times B}(x_1, y_1), \mu_{A \times B}(x_2, y_2))$. Also, $\mu_{A \times B}[(x_1, y_1)(x_2, y_2)] = \mu_{A \times B}(x_1 x_2, y_1 y_2) = \max\{T(\mu_A(x_1 x_2), \mu_B(y_1 y_2))\} = \max\{\mu_A(x_2 x_1), \mu_B(y_2 y_1)\} = \mu_{A \times B}(x_2 x_1, y_2 y_1) = \mu_{A \times B}[(x_2, y_2)(x_1, y_1)]$. Therefore, $\mu_{A \times B}[(x_1, y_1)(x_2, y_2)] \leq \mu_{A \times B}[(x_2, y_2)(x_1, y_1)]$. Hence $A \times B$ is an anti (T, S) –fuzzy ideal of $G \times H$.

2.4 Theorem: If A is an anti (T, S) –fuzzy subhemiring of a hemiring $(R, +, \cdot)$, then $\mu_A(x) \geq \mu_A(0)$, for $x \in R$, the zero $0 \in R$.

Proof: For $x \in R$, and 0 is the zero element of R . Now, $\mu_A(x) = \mu_A(x + 0) \leq S(\mu_A(x), \mu_A(0))$, for all $x \in R$. So, $\mu_A(x) \geq \mu_A(0)$ is only possible.

2.5 Theorem: Let A and B be anti (T, S) –fuzzy ideal of the hemirings R_1 and R_2 respectively. Suppose that 0_1 and 0_2 are the zero element of R_1 and R_2 respectively. If $A \times B$ is an anti (T, S) –fuzzy ideal of $R_1 \times R_2$, then at least one of the following two statements must hold.

(i) $B(0_2) \leq A(x)$, for all x in R_1 ,

(ii) $A(0_1) \leq B(y)$, for all y in R_2 .

Proof: Let $A \times B$ be an anti (T, S) –fuzzy ideal of $R_1 \times R_2$. By contraposition, suppose that none of the statements (i) and (ii) holds. Then we can find a in R_1 and b in R_2 such that $A(a) < B(0_2)$ and $B(b) < A(0_1)$. We have, $A \times B(a, b) = \max\{A(a), B(b)\} < \max\{B(0_2), A(0_1)\} = \max\{A(0_1), B(0_2)\} = A \times B(0_1, 0_2)$. Thus $A \times B$ is not an anti (T, S) –fuzzy ideal of $R_1 \times R_2$. Hence either $B(0_2) \leq A(x)$, for all x in R_1 or $A(0_1) \leq B(y)$, for all $y \in R_2$.

2.6 Theorem: Let A and B be two fuzzy subsets of the hemirings R_1 and R_2 respectively and $A \times B$ is an anti (T, S) –fuzzy ideal of $R_1 \times R_2$. Then the following are true:

(i) if $A(x) \geq B(0_2)$, then A is an anti (T, S) –fuzzy ideal of R_1 .

(ii) if $B(x) \geq A(0_1)$, then B is an anti (T, S) –fuzzy ideal of R_2 .

(iii) either A is an anti (T, S) –fuzzy ideal of R_1 or B is an anti (T, S) –fuzzy ideal of R_2 .

