

Dynamic Response of a Timoshenko Beam on an Elastic Foundation Subjected to Multiple Distributed Moving Loads

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ABSTRACT

This study develops and verifies a Galerkin formulation for the transient response of a simply supported Timoshenko beam on a Winkler foundation under two finite-length distributed loads. The model includes shear deformation, rotary inertia, a uniform tensile prestress, foundation stiffness, load spacing, travelling velocity, and constant or harmonic load amplitudes. Unlike a semi-infinite Heaviside step, each load is represented by the difference of two Heaviside functions, so that its physical footprint and entry and exit from the span are treated explicitly. Sine and cosine basis functions satisfying the simply supported conditions reduce the governing equations to independent two-degree-of-freedom modal systems. Laplace transformation and partial-fraction inversion give an explicit two-branch impulse-response solution; convolution with the projected moving loads then gives the final analytical displacement and rotation fields. A state-transition convolution scheme also evaluates the same retained-mode response, and the two forms agree to machine precision: both predict a peak midspan deflection of 2.9226547 mm at 2.400 s, with a maximum time-history difference of 3.27×10^{-13} mm. A systematic truncation study shows that 12 Galerkin modes differ from the 16-mode reference by only 0.023% in peak response and 0.012% in normalized root-mean-square error. An independent 100-element Timoshenko finite-element model gives 2.922 mm; the peak difference is 0.023%. Increasing the Winkler stiffness from 0 to 2.0 MN/m² reduces the peak response by 86.4%, while increasing tensile prestress from 0 to 120 MN produces a 42.9% reduction. Load spacing produces non-monotonic modal interference, and the velocity sweep gives a maximum dynamic amplification factor of 1.619 at 125 m/s for the selected parameters. Under undamped harmonic loading, excitation at the first flexural frequency produces the expected resonant growth during load passage. These verified results clarify the separate and interacting effects of foundation stiffness, tensile prestress, load magnitude, spacing, speed, and excitation frequency.

Keywords: Timoshenko beam; Winkler foundation; distributed moving load; Galerkin method; finite-element verification; dynamic amplification; resonance.

INTRODUCTION

Structural members in bridges, railway systems, crane girders, conveyor supports, and similar transport-related facilities are commonly subjected to loads that move over time. Their response differs from the corresponding static response because the travelling excitation interacts with the natural frequencies and mode shapes of the structure. The resulting vibration depends not only on load magnitude but also on velocity, footprint, spacing, excitation frequency, support condition, and the stiffness and inertia of the structural member [1–4]. A reliable

moving-load model must therefore retain the features that control this interaction and must be supported by convergence and independent verification.

Euler–Bernoulli theory is adequate for slender members at relatively low frequencies, but it neglects shear deformation and rotary inertia. Timoshenko beam theory retains both effects and is consequently more appropriate for deep beams, higher modes, and dynamic problems in which short wavelengths may contribute materially to the response [1, 2]. The foundation also changes the modal spectrum. The Winkler idealization represents the support as independent springs and remains useful when the principal concern is the first-order influence of local foundation stiffness [4, 5, 17]. Axial force provides a further stiffness mechanism: a tensile force increases lateral stiffness, whereas a compressive force decreases it and may approach a buckling condition.

Moving-load studies have progressed from single concentrated forces to moving masses, finite distributed loads, multi-span systems, and finite-element formulations [6–10]. Wang [6] examined Timoshenko beams under a moving force, while Esmailzadeh and Ghorashi [7] treated uniformly distributed moving masses. Xu et al. [8] developed a three-dimensional moving Timoshenko beam element, and Assie et al. [9] investigated thick perforated beams under moving loads. Related studies have considered variable prestress, non-prismatic geometry, variable velocity, damping, and more elaborate foundations [10–14, 18]. These works establish that shear deformation, support stiffness, prestress, and load inertia can change both resonance conditions and response amplitudes.

Two issues remain important for the present class of problems. First, several analytical formulations describe a “distributed” moving load with a single Heaviside function. That representation activates the entire part of the beam behind a moving front and does not describe a load with a finite physical footprint. Second, lengthy closed-form derivations are sometimes presented without demonstrating modal convergence or comparison with an independent solution. Such omissions make it difficult to determine whether an observed parameter trend is physical or is a consequence of truncation, inconsistent boundary conditions, or an implementation error.

This study addresses these issues for a simply supported, uniform Timoshenko beam on a Winkler foundation under two finite distributed moving loads. The specific contributions are: (i) a dimensionally consistent finite-footprint load model; (ii) a Galerkin reduction whose trial functions satisfy the stated boundary conditions; (iii) an explicit Laplace-transform and convolution solution for both displacement and rotation, with constant- and harmonic-load specializations; (iv) a direct demonstration that the analytical convolution and state-transition forms are algebraically and numerically equivalent for the retained modes; (v) explicit characterization of the result as an exact time solution of a truncated modal system, but an approximate spatial solution of the beam problem; (vi) a modal convergence study and an independent finite-element and Newmark time-integration verification; and (vii) quantitative assessment of foundation stiffness, tensile prestress, individual load magnitude, load spacing, travelling velocity, harmonic frequency, peak deflection, and dynamic amplification.

Mathematical Formulation

Beam model and assumptions

Consider a homogeneous prismatic beam of length L , cross-sectional area A , second moment of area I , Young’s modulus E , shear modulus G , mass density ρ , and shear-correction factor κ . The transverse displacement is $w(x, t)$ and the cross-section rotation is $\theta(x, t)$. The beam rests on a Winkler foundation of line stiffness k_w and carries a uniform tensile prestress $N_T \geq 0$. Small displacement, linear elasticity, perfect contact with the foundation, and prescribed moving forces are assumed. The inertia of the travelling bodies is therefore not included.

With positive N_T denoting tension, the coupled Timoshenko equations are

$$\rho A \frac{\partial^2 w}{\partial t^2} - \kappa GA \left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta}{\partial x} \right) - N_T \frac{\partial^2 w}{\partial x^2} + k_w w = q(x, t) \quad (1)$$

and

$$\rho I \frac{\partial^2 \theta}{\partial t^2} - EI \frac{\partial^2 \theta}{\partial x^2} + \kappa GA \left(\theta - \frac{\partial w}{\partial x} \right) = 0. \quad (2)$$

Every term in (1) has units of force per unit length, while every term in (2) has units of force. For a simply supported Timoshenko beam, zero transverse displacement and zero bending moment are imposed at both ends:

$$w(0, t) = w(L, t) = 0, \quad EI \frac{\partial \theta}{\partial x}(0, t) = EI \frac{\partial \theta}{\partial x}(L, t) = 0. \quad (3)$$

The beam is initially at rest:

$$w(x, 0) = \frac{\partial w}{\partial t}(x, 0) = \theta(x, 0) = \frac{\partial \theta}{\partial t}(x, 0) = 0. \quad (4)$$

These conditions are consistent with the sine–cosine functions used in the Galerkin expansion below. In particular, imposing both $w = 0$ and $\partial w / \partial x = 0$ would describe a clamped displacement field and would be inconsistent with a simply supported sine basis.

Finite distributed moving loads

The right edge of the j th load footprint is

$$\xi_j(t) = vt - (j - 1)d, \quad j = 1, 2, \quad (5)$$

where v is the common travelling velocity and d is the spacing between the corresponding edges of the two loads. If the footprint length and intensity are ℓ_j and p_j , respectively, the load distribution is

$$q(x, t) = \sum_{j=1}^2 p_j s(t) \chi_j(x, t), \quad (6)$$

with

$$\chi_j(x, t) = H[x - (\xi_j - \ell_j)] - H(x - \xi_j). \quad (7)$$

Here $H(\cdot)$ is the Heaviside function. Equation (7) equals one only over the instantaneous footprint $\xi_j - \ell_j < x < \xi_j$ and equals zero elsewhere. Thus, the resultant of a fully supported load is $P_j = p_j \ell_j$. Constant moving loads are obtained with $s(t) = 1$, while harmonic moving loads are represented by $s(t) = \sin(\Omega t)$, where Ω is the imposed circular frequency. The spatial integration over $0 \leq x \leq L$ automatically accounts for partial entry and exit.

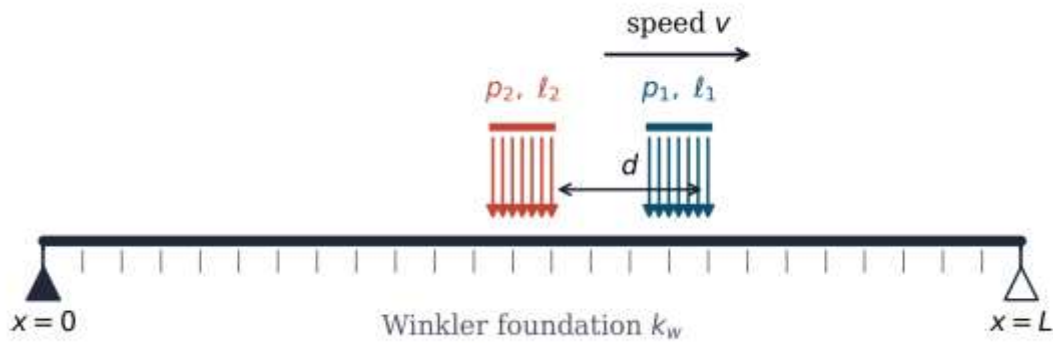


Figure 1. Configuration of the simply supported Timoshenko beam, Winkler foundation, and two finite distributed loads.