Proof: Let $A \times B$ be an anti (T, S) –fuzzy ideal of $R_1 \times R_2$ and $x, y \in R_1$ and $0_2 \in R_2$. Then $(x, 0_2)$ and $(y, 0_2) \in R_1 \times R_2$. Now, using the property that $A(x) \geq B(0_2)$, for all $x \in R_1$. We get, $A(x + y) = \max\{A(x + y), B(0_2 + 0_2)\} = A \times B((x + y), (0_2 + 0_2)) = A \times B[(x, 0_2) + (y, 0_2)] \leq S(A \times B(x, 0_2), A \times B(y, 0_2)) = S(\max\{A(x), B(0_2)\}, \max\{A(y), B(0_2)\}) = S(A(x), A(y))$. Therefore, $A(x + y) \leq S(A(x), A(y))$, for all x and $y \in R_1$. Also, $A(xy) = \max\{A(xy), B(0_2 0_2)\} = A \times B((xy), (0_2 0_2)) = A \times B[(x, 0_2)(y, 0_2)] \leq T(A \times B(x, 0_2), A \times B(y, 0_2)) = T(\max\{A(x), B(0_2)\}, \max\{A(y), B(0_2)\}) = T(A(x), A(y))$. Therefore, $A(xy) \leq T(A(x), A(y))$, for all x and $y \in R_1$. Hence A is an anti (T, S) –fuzzy ideal of R_1 . Thus (i) is proved. Now, $B(x) \geq A(0_2)$, for all $x \in R_2$, let x and y in R_2 and $0_1 \in R_1$. Then $(0_1, x)$ and $(0_1, y) \in R_1 \times R_2$. We get, $B(x + y) = \max\{B(x + y), A(0_1 + 0_1)\} = \max\{A(0_1 + 0_1), B(x + y)\} = A \times B((0_1 + 0_1), (x + y)) = A \times B[(0_1, x) + (0_1, y)] \leq S(A \times B(0_1, x), A \times B(0_1, y)) = S(\max\{A(0_1), B(x)\}, \max\{A(0_1), B(y)\}) = S(B(x), B(y))$. Therefore, $B(x + y) \leq S(B(x), B(y))$, for all x and $y \in R_2$. Also, $B(xy) = \max\{B(xy), A(0_1 0_1)\} = \max\{A(0_1 0_1), B(xy)\} = A \times B((0_1 0_1), (xy)) = A \times B[(0_1, x)(0_1, y)] \leq T(A \times B(0_1, x), A \times B(0_1, y)) = T(\max\{A(0_1), B(x)\}, \max\{A(0_1), B(y)\}) = T(B(x), B(y))$. Therefore, $B(xy) \leq T(B(x), B(y))$, for all x and $y \in R_2$. Hence B is an anti (T, S) –fuzzy ideal of a hemiring R_2 . Thus (ii) is proved. (iii) is clear.

2.7 Theorem: Let A be a fuzzy subset of a hemiring R and V be the anti-strongest fuzzy relation on R . Then A is an anti (T, S) –fuzzy ideal of R if and only if V is an anti (T, S) –fuzzy ideal of $R \times R$.

Proof: Suppose that A is an anti (T, S) –fuzzy ideal of a hemiring R . Then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R \times R$. We have, $\mu_V(x + y) = \mu_V[(x_1, x_2) + (y_1, y_2)] = \mu_V(x_1 + y_1, x_2 + y_2) = \max\{\mu_A(x_1 + y_1), \mu_A(x_2 + y_2)\} \leq \max\{S(\mu_A(x_1), \mu_A(y_1)), S(\mu_A(x_2), \mu_A(y_2))\} \leq S(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}) = S(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) = S(\mu_V(x), \mu_V(y))$. Therefore, $\mu_V(x + y) \leq S(\mu_V(x), \mu_V(y))$, for all x and $y \in R \times R$. And, $\mu_V(xy) = \mu_V[(x_1, x_2)(y_1, y_2)] =$

$\mu_V(x_1y_1, x_2y_2) = \max\{\mu_A(x_1y_1), \mu_A(x_2y_2)\} \leq \max\{T(\mu_A(x_1), \mu_A(y_1)), T(\mu_A(x_2), \mu_A(y_2))\} \leq T(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\}) = T(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) = T(\mu_V(x), \mu_V(y))$.
Therefore, $\mu_V(xy) \leq T(\mu_V(x), \mu_V(y))$, for all x and $y \in R \times R$. This proves that V is an anti (T, S) –fuzzy ideal of $R \times R$. Conversely assume that V is an anti (T, S) –fuzzy ideal of $R \times R$, then for any $x = (x_1, x_2)$ and $y = (y_1, y_2) \in R \times R$, we have, $\max\{\mu_A(x_1 + y_1), \mu_A(x_2 + y_2)\} = \mu_V(x_1 + y_1, x_2 + y_2) = \mu_V[(x_1, x_2) + (y_1, y_2)] = \mu_V(x + y) \leq S(\mu_V(x), \mu_V(y)) = S(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) = S(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\})$. If $x_2 = 0, y_2 = 0$, we get, $\mu_A(x_1 + y_1) \leq S(\mu_A(x_1), \mu_A(y_1))$, for all x_1 and $y_1 \in R$. And, $\max\{\mu_A(x_1y_1), \mu_A(x_2y_2)\} = \mu_V(x_1y_1, x_2y_2) = \mu_V[(x_1, x_2)(y_1, y_2)] = \mu_V(xy) \leq T(\mu_V(x), \mu_V(y)) = T(\mu_V(x_1, x_2), \mu_V(y_1, y_2)) = T(\max\{\mu_A(x_1), \mu_A(x_2)\}, \max\{\mu_A(y_1), \mu_A(y_2)\})$. If $x_2 = 0, y_2 = 0$, we get $\mu_A(x_1y_1) \leq T(\mu_A(x_1), \mu_A(y_1))$, for all x_1 and $y_1 \in R$. Therefore A is an anti (T, S) –fuzzy ideal of R .

2.8 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring $(R, +, \cdot)$. If $\mu_A(x + y) = 1$, then either $\mu_A(x) = 1$ or $\mu_A(y) = 1$, for all x and $y \in R$.

Proof: The argument is trivial.

In the following theorems $\circ$ is the composition operation of functions:

2.9 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring H and f be an isomorphism from a hemiring R onto H . Then $A \circ f$ is an anti (T, S) –fuzzy ideal of the hemiring R .

Proof: Let x, y in R and A be an anti (T, S) –fuzzy ideal of a hemiring H . Then we have, $(\mu_{A \circ f})(x + y) = \mu_A(f(x + y)) = \mu_A(f(x) + f(y)) \leq S(\mu_A(f(x)), \mu_A(f(y))) \leq S((\mu_{A \circ f})(x), (\mu_{A \circ f})(y))$. And, $\mu_{A \circ f}(xy) = \mu_A(f(xy)) = \mu_A(f(x)f(y)) \leq T(\mu_A(f(x)), \mu_A(f(y))) \leq T((\mu_{A \circ f})(x), (\mu_{A \circ f})(y))$. Then, clearly $A \circ f$ is an anti (T, S) –fuzzy ideal of a hemiring R .

2.10 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring H and f be an anti-isomorphism from a hemiring R onto H . Then $A \circ f$ is an anti (T, S) –fuzzy ideal of the hemiring R .

Proof: Let x, y in R and A be an anti (T, S) –fuzzy ideal of a hemiring H . Then we have, $\mu_{A \circ f}(x + y) = \mu_A(f(x + y)) = \mu_A(f(y) + f(x)) \leq S(\mu_A(f(x)), \mu_A(f(y))) \leq S((\mu_{A \circ f})(x), (\mu_{A \circ f})(y))$. Now, $\mu_{A \circ f}(xy) = \mu_A(f(xy)) = \mu_A(f(y)f(x)) \leq T(\mu_A(f(x)), \mu_A(f(y))) \leq T((\mu_{A \circ f})(x), (\mu_{A \circ f})(y))$. Then, clearly $A \circ f$ is anti (T, S) –fuzzy ideal of the hemiring R .

2.11 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring $(R, +, \cdot)$. Then the pseudo anti (T, S) –fuzzy coset $(aA)^P$ is an anti (T, S) –fuzzy ideal of a hemiring R , for every a in R .