Galerkin reduction

The expansions

$$w(x, t) = \sum_{n=1}^{N_m} q_n(t) \sin(\beta_n x), \quad \theta(x, t) = \sum_{n=1}^{N_m} r_n(t) \cos(\beta_n x), \quad \beta_n = \frac{n\pi}{L} \quad (8)$$

satisfy (3). Substitution into (1) and (2), multiplication by the corresponding basis functions, and integration over the beam length give, for each spatial mode,

$$\mathbf{M} \ddot{\mathbf{y}}_n + \mathbf{K}_n \mathbf{y}_n = \mathbf{f}_n(t), \quad \mathbf{y}_n = [q_n \quad r_n]^T, \quad (9)$$

where

$$\mathbf{M} = \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \quad (10)$$

and

$$\mathbf{K}_n = \begin{bmatrix} (\kappa GA + N_T) \beta_n^2 + k_w & -\kappa GA \beta_n \\ -\kappa GA \beta_n & EI \beta_n^2 + \kappa GA \end{bmatrix}. \quad (11)$$

Let $a_j(t) = \max[0, \xi_j(t) - \ell_j]$ and $b_j(t) = \min[L, \xi_j(t)]$. When $b_j > a_j$, the generalized force is

$$\mathbf{f}_n(t) = \begin{bmatrix} F_n(t) \\ 0 \end{bmatrix}, \quad F_n(t) = \frac{2s(t)}{L\beta_n} \sum_{j=1}^2 p_j [\cos(\beta_n a_j) - \cos(\beta_n b_j)], \quad (12)$$

and the contribution is set to zero when a footprint does not intersect the span. The two natural-frequency branches for each n follow from

$$\det(\mathbf{K}_n - \omega^2 \mathbf{M}) = 0. \quad (13)$$

For time integration, define $\mathbf{z}_n = [\mathbf{y}_n^T, \dot{\mathbf{y}}_n^T]^T$. The zero-initial-condition response is evaluated from

$$\mathbf{z}_n(t) = \int_0^t \exp[\mathbf{A}_n(t - \tau)] \mathbf{B} \mathbf{f}_n(\tau) d\tau, \quad (14)$$

where $\mathbf{A}_n = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K}_n & \mathbf{0} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix}$. A state-transition matrix with midpoint load evaluation is used over each time step. Equation (14) is an exact convolution for the retained modal system when the load is represented exactly within a step; the complete beam solution remains a Galerkin approximation because the spatial series is truncated at N_m .

Explicit analytical solution by Laplace transformation

Equation (14) is compact and convenient for numerical evaluation, but the analytical steps can be shown explicitly. Introduce the modal mass and stiffness coefficients

$$m_w = \rho A, \quad m_\theta = \rho I, \quad c_n = \kappa G A \beta_n, \quad (15)$$

and

$$k_{11,n} = (\kappa G A + N_T) \beta_n^2 + k_w, \quad k_{22,n} = E I \beta_n^2 + \kappa G A. \quad (16)$$

Let $Q_n(s)$, $R_n(s)$, and $\hat{F}_n(s)$ be the Laplace transforms of $q_n(t)$, $r_n(t)$, and $F_n(t)$, respectively. Applying the zero initial conditions in (4) to the modal equations in (9) gives

$$\begin{bmatrix} m_w s^2 + k_{11,n} & -c_n \\ -c_n & m_\theta s^2 + k_{22,n} \end{bmatrix} \begin{bmatrix} Q_n(s) \\ R_n(s) \end{bmatrix} = \begin{bmatrix} \hat{F}_n(s) \\ 0 \end{bmatrix}. \quad (17)$$

The determinant of the transformed coefficient matrix is a biquadratic polynomial that factors into the flexural and shear branches:

$$\Delta_n(s) = m_w m_\theta s^4 + a_n s^2 + b_n = m_w m_\theta (s^2 + \omega_{n-}^2)(s^2 + \omega_{n+}^2), \quad (18)$$

where

$$a_n = m_w k_{22,n} + m_\theta k_{11,n}, \quad b_n = k_{11,n} k_{22,n} - c_n^2, \quad (19)$$

and

$$\omega_{n\pm}^2 = \frac{a_n \pm \sqrt{a_n^2 - 4m_w m_\theta b_n}}{2m_w m_\theta}. \quad (20)$$

The lower sign branch ω_{n-} is flexure dominated, whereas ω_{n+} is shear dominated. Direct inversion of the matrix in (17) gives

$$Q_n(s) = \frac{m_\theta s^2 + k_{22,n}}{\Delta_n(s)} \hat{F}_n(s), \quad R_n(s) = \frac{c_n}{\Delta_n(s)} \hat{F}_n(s). \quad (21)$$

Partial-fraction expansion separates the two natural-frequency branches. Define

$$A_{n-} = \frac{k_{22,n}/m_\theta - \omega_{n-}^2}{m_w(\omega_{n+}^2 - \omega_{n-}^2)}, \quad A_{n+} = \frac{\omega_{n+}^2 - k_{22,n}/m_\theta}{m_w(\omega_{n+}^2 - \omega_{n-}^2)}, \quad B_n = \frac{c_n}{m_w m_\theta (\omega_{n+}^2 - \omega_{n-}^2)}. \quad (22)$$

The corresponding modal impulse-response functions are

$$h_n^{(q)}(t) = A_{n-} \frac{\sin(\omega_{n-} t)}{\omega_{n-}} + A_{n+} \frac{\sin(\omega_{n+} t)}{\omega_{n+}}, \quad h_n^{(r)}(t) = B_n \left[\frac{\sin(\omega_{n-} t)}{\omega_{n-}} - \frac{\sin(\omega_{n+} t)}{\omega_{n+}} \right]. \quad (23)$$

By the convolution theorem, the zero-initial-condition modal coordinates are therefore

$$q_n(t) = \int_0^t h_n^{(q)}(t - \tau) F_n(\tau) d\tau, \quad r_n(t) = \int_0^t h_n^{(r)}(t - \tau) F_n(\tau) d\tau. \quad (24)$$

Substitution into the Galerkin expansion gives the final analytical solution:

$$w(x, t) = \sum_{n=1}^{N_m} \sin(\beta_n x) \int_0^t h_n^{(q)}(t - \tau) F_n(\tau) d\tau, \quad \theta(x, t) = \sum_{n=1}^{N_m} \cos(\beta_n x) \int_0^t h_n^{(r)}(t - \tau) F_n(\tau) d\tau. \quad (25)$$

Equation (25), together with the clipped-footprint generalized force in (12), is the most useful analytical form because it remains valid during partial entry, full traversal, and partial exit of either load. It also shows precisely what is exact and what is approximate: the time-domain convolution is exact for every retained modal system, whereas truncation at N_m makes the spatial beam representation approximate.

For comparison with conventional closed forms, consider the forcing components that apply during an interval in which both footprints are completely on the span. Let $\Omega_n = \beta_n v$, $\gamma_{nj} = \beta_n(j - 1)d$, and $\lambda_{nj} = \beta_n \ell_j$. For constant load amplitudes, (12) then reduces to

$$F_n(t) = C_n \cos(\Omega_n t) + D_n \sin(\Omega_n t), \quad (26)$$

where

$$C_n = \sum_{j=1}^2 \frac{2p_j}{L\beta_n} [(\cos\lambda_{nj} - 1)\cos\gamma_{nj} - \sin\lambda_{nj}\sin\gamma_{nj}], \quad (27)$$

and

$$D_n = \sum_{j=1}^2 \frac{2p_j}{L\beta_n} [(\cos\lambda_{nj} - 1)\sin\gamma_{nj} + \sin\lambda_{nj}\cos\gamma_{nj}]. \quad (28)$$

Define the zero-initial-condition cosine and sine response kernels

$$\mathcal{C}(\omega, \lambda, t) = \frac{\cos(\lambda t) - \cos(\omega t)}{\omega^2 - \lambda^2}, \quad \mathcal{S}(\omega, \lambda, t) = \frac{\sin(\lambda t) - (\lambda/\omega)\sin(\omega t)}{\omega^2 - \lambda^2}. \quad (29)$$