Proof: Let A be an anti (T, S) –fuzzy ideal of a hemiring R . For every x and $y \in R$, we have, $((a\mu_A)^P)(x + y) = p(a)\mu_A(x + y) \leq p(a)S(\mu_A(x), \mu_A(y)) = S(p(a)\mu_A(x), p(a)\mu_A(y)) = S(((a\mu_A)^P)(x), ((a\mu_A)^P)(y))$. Therefore, $((a\mu_A)^P)(x + y) \leq S(((a\mu_A)^P)(x), ((a\mu_A)^P)(y))$. Now, $((a\mu_A)^P)(xy) = p(a)\mu_A(xy) \leq p(a)T(\mu_A(x), \mu_A(y)) = T(p(a)\mu_A(x), p(a)\mu_A(y)) = T(((a\mu_A)^P)(x), ((a\mu_A)^P)(y))$. Therefore, $((a\mu_A)^P)(xy) \leq T(((a\mu_A)^P)(x), ((a\mu_A)^P)(y))$. Hence $(aA)^P$ is an anti (T, S) –fuzzy ideal of a hemiring R .

2.12 Theorem: The homomorphic image of an anti (T, S) –fuzzy ideal of a hemiring R is an anti (T, S) –fuzzy ideal of a hemiring R' .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Let $f: R \rightarrow R'$ be a homomorphism. Then, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where A is an anti

(T, S) –fuzzy ideal of R . We have to prove that V is an anti (T, S) –fuzzy ideal of R' . Now, for $(x), f(y) \in R'$, $\mu_V(f(x) + f(y)) = \mu_V(f(x + y)) \leq \mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$, which implies that $\mu_V(f(x) + f(y)) \leq S(\mu_V(f(x)), \mu_V(f(y)))$. Again, $\mu_V(f(x)f(y)) = \mu_V(f(xy)) \leq \mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$, which implies that $\mu_V(f(x)f(y)) \leq T(\mu_V(f(x)), \mu_V(f(y)))$. Hence V is an anti (T, S) –fuzzy ideal of R' .

2.13 Theorem: The homomorphic pre-image of an anti (T, S) –fuzzy ideal of a hemiring R' is an anti (T, S) –fuzzy ideal of a hemiring R .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Let $f: R \rightarrow R'$ be a homomorphism. Then, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where V is an anti (T, S) –fuzzy ideal of R' . We have to prove that A is an anti (T, S) –fuzzy ideal of R . Let x and $y \in R$. Then, $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(x) + f(y)) \leq S(\mu_V(f(x)), \mu_V(f(y))) = S(\mu_A(x), \mu_A(y))$, which implies that $\mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$. Again, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(x)f(y)) \leq T(\mu_V(f(x)), \mu_V(f(y))) = T(\mu_A(x), \mu_A(y))$, which implies that $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$. Hence A is an anti (T, S) –fuzzy ideal of R .

2.14 Theorem: The anti-homomorphic image of an anti (T, S) –fuzzy ideal of a hemiring R is an anti (T, S) –fuzzy ideal of a hemiring R' .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Let $f: R \rightarrow R'$ be an anti-homomorphism. Then, $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all x and y in R . Let $V = f(A)$, where A is an anti (T, S) –fuzzy ideal of R . We have to prove that V is an anti (T, S) –fuzzy ideal of R' . Now, for $f(x), f(y)$ in R' , $\mu_V(f(x) + f(y)) = \mu_V(f(y) + f(x)) \leq \mu_A(y + x) \leq S(\mu_A(y), \mu_A(x)) = S(\mu_A(x), \mu_A(y))$, which implies that $\mu_V(f(x) + f(y)) \leq S(\mu_V(f(x)), \mu_V(f(y)))$. Again, $\mu_V(f(x)f(y)) = \mu_V(f(yx)) \leq \mu_A(yx) \leq T(\mu_A(y), \mu_A(x)) = T(\mu_A(x), \mu_A(y))$, which implies that $\mu_V(f(x)f(y)) \leq T(\mu_V(f(x)), \mu_V(f(y)))$. Hence V is an anti (T, S) –fuzzy ideal of R' .