At $\lambda = \omega$, the removable limits are $\mathcal{C} = t \sin(\omega t)/(2\omega)$ and $\mathcal{S} = [\sin(\omega t) - \omega t \cos(\omega t)]/(2\omega^2)$. If local time is measured from the beginning of a fully supported segment and the zero-state forced contribution is required, evaluation of the corresponding integrals gives the constant-load closed form

$$q_n^{(C)}(t) = \sum_{\sigma \in \{-, +\}} A_{n\sigma} [C_n \mathcal{C}(\omega_{n\sigma}, \Omega_n, t) + D_n \mathcal{S}(\omega_{n\sigma}, \Omega_n, t)], \quad (30)$$

and

$$r_n^{(C)}(t) = B_n \{C_n [\mathcal{C}(\omega_{n-}, \Omega_n, t) - \mathcal{C}(\omega_{n+}, \Omega_n, t)] + D_n [\mathcal{S}(\omega_{n-}, \Omega_n, t) - \mathcal{S}(\omega_{n+}, \Omega_n, t)]\}. \quad (31)$$

For $s(t) = \sin(\Omega t)$, multiplication of (26) by the imposed harmonic factor produces two travelling sidebands:

$$F_n^{(H)}(t) = \frac{C_n}{2} [\sin(v_+ t) + \sin(v_- t)] + \frac{D_n}{2} [\cos(v_- t) - \cos(v_+ t)], \quad v_{\pm} = \Omega \pm \Omega_n. \quad (32)$$

Using $\varepsilon_- = 1$ and $\varepsilon_+ = -1$, its explicit modal response is

$$q_n^{(H)}(t) = \frac{1}{2} \sum_{\sigma \in \{-, +\}} A_{n\sigma} \{C_n [\mathcal{S}(\omega_{n\sigma}, v_+, t) + \mathcal{S}(\omega_{n\sigma}, v_-, t)] + D_n [\mathcal{C}(\omega_{n\sigma}, v_-, t) - \mathcal{C}(\omega_{n\sigma}, v_+, t)]\}, \quad (33)$$

and

$$r_n^{(H)}(t) = \frac{B_n}{2} \sum_{\sigma \in \{-, +\}} \varepsilon_{\sigma} \{C_n [\mathcal{S}(\omega_{n\sigma}, v_+, t) + \mathcal{S}(\omega_{n\sigma}, v_-, t)] + D_n [\mathcal{C}(\omega_{n\sigma}, v_-, t) - \mathcal{C}(\omega_{n\sigma}, v_+, t)]\}. \quad (34)$$

Equations (30)–(34) are zero-state closed-form building blocks for a full-footprint segment. In a continuing traversal, the homogeneous response associated with displacement and velocity at the segment boundary must also be added so that q_n , r_n , and their first derivatives remain continuous. At entry and exit, the limits $a_j(t)$ and $b_j(t)$ change piecewise; the general analytical result (25) with (12) automatically includes the complete loading history and avoids a semi-infinite load. The displacement block of $\exp[\mathbf{A}_n(t - \tau)]\mathbf{B}$ in (14) is exactly the pair of impulse responses in (23), so the Laplace-transform and state-transition forms are two representations of the same retained-mode solution.

Convergence, verification, and response measures

Modal convergence is evaluated against a 16-mode reference. For a response sampled at N_t time points, the normalized root-mean-square error is

$$NRMSE = \frac{\left[\frac{1}{N_t} \sum_{k=1}^{N_t} (w_k - w_k^{ref})^2 \right]^{1/2}}{\max_k |w_k^{ref}|} \times 100\%. \quad (35)$$

The peak-response difference between two methods is

$$e_{peak} = \frac{\max |w|_G - \max |w|_{FE}}{\max |w|_{FE}} \times 100\%. \quad (36)$$

An independent finite-element model is used for verification. It comprises 100 two-node Timoshenko elements, linear interpolation of w and θ , selective reduced integration of the shear term to avoid locking, consistent translational and rotary mass matrices, exact element integration of the overlapping load footprint, and the average-acceleration Newmark scheme with $\gamma = 1/2$, $\beta = 1/4$, and $\Delta t = 0.002$ s [15, 16]. Only the transverse displacement is restrained at the two supports; end rotations remain free.

The dynamic amplification factor is defined relative to the largest quasi-static midspan response attained as the same pair of loads traverses the beam:

$$DAF = \frac{\max_{0 \leq t \leq t_p} |w_{dyn}(L/2, t)|}{\max_{0 \leq t \leq t_p} |w_{qs}(L/2, t)|}, \quad K_n y_{n,qs} = f_n(t), \quad (37)$$

where $t_p = (L + d + \ell)/v$ is the time at which the second load has completely left the span for equal footprint lengths.

Numerical implementation

Table 1 gives the baseline parameters. The values define a representative steel Timoshenko beam and are used consistently in the Galerkin and finite-element models. Foundation stiffness is stated as a line-foundation coefficient: because $k_w w$ is force per unit length, k_w has units N/m². Prestress is a tensile resultant in newtons, and the moving-load parameters p_1 and p_2 are intensities in N/m.

Table 1. Baseline beam, foundation, load, and numerical parameters

Quantity	Symbol	Value	SI unit
Beam length	L	50.0	m
Rectangular width	b	0.50	m
Rectangular depth	h	1.00	m
Cross-sectional area	A	0.50	m ²
Second moment of area	I	0.041667	m ⁴
Young's modulus	E	210	GPa
Poisson's ratio	ν	0.30	—
Shear modulus	G	80.769	GPa
Mass density	ρ	7850	kg/m ³
Shear-correction factor	κ	5/6	—
Winkler stiffness	k_w	0.50	MN/m ²
Tensile prestress	N_T	40.0	MN
Load intensities	p_1, p_2	9.0, 10.0	kN/m
Footprint lengths	ℓ_1, ℓ_2	3.0, 3.0	m
Load spacing	d	8.0	m
Travelling velocity	v	12.5	m/s

Quantity	Symbol	Value	SI unit
Time increment	Δt	0.002	s
Retained Galerkin modes	N_m	12	—
Observation point	x_o	$L/2$	m

The first four flexural frequencies of the baseline system are 2.263, 4.615, 9.099, and 15.491 Hz. The corresponding shear-dominated branches begin at approximately 1.615 kHz and contribute negligibly to the reported midspan displacement, although they are retained in each two-degree-of-freedom modal system.

RESULTS AND DISCUSSION

Modal convergence

Table 2 and Figure 2(a) demonstrate convergence of the Galerkin expansion. One spatial mode underestimates the 16-mode peak by 4.400%. The three-mode result differs by only -0.104%, and the 12-mode result differs by 0.023%, with an NRMSE of 0.0116%. Even spatial modes make no direct contribution at exact midspan because $\sin(n\pi/2) = 0$ for even n ; they are nevertheless retained for response evaluation away from midspan. Twelve modes were therefore used in all subsequent calculations.

Table 2. Galerkin convergence at midspan relative to the 16-mode solution

Modes	Peak deflection (mm)	Peak difference (%)	NRMSE (%)
1	2.7934	-4.3998	2.7363
3	2.9189	-0.1043	0.2091
5	2.9264	0.1499	0.0556
7	2.9251	0.1059	0.0397
10	2.9236	0.0547	0.0231
12	2.9227	0.0229	0.0116
16	2.9220	0.0000	0.0000

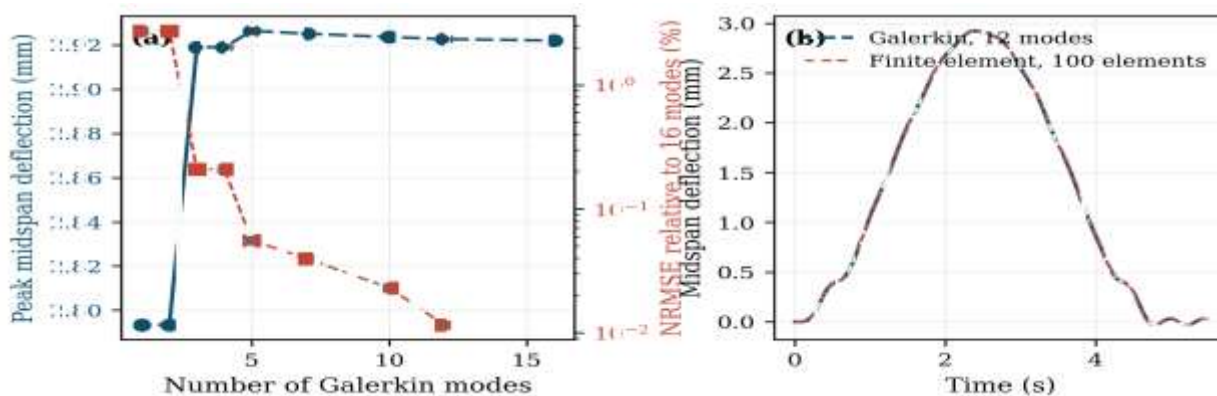


Figure 2. Galerkin convergence and independent finite-element verification: (a) peak response and NRMSE versus modal truncation; (b) midspan time histories from the 12-mode Galerkin and 100-element finite-element models.