2.15 Theorem: The anti-homomorphic pre-image of an anti (T, S) –fuzzy ideal of a hemiring R' is an anti (T, S) –fuzzy ideal of a hemiring R .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings. Let $f: R \rightarrow R'$ be an anti-homomorphism. Then, $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all x and $y \in R$. Let $V = f(A)$, where V is an anti (T, S) –fuzzy ideal of R' . We have to prove that A is an anti (T, S) –fuzzy ideal of R . Let x and $y \in R$. Then, $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(y) + f(x)) \leq S(\mu_V(f(y)), \mu_V(f(x))) = S(\mu_V(f(x)), \mu_V(f(y))) = S(\mu_A(x), \mu_A(y))$, which implies that $\mu_A(x + y) \leq S(\mu_A(x), \mu_A(y))$. Again, $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(y)f(x)) \leq T(\mu_V(f(y)), \mu_V(f(x))) = T(\mu_V(f(x)), \mu_V(f(y))) = T(\mu_A(x), \mu_A(y))$, which implies that $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y))$. Hence A is an anti (T, S) –fuzzy ideal of R .

2.16 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R , A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all x in R , then A^+ is an anti (T, S) –fuzzy ideal of a hemiring R .

Proof: Let x and y in R . We have, $A^+(x + y) = A(x + y) + 1 - A(0) \leq S(A(x), A(y)) + 1 - A(0) \leq S((A(x) + 1 - A(0)), (A(y) + 1 - A(0))) = S(A^+(x), A^+(y))$. Therefore, $A^+(x + y) \leq S(A^+(x), A^+(y))$, for all $x, y \in R$. Similarly, $A^+(xy) = A(xy) + 1 - A(0) \leq T(A(x), A(y)) + 1 - A(0) \leq T((A(x) + 1 - A(0)), (A(y) + 1 - A(0))) = T(A^+(x), A^+(y))$. Therefore, $A^+(xy) \leq T(A^+(x), A^+(y))$, for all $x, y \in R$. Hence A^+ is an anti (T, S) –fuzzy ideal of a hemiring R .

2.17 Theorem : Let A be an anti (T, S) –fuzzy ideal of a hemiring R , A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$, then there exists $0 \in R$ such that $A(0) = 1$ if and only if $A^+(x) = A(x)$.

Proof: The argument is trivial.

2.18 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R , A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then there exists $x \in R$ such that $A^+(x) = 1$ if and only if $x = 0$.

Proof: The argument is trivial.

2.19 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R , A^+ be a fuzzy set in R defined by $A^+(x) = A(x) + 1 - A(0)$, for all $x \in R$. Then $(A^+)^+ = A^+$.

Proof: Let x and y in R . We have, $(A^+)^+(x) = A^+(x) + 1 - A^+(0) = \{A(x) + 1 - A(0)\} + 1 - \{A(0) + 1 - A(0)\} = A(x) + 1 - A(0) = A^+(x)$. Hence $(A^+)^+ = A^+$.

2.20 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R . Then A^0 is an anti (T, S) –fuzzy ideal of the hemiring R .

Proof: For any $x \in R$, we have $A^0(x + y) = A(x + y)A(0) \leq [A(0)]S(A(x), A(y)) \leq S([A(x)A(0)], [A(y)A(0)]) = S(A^0(x), A^0(y))$. That is, $A^0(x + y) \leq S(A^0(x), A^0(y))$, for all $x, y \in R$. Similarly, $A^0(xy) = A(xy)A(0) \leq [A(0)]T(A(x), A(y)) \leq T([A(x)A(0)], [A(y)A(0)]) = T(A^0(x), A^0(y))$. That is, $A^0(xy) \leq T(A^0(x), A^0(y))$, for all $x, y \in R$. Hence A^0 is an anti (T, S) –fuzzy ideal of the hemiring R .

2.21 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R . Then for α in $[0, 1]$ such that $\mu_A(0) \leq \alpha$, A_α is a lower level ideal of R .