Independent finite-element verification

Figure 2(b) shows that the modal and finite-element time histories are visually indistinguishable at the plotted scale. The quantitative comparison in Table 3 gives a Galerkin peak of 2.9227 mm and a finite-element peak of 2.9220 mm. The peak difference is 0.0232%, the time-history NRMSE is 0.0121%, and the linear correlation coefficient is 0.99999995. The peak times differ by 0.006 s, corresponding to three time increments. This agreement independently verifies the load projection, foundation and tensile-stiffness contributions, support conditions, and transient integration.

Table 3. Galerkin and finite-element verification for the baseline case

Indicator	Galerkin model	Finite-element model	Difference
Spatial discretization	12 modes	100 elements	–
Peak midspan deflection	2.9227 mm	2.9220 mm	0.0232%
Time of peak	2.400 s	2.394 s	0.006 s
Time-history NRMSE	–	–	0.0121%
Correlation coefficient	–	–	0.99999995

Foundation stiffness and tensile prestress

Figure 3(a) shows a strong, monotonic reduction in response as k_w increases. With no foundation, the peak midspan displacement is 7.975 mm. It decreases to 4.277, 2.923, 1.859, and 1.082 mm for $k_w = 0.25, 0.50, 1.00$, and 2.00 MN/m², respectively. Thus, the largest investigated stiffness reduces the unsupported-foundation response by 86.4%. The decrease is nonlinear because the foundation term shifts the modal frequencies as well as the quasi-static stiffness.

Figure 3(b) isolates the effect of tensile prestress. Increasing N_T from 0 to 20, 40, 80, and 120 MN reduces the peak response from 3.706 mm to 3.277, 2.923, 2.448, and 2.116 mm. The total reduction is 42.9%. These results apply to tensile prestress. A compressive force would enter (1) with the opposite sign and would reduce the effective lateral stiffness rather than increase it.

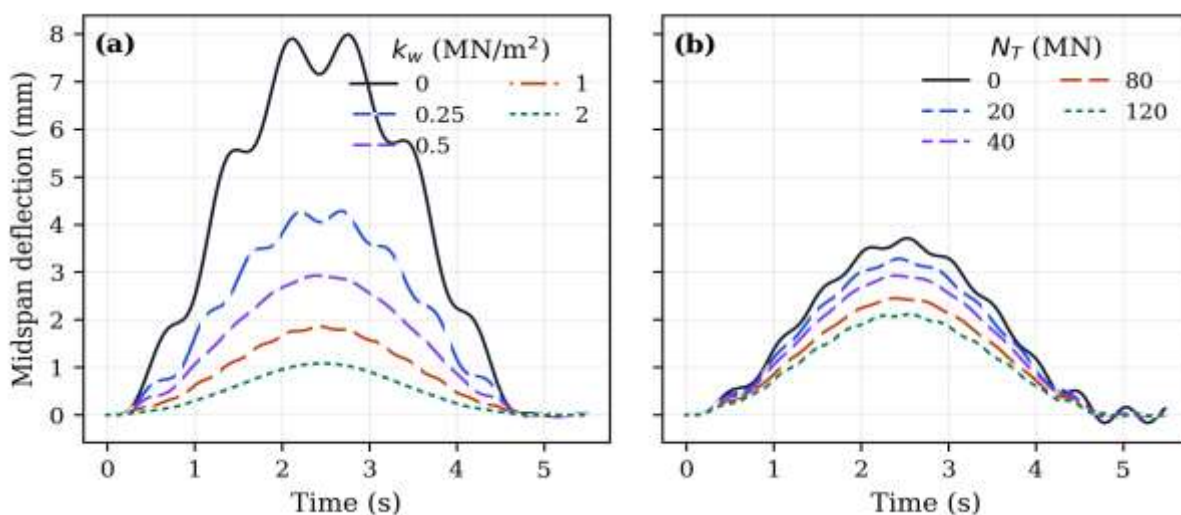


Figure 3. Influence of (a) Winkler foundation stiffness and (b) uniform tensile prestress on the midspan response under two constant distributed moving loads.

Individual load magnitudes

The response varies approximately linearly with each load intensity over the investigated range, as expected for the linear beam model. Increasing p_1 from 5 to 11 kN/m raises the peak from 2.325 to 3.231 mm, an increase of 39.0%. Increasing p_2 from 6 to 12 kN/m raises the peak from 2.321 to 3.237 mm, an increase of 39.5%. The small differences between the two families arise from their distinct entry times and positions relative to the midspan mode amplitudes. The results confirm that the two loads should be projected and superposed at their actual positions; treating the pair as a single stationary resultant would discard this timing information.

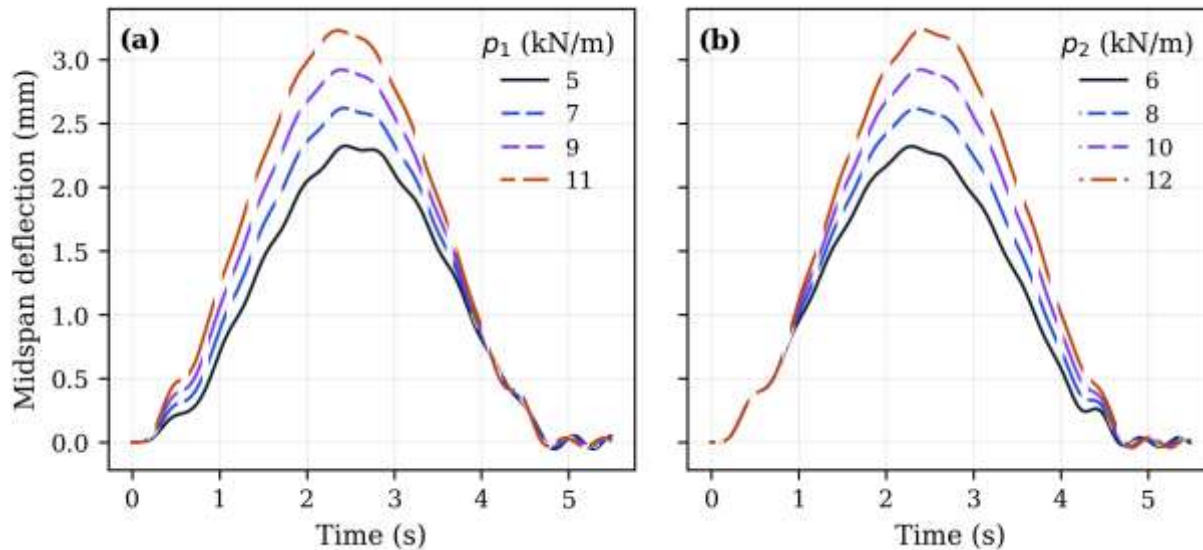


Figure 4. Sensitivity of the midspan response to the intensity of (a) the leading load and (b) the trailing load.

Load spacing and travelling velocity

Load spacing controls the phase with which the two moving footprints excite each mode. Figure 5(a) therefore shows a non-monotonic response rather than a simple proportional trend. The peak increases slightly from 3.057 mm at $d = 3$ m to 3.083 mm at $d = 5$ m, then falls to 2.923, 2.810, 2.575, and 2.287 mm at spacings of 8, 12, 16, and 20 m. The 5 m configuration gives the largest constructive interaction among the investigated spacings. At larger separation, the two loads spend less time acting simultaneously near the high-amplitude portion of the first mode.

Figure 5(b) shows that velocity changes both the peak response and DAF. The baseline $v = 12.5$ m/s case has a quasi-static peak of 2.909 mm, a dynamic peak of 2.923 mm, and $DAF = 1.005$. The DAF rises to 1.383 at 75 m/s and reaches 1.619 at 125 m/s before decreasing to 1.549 at 225 m/s and 1.348 at 275 m/s. The nominal first-mode critical speed for a single moving sinusoidal excitation is $v_{c1} = \omega_1 L / \pi = 226.3$ m/s. The maximum for the present two-load, finite-footprint case occurs below this value because the response also depends on footprint filtering, finite residence time, the 8 m spacing, and the phase difference between the two load projections. This result demonstrates why velocity and spacing should be studied together rather than inferred from a single-load resonance condition.

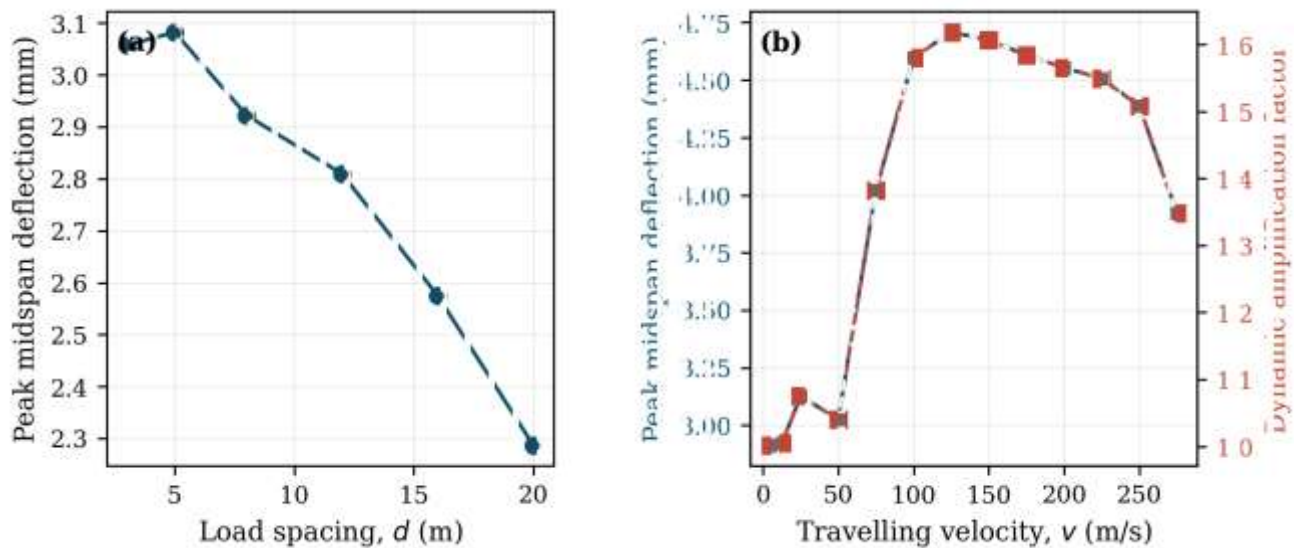


Figure 5. Effects of (a) load spacing and (b) travelling velocity on peak midspan response and dynamic amplification.

Harmonic excitation frequency

For harmonic loading, the forcing contains the imposed frequency Ω in addition to the frequencies created by spatial translation. Figure 6(a) shows a sharp maximum at $\Omega/\omega_1 = 1$. The peak responses at ratios 0.50, 0.75, 1.00, 1.25, and 1.50 are 4.060, 6.689, 51.988, 5.224, and 2.286 mm, respectively. The resonant value is bounded because the loads remain on the beam for a finite time, but its magnitude is high because the mathematical model is undamped. Figure 6(b) confirms the accumulation of first-mode oscillation during passage when $\Omega = \omega_1$. In a physical beam, material, structural, and foundation damping would limit this growth. The resonant result should therefore be interpreted as a conservative undamped benchmark rather than a direct prediction for a specific bridge.

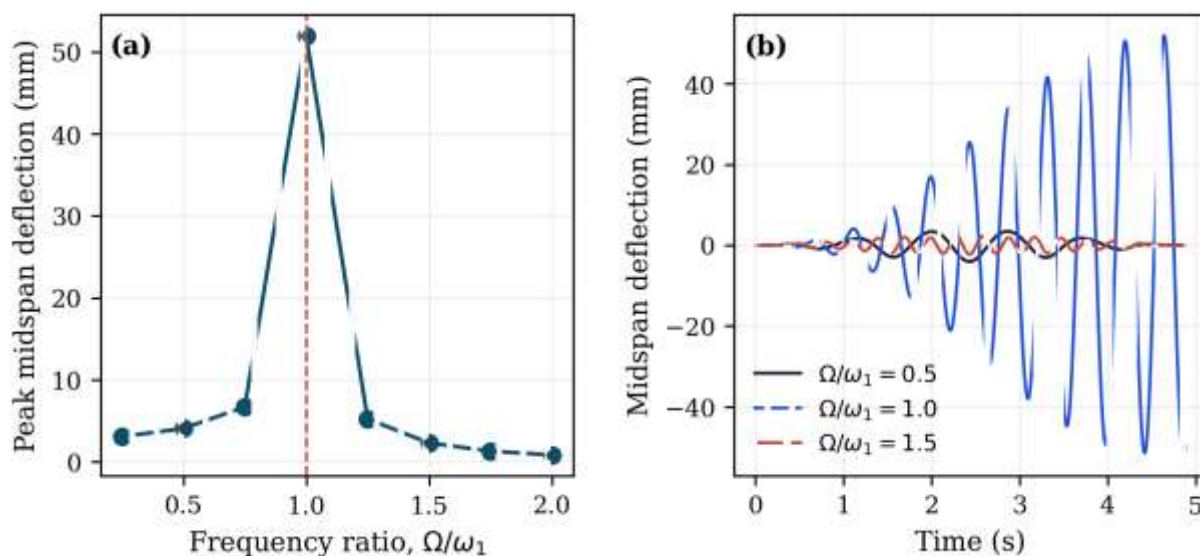


Figure 6. Harmonic-load sensitivity: (a) peak response versus imposed frequency ratio and (b) representative midspan time histories.

Table 4 consolidates the principal quantitative sensitivities. Foundation stiffness has the largest monotonic suppressive effect within the selected ranges. Tensile prestress also reduces response, whereas increasing either load intensity increases it. Spacing and velocity are interaction parameters and consequently show non-

monotonic behavior. Harmonic tuning to the first flexural frequency produces the strongest amplification in the undamped model.

Table 4. Quantitative summary of the principal parameter effects

Parameter change	Initial peak (mm)	Final or maximum peak (mm)	Quantitative effect
k_w : 0 to 2.0 MN/m ²	7.975	1.082	86.4% reduction
N_T : 0 to 120 MN	3.706	2.116	42.9% reduction
p_1 : 5 to 11 kN/m	2.325	3.231	39.0% increase
p_2 : 6 to 12 kN/m	2.321	3.237	39.5% increase
d : 3 to 20 m	3.057	3.083 at 5 m	Non-monotonic; 2.287 mm at 20 m
v : 5 to 275 m/s	DAF 1.002	DAF 1.619 at 125 m/s	Non-monotonic dynamic amplification
Ω/ω_1 : 0.25 to 2.00	3.068	51.988 at 1.00	Undamped resonance at first mode

Equivalence of the explicit analytical and state-transition solutions

The explicit solution in (24)–(25) was evaluated independently by integrating the two analytical impulse-response branches over every midpoint-force interval. The state-transition result in (14) was evaluated with the matrix exponential over the same intervals. Table 5 presents the comparison for the 12-mode baseline case. The identical peak value and peak time, together with an NRMSE of $6.79 \times 10^{-12}\%$ and a correlation coefficient of 1.000000000, show that the previously presented state-transition solution did not replace a different physical model; it is the matrix form of the restored Laplace–convolution solution.

Table 5. Direct comparison of the two equivalent retained-mode solution forms

Indicator	Explicit analytical convolution	State-transition convolution	Comparison
Peak midspan deflection	2.9226547 mm	2.9226547 mm	Same to 7 decimal places
Time of peak	2.400 s	2.400 s	0.000 s
Maximum absolute time-history difference	–	–	3.27×10^{-13} mm

Figure 7(a) superposes the two midspan histories, which are visually indistinguishable. Figure 7(b) magnifies their pointwise difference by 10^{12} ; the residual remains at floating-point round-off level. This comparison verifies both the partial-fraction coefficients in (22)–(23) and the matrix-exponential implementation in (14).

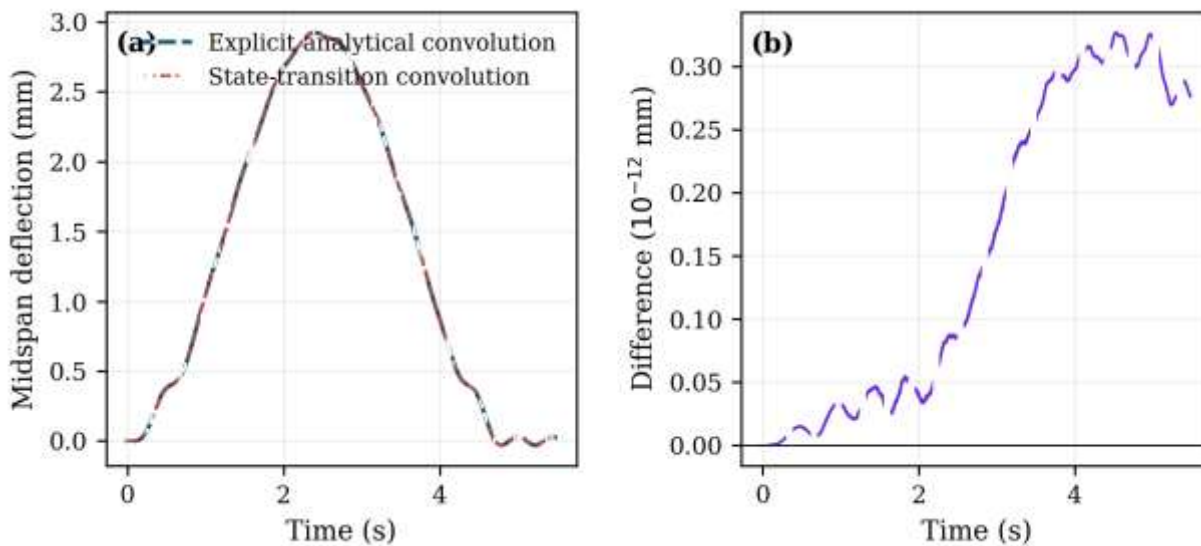


Figure 7. Equivalence of the restored analytical solution and the state-transition solution: (a) superposed midspan responses and (b) floating-point-level residual after magnification.