Proof: For all x and y in A_α , we have, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$. Now, $\mu_A(x - y) \leq S(\mu_A(x), \mu_A(y)) \leq S(\alpha, \alpha) = \alpha$, which implies that $\mu_A(x - y) \leq \alpha$. And, $\mu_A(xy) \leq T(\mu_A(x), \mu_A(y)) \leq T(\alpha, \alpha) = \alpha$, which implies that $\mu_A(xy) \leq \alpha$. Therefore, $\mu_A(x - y) \leq \alpha$ and $\mu_A(xy) \leq \alpha$. Therefore, $x - y$ and xy in A_α . Hence A_α is a lower level ideal of a hemiring R .

2.22 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R . If any two lower level ideals of A belongs to R , then their intersection is also lower level ideal of $A \in R$.

Proof: Let $\alpha_1, \alpha_2 \in [0, 1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha_1} \subseteq A_{\alpha_2}$. Therefore, $A_{\alpha_1} \cap A_{\alpha_2} = A_{\alpha_1}$, but A_{α_1} is a lower level ideal of A .

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$. Therefore, $A_{\alpha_1} \cap A_{\alpha_2} = A_{\alpha_2}$, but A_{α_2} is a lower level ideal of A .

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha_1} = A_{\alpha_2}$.

Hence, intersection of any two lower level ideals is a lower level ideal of A .

2.23 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R . If $\alpha_i \in [0, 1]$ and A_{α_i} , $i \in I$ is a collection of lower level ideals of A , then their intersection is also a lower level ideal of A .

Proof: The argument is trivial.

2.24 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R . If any two lower level ideals of A belongs to R , then their union is also a lower level ideal of A in R .

Proof: Let $\alpha_1, \alpha_2 \in [0,1]$.

Case (i): If $\alpha_1 < \mu_A(x) < \alpha_2$, then $A_{\alpha_1} \subseteq A_{\alpha_2}$. Therefore, $A_{\alpha_1} \cup A_{\alpha_2} = A_{\alpha_2}$, but A_{α_2} is a lower level ideal of A .

Case (ii): If $\alpha_1 > \mu_A(x) > \alpha_2$, then $A_{\alpha_2} \subseteq A_{\alpha_1}$. Therefore, $A_{\alpha_1} \cup A_{\alpha_2} = A_{\alpha_1}$, but A_{α_1} is a lower level ideal of A .

Case (iii): If $\alpha_1 = \alpha_2$, then $A_{\alpha_1} = A_{\alpha_2}$.

Hence, union of any two lower level ideal is also a lower level ideal of A .

2.25 Theorem: Let A be an anti (T, S) –fuzzy ideal of a hemiring R . If $\alpha_i \in [0,1]$ and $A_{\alpha_i}, i \in I$ is a collection of lower level ideals of A , then their union is also a lower level ideal of A .

Proof: The argument is trivial.

2.26 Theorem: The homomorphic image of a lower level ideal of an anti (T, S) –fuzzy ideal of a hemiring R is a lower level ideal of an anti (T, S) –fuzzy ideal of a hemiring R' .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings and $f: R \rightarrow R'$ be a homomorphism. That is, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$, for all x and $y \in R$. Let $V = f(A)$, where A is an anti (T, S) –fuzzy ideal of a hemiring R . Clearly V is an anti (T, S) –fuzzy ideal of a hemiring R' . Let x and $y \in R$, implies $f(x)$ and $f(y) \in R'$. Let A_α is a lower level ideal of A . That is, $\mu_V(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$; $\mu_A(x + y) \leq \alpha$ and $\mu_A(xy) \leq \alpha$. We have to prove that $f(A_\alpha)$ is a lower level ideal of V . Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$ and $\mu_V(f(y)) \leq \mu_A(y) \leq \alpha$. Also, $\mu_V(f(x) + f(y)) = \mu_V(f(x + y)) \leq \mu_A(x + y) \leq \alpha$ and $\mu_V(f(x)f(y)) = \mu_V(f(xy)) \leq \mu_A(xy) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence $f(A_\alpha)$ is a lower level ideal of an anti (T, S) -fuzzy ideal V of R' .