Recovered curves from the earlier analytical dataset

The eight native charts in the earlier manuscript were recovered from their embedded numerical caches and replotted without changing their data. They are included here to preserve the authors' original numerical evidence and to permit direct comparison with the revised analysis. The cached responses are reported in micrometres because the stored ordinates are of order 10^{-5} m. The symbols F_D , N_F , L_1 , and L_2 reproduce the labels in the original charts. Because the earlier manuscript did not define all chart units and load-footprint parameters consistently, these curves are used for qualitative traceability rather than as an independent quantitative validation of the corrected model.

Table 6 summarizes the peak values extracted from the recovered curves. It shows the same directional behavior as the verified formulation: larger foundation and tensile-prestress parameters reduce response, whereas larger moving-load magnitudes increase response. The absolute values should not be compared directly with Tables 2–5 because the earlier and revised parameter sets are not dimensionally identical.

Table 6. Peak-response trends recovered from the eight original chart datasets

Loading case	Varied parameter	First peak (μm)	Last peak (μm)	Change
Constant	F_D : 0 to 1,000,000	20.877	6.979	66.6% reduction
Constant	N_F : 0 to 800,000	12.643	3.226	74.5% reduction
Constant	L_1 : 5,000 to 11,000	13.720	22.930	67.1% increase
Constant	L_2 : 6,000 to 12,000	19.534	26.788	37.1% increase
Harmonic	F_D : 0 to 1,000,000	16.591	5.417	67.3% reduction
Harmonic	N_F : 0 to 800,000	8.930	2.279	74.5% reduction
Harmonic	L_1 : 5,000 to 11,000	3.636	5.909	62.5% increase
Harmonic	L_2 : 6,000 to 12,000	13.895	19.601	41.1% increase

For the constant-load dataset in Figure 8, the peak response decreases from 20.877 to 6.979 μm as the original foundation parameter F_D rises from 0 to 1,000,000. The 66.6% reduction is monotonic and is consistent with the increased restoring stiffness predicted by (1) and by the verified foundation sweep in Figure 3(a).

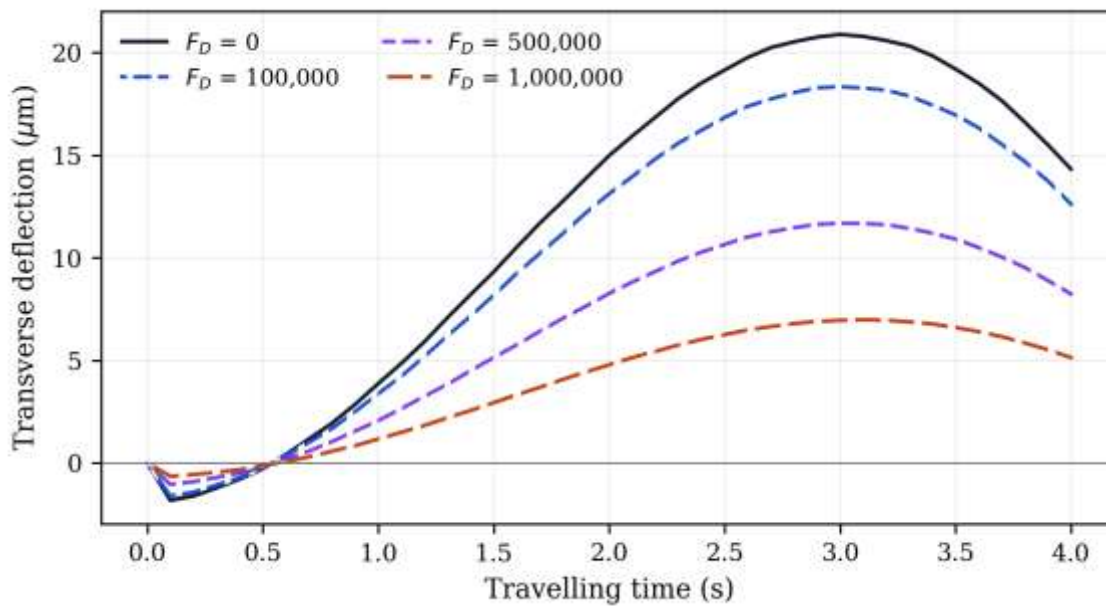


Figure 8. Recovered original constant-load response for the four foundation-parameter values stored in the source chart.

Figure 9 isolates the original prestress sweep under constant loads. The peak decreases from 12.643 to 3.226 μm between $N_F = 0$ and 800,000, a 74.5% reduction. This trend agrees with the tensile-stiffening term in $k_{11,n}$; it should not be generalized to compressive axial force, which has the opposite stiffness effect.

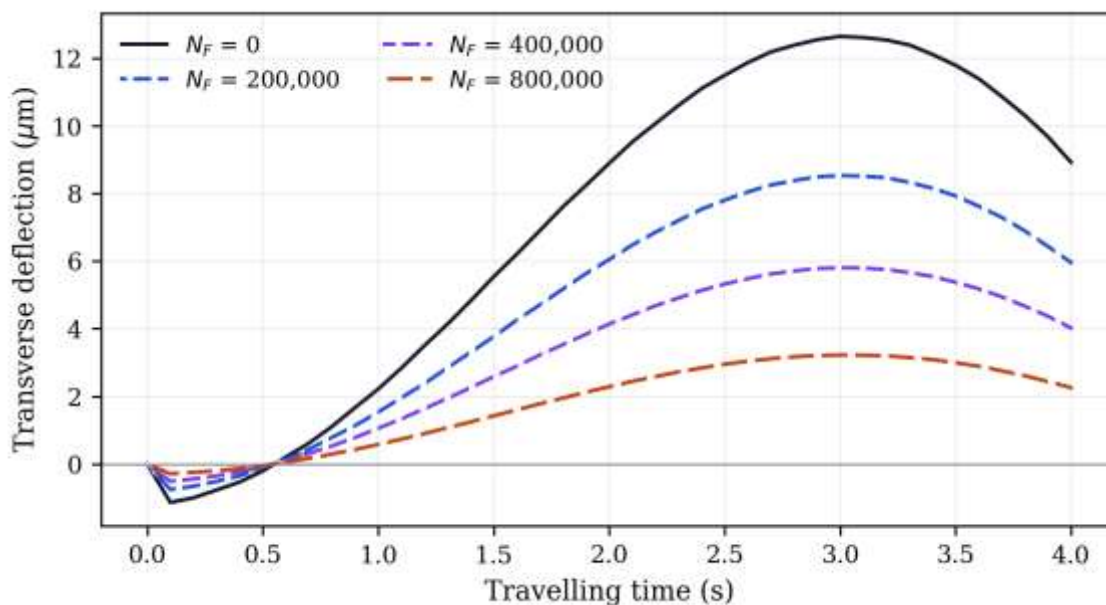


Figure 9. Recovered original constant-load response for the four prestress values stored in the source chart.

Figure 10 shows the recovered sensitivity to the first load magnitude under constant excitation. Raising L_1 from 5,000 to 11,000 increases the peak from 13.720 to 22.930 μm , or 67.1%. The increasing family of curves is physically consistent with linear superposition, while differences in waveform reflect the fixed contribution and phase of the second moving load.

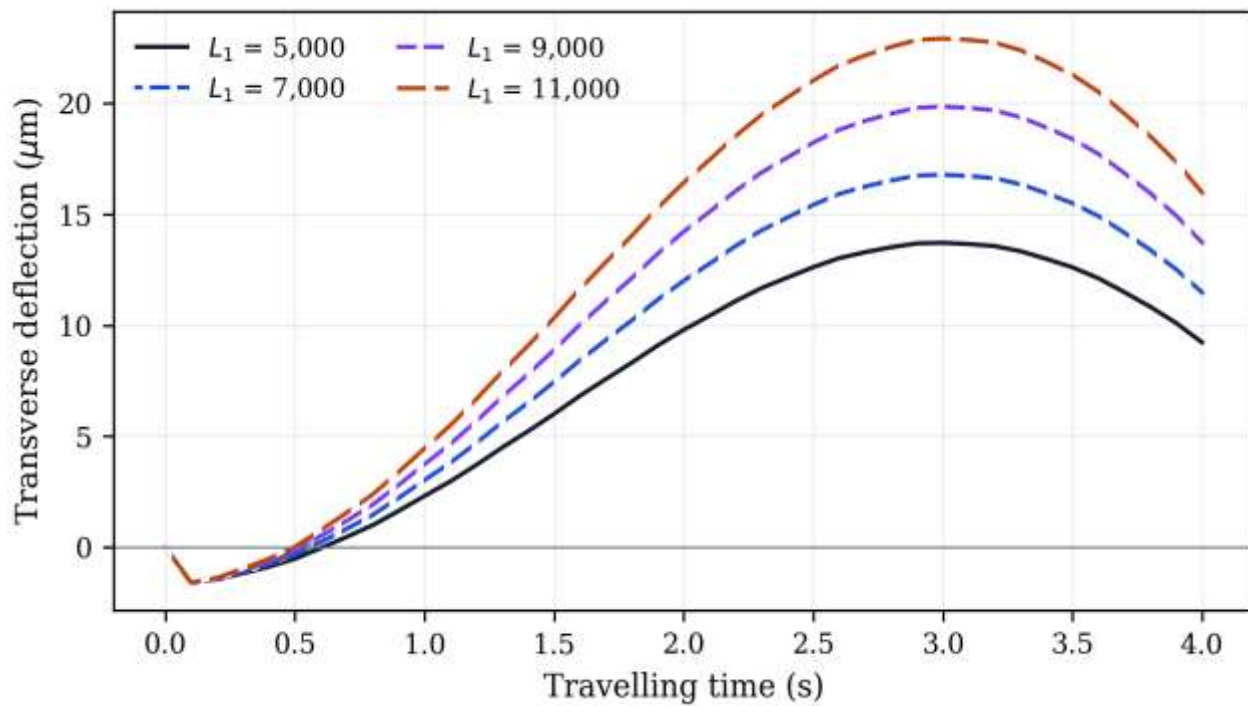


Figure 10. Recovered original constant-load response as the first moving-load magnitude is varied.

Figure 11 presents the corresponding constant-excitation sweep for the second load. Its peak rises from 19.534 to 26.788 μm as L_2 increases from 6,000 to 12,000, representing a 37.1% increase. The smaller percentage than in Figure 10 is specific to the original baseline magnitudes, positions, and passage timing; it does not imply a different constitutive response to the two loads.

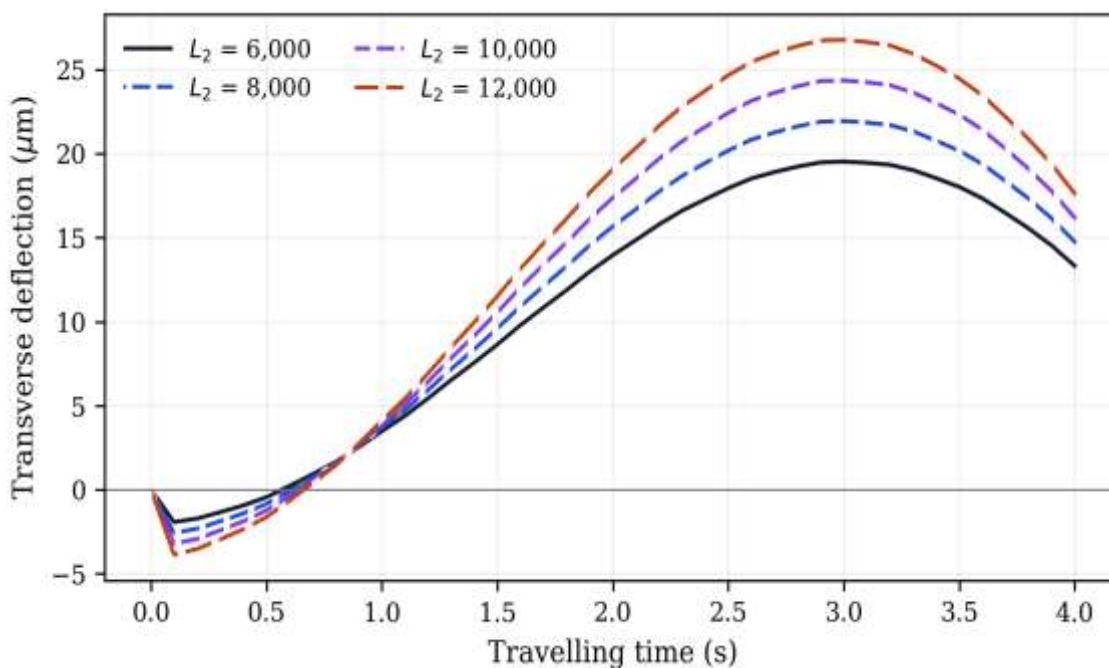


Figure 11. Recovered original constant-load response as the second moving-load magnitude is varied.

For the harmonic dataset in Figure 12, increasing F_D from 0 to 1,000,000 reduces the peak from 16.591 to 5.417 μm , a 67.3% decrease. The foundation suppresses the oscillatory response in the earlier data in the same direction as it suppresses the constant-load response, although the imposed harmonic factor changes the waveform and phase.

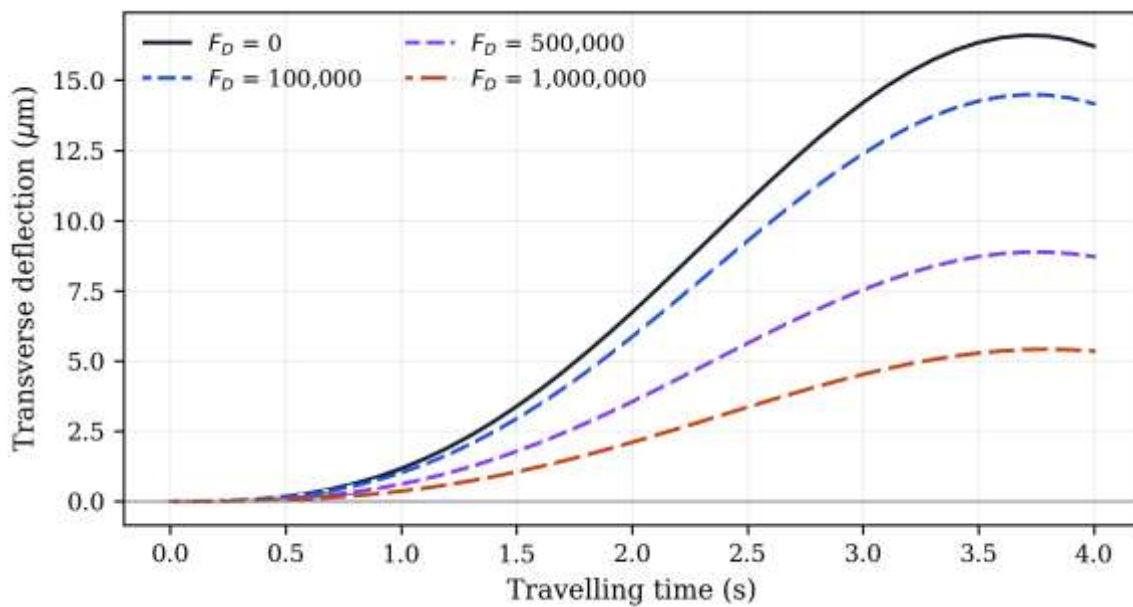


Figure 12. Recovered original harmonic-load response for the four foundation-parameter values stored in the source chart.

Figure 13 shows the original harmonic prestress sweep. The peak falls from 8.930 to 2.279 μm as N_F increases from 0 to 800,000, a 74.5% reduction. This again supports the qualitative conclusion that tensile prestress raises effective lateral stiffness and reduces transverse vibration for the sign convention adopted in the revised formulation.

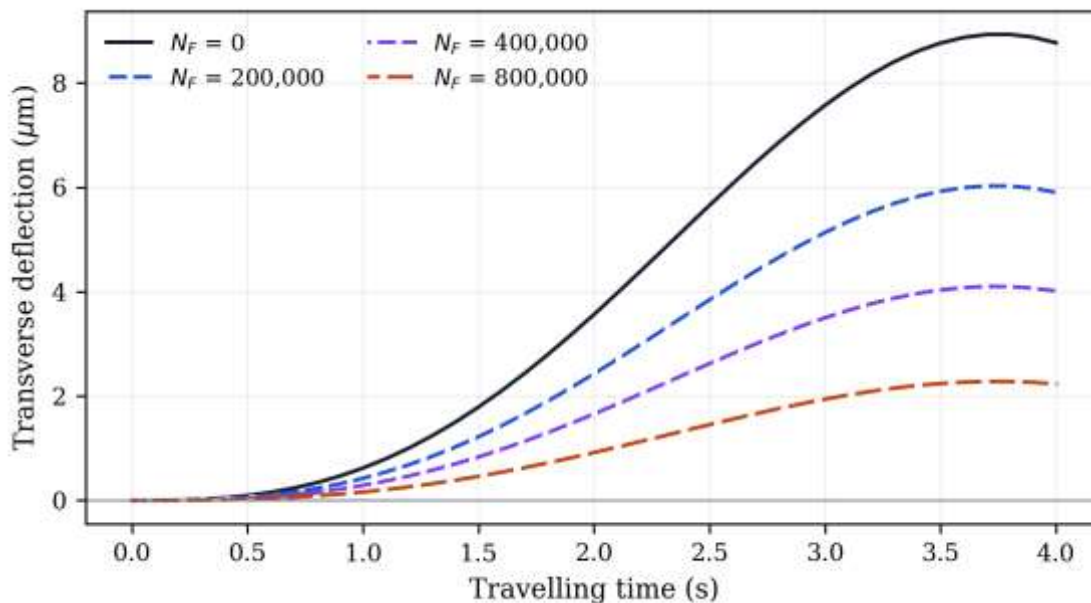


Figure 13. Recovered original harmonic-load response for the four prestress values stored in the source chart.