2.27 Theorem: The homomorphic pre-image of a lower level ideal of an anti (T, S) –fuzzy ideal of a hemiring R' is a lower level ideal of an anti (T, S) –fuzzy ideal of a hemiring R .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings and $f: R \rightarrow R'$ be a homomorphism. That is, $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x and $y \in R$. Let $V = f(A)$, where V is an anti (T, S) –fuzzy ideal of a hemiring R' . Clearly A is an anti (T, S) –fuzzy ideal of a hemiring R . Let $f(x)$ and $f(y) \in R'$, implies x and $y \in R$. Let $f(A_\alpha)$ is a lower level ideal of V . That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. We have to prove that A_α is a lower level ideal of A . Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$ and $\mu_A(y) = \mu_V(f(y)) \leq \alpha$. Also, $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(x)f(y)) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence, A_α is a lower level ideal of an anti (T, S) –fuzzy ideal A of R .

2.28 Theorem: The anti homomorphic image of a lower level ideal of an anti (T, S) –fuzzy ideal of a hemiring R is a lower level ideal of an anti (T, S) -fuzzy ideal of a hemiring R' .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings and $f: R \rightarrow R'$ be an anti-homomorphism. That is, $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all x and y in R . Let $V = f(A)$, where A is an anti (T, S) –fuzzy ideal of R . Clearly V is an anti (T, S) –fuzzy ideal of R' . Let x and $y \in R$, implies $f(x)$ and $f(y) \in R'$. Let A_α is a lower level ideal of A . That is, $\mu_A(x) \leq \alpha$ and $\mu_A(y) \leq \alpha$; $\mu_A(y + x) \leq \alpha$ and $\mu_A(yx) \leq \alpha$. We have to prove that $f(A_\alpha)$ is a lower level ideal of V . Now, $\mu_V(f(x)) \leq \mu_A(x) \leq \alpha$ and $\mu_V(f(y)) \leq \mu_A(y) \leq \alpha$. Also, $\mu_V(f(x) + f(y)) = \mu_V(f(y + x)) \leq \mu_A(y + x) \leq \alpha$ and

$\mu_V(f(x)f(y)) = \mu_V(f(yx)) \leq \mu_A(yx) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence $f(A_\alpha)$ is a lower level ideal of an anti (T, S) –fuzzy ideal V of R' .

2.29 Theorem: The anti-homomorphic pre-image of a lower level ideal of an anti (T, S) –fuzzy ideal of a hemiring R' is a lower level ideal of an anti (T, S) –fuzzy ideal of a hemiring R .

Proof: Let $(R, +, \cdot)$ and $(R', +, \cdot)$ be any two hemirings and $f : R \rightarrow R'$ be an anti-homomorphism. That is, $f(x + y) = f(y) + f(x)$ and $f(xy) = f(y)f(x)$, for all x and y in R . Let $V = f(A)$, where V is an anti (T, S) –fuzzy ideal of a hemiring R' . Clearly A is an anti (T, S) –fuzzy ideal of a hemiring R . Let $f(x)$ and $f(y)$ in R' , implies x and y in R . Let $f(A_\alpha)$ is a lower level ideal of V . That is, $\mu_V(f(x)) \leq \alpha$ and $\mu_V(f(y)) \leq \alpha$; $\mu_V(f(y) + f(x)) \leq \alpha$ and $\mu_V(f(y)f(x)) \leq \alpha$. We have to prove that A_α is a lower level ideal of A . Now, $\mu_A(x) = \mu_V(f(x)) \leq \alpha$ and $\mu_A(y) = \mu_V(f(y)) \leq \alpha$. Also, $\mu_A(x + y) = \mu_V(f(x + y)) = \mu_V(f(y) + f(x)) \leq \alpha$ and $\mu_A(xy) = \mu_V(f(xy)) = \mu_V(f(y)f(x)) \leq \alpha$. Therefore, $\mu_V(f(x) + f(y)) \leq \alpha$ and $\mu_V(f(x)f(y)) \leq \alpha$. Hence A_α is a lower level ideal of an anti (T, S) –fuzzy ideal A of R .

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