Figure 14 recovers the harmonic response while varying the first moving load. Increasing L_1 from 5,000 to 11,000 raises the peak from 3.636 to 5.909 μm , a 62.5% increase. The curves retain the same oscillatory phase pattern because the harmonic frequency is unchanged, while their amplitudes grow with the applied load.

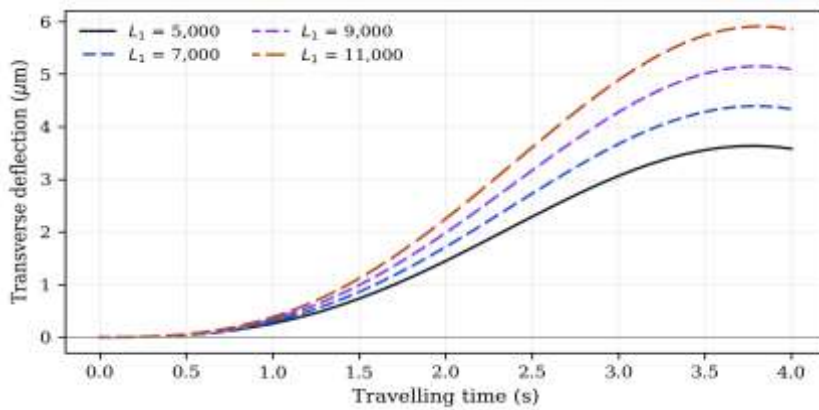


Figure 14. Recovered original harmonic-load response as the first moving-load magnitude is varied.

Finally, Figure 15 shows the original harmonic sweep for the second moving load. The peak increases from 13.895 to 19.601 μm between $L_2 = 6,000$ and 12,000, a 41.1% increase. Together, Figures 14 and 15 confirm the expected load-amplitude trend, but their recovered values remain supporting historical results rather than substitutes for convergence or finite-element verification.

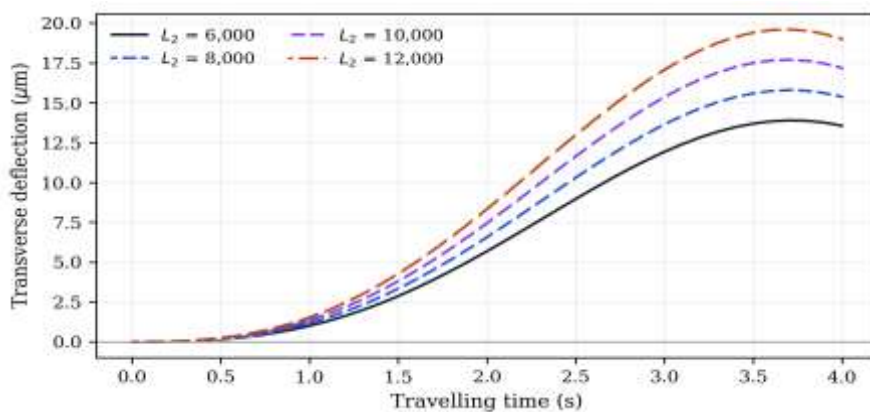


Figure 15. Recovered original harmonic-load response as the second moving-load magnitude is varied.

Scope and limitations

The present formulation concerns prescribed moving forces. It does not include the inertia of moving masses, Coriolis and centripetal terms associated with mass–beam interaction, vehicle suspension, loss of contact, or surface irregularity. The beam, foundation, and prestress are uniform; the response is linear; and damping is omitted so that stiffness, spacing, velocity, and frequency effects can be isolated. The numerical parameters are representative and are not a calibration to a particular bridge. Consequently, the reported magnitudes should be used as a verified benchmark for the stated model rather than as site-specific design values. The recovered curves in Section 4.8 retain the earlier manuscript’s cached numerical values, but incomplete unit and footprint information prevents them from serving as quantitative validation; that role is instead provided by the convergence, analytical-equivalence, and finite-element studies.

Extensions should include viscous or viscoelastic foundation damping, moving-mass coupling, compressive prestress and stability limits, non-uniform foundations, measured vehicle axle sequences, and comparison with laboratory or field data. Damping is particularly important near the harmonic resonance identified in Section 4.6.

CONCLUSIONS

This work reformulated and verified the dynamic response of a simply supported Timoshenko beam on a Winkler

foundation under two finite distributed moving loads. The following conclusions are supported by the numerical evidence:

- i. Representing each moving load by the difference of two Heaviside functions provides a finite footprint and treats partial entry and exit consistently. The sine–cosine Galerkin basis satisfies the simply supported displacement and zero-moment conditions.
- ii. Laplace transformation, matrix inversion, partial-fraction separation of the flexural and shear branches, and convolution with the generalized force lead directly to the final analytical displacement and rotation fields in (25). The constant- and harmonic-load closed forms in (30)–(34) are specializations of this general result.
- iii. The explicit analytical convolution and state-transition convolution are equivalent representations of the retained modal system. Both give a peak of 2.9226547 mm at 2.400 s; their maximum time-history difference is 3.27×10^{-13} mm and their correlation coefficient is 1.000000000.
- iv. The temporal solution is exact for each retained mode, but the full beam solution is a spatial Galerkin approximation rather than an exact beam solution. Twelve modes reduce the midspan NRMSE to 0.0116% relative to a 16-mode reference, and the peak differs by only 0.0229%.
- v. Independent verification with a 100-element Timoshenko finite-element model gives a peak difference of 0.0232% and a time-history NRMSE of 0.0121%. This agreement supports the analytical projection, boundary conditions, and transient implementation.
- vi. Increasing k_w from 0 to 2.0 MN/m² reduces peak deflection by 86.4%, while increasing tensile prestress from 0 to 120 MN produces a 42.9% reduction. Increasing either load intensity increases the response approximately linearly for the linear model.
- vii. Load spacing and velocity produce non-monotonic effects through modal phase and finite residence time. For the selected configuration, the maximum DAF is 1.619 at 125 m/s, and a 5 m spacing gives the largest response among the spacings examined. Harmonic forcing at the first flexural frequency produces pronounced resonant growth in the undamped model.
- viii. The eight curves recovered from the earlier manuscript confirm the same qualitative effects of foundation stiffness, tensile prestress, and load magnitude. They are retained for traceability, while the convergence and independent finite-element results provide the quantitative verification.

The restored analytical derivation, direct solution-form comparison, convergence study, finite-element verification, and quantitative sensitivity results provide a stronger basis for applying the model to transportation structures than qualitative curve comparison alone.

Nomenclature

Symbol	Definition	SI unit
A	Cross-sectional area	m ²
A_{n-}, A_{n+}, B_n	Partial-fraction coefficients of the two modal branches	Varies
a_n, b_n	Coefficients of the modal characteristic polynomial	Varies
c_n	Modal shear-coupling coefficient	N/m
C_n, D_n	Cosine and sine coefficients of the fully supported generalized force	N/m

Symbol	Definition	SI unit
d	Spacing between corresponding edges of the loads	m
E, G	Young's and shear moduli	Pa
F_n	Generalized modal force	N/m
H	Heaviside function	–
I	Second moment of cross-sectional area	m^4
$k_{11,n}, k_{22,n}$	Modal direct-stiffness coefficients	$N/m^2, N$
k_w	Winkler line-foundation stiffness	N/m^2
L	Beam length	m
m_w, m_θ	Translational and rotary modal mass coefficients	kg/m, kg·m
N_m	Number of retained spatial modes	–
N_T	Uniform tensile prestress resultant	N
p_j	Intensity of the j th distributed load	N/m
$q(x, t)$	Total transverse load per unit beam length	N/m
q_n, r_n	Modal displacement and rotation coordinates	m, rad
Q_n, R_n	Laplace transforms of q_n and r_n	$m \cdot s, rad \cdot s$
t	Time	s
v	Travelling velocity	m/s
w	Transverse beam displacement	m
x	Axial coordinate	m
β_n	Spatial wave number $n\pi/L$	1/m
$\mathcal{C}, \mathcal{S}$	Cosine- and sine-forcing response kernels	s^2
κ	Shear-correction factor	–
ℓ_j	Footprint length of the j th load	m
$v_\pm$	Harmonic travelling sideband frequencies	rad/s
ρ	Mass density	kg/m^3
θ	Cross-section rotation	rad
ω_{n-}, ω_{n+}	Flexural- and shear-dominated modal frequencies	rad/s
Ω	Imposed harmonic circular frequency	rad/s
Ω_n	Travelling modal frequency $\beta_n v$	rad/s

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