

Dynamic Response of a Double Euler–Bernoulli Beam System Interconnected by a Viscoelastic Pasternak Layer under a Moving Mass

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ABSTRACT

This paper analyzes the transverse vibration of two identical, supported Euler–Bernoulli beams connected by a viscoelastic Pasternak-type layer and traversed by a concentrated mass moving at constant speed. The coupled governing equations are reduced by a Galerkin modal expansion in which the inertia of the moving mass, including its Coriolis and centripetal components, is retained. Struble’s asymptotic technique is then used to derive a closed-form modified frequency that accounts for the load inertia, and the resulting coupled modal equations are solved with a multi-step differential transform method (MsDTM). For the moving-force limit, an exact solution is obtained by transforming to in-phase and out-of-phase coordinates; this solution is used to verify the MsDTM, which agrees with it to about 10^{-14} . It was also shown that a single, truncated differential transform series diverges well before the load leaves the span. The Struble-reduced model was further checked against a full time-varying-inertia model and reproduces its peak mid-span deflections to within 5% for speed ratios up to 0.5 and mass ratios up to 0.25. The parametric study shows that load inertia increases the upper-beam dynamic amplification factor relative to the moving-force approximation, by about 5% at a speed ratio $\alpha = 0.3$ and about 10% at $\alpha = 0.5$ for a mass ratio of 0.25, and delays the time of peak response, whereas the lower beam is much less affected. Increasing the Winkler stiffness or the shear parameter of the layer reduces the upper-beam response and increases the lower-beam response, whereas the layer damping has only a weak influence on the peak deflection during a single passage. The moving-force idealization is therefore non-conservative for the upper beam once the speed ratio exceeds about 0.2. A simple inertia amplification factor of about $1 + 0.4\epsilon$ reproduces the full-model upper-beam peak for $\alpha \leq 0.5$, whereas above this speed the first-order reduction should be verified against a full model.

Keywords: Double-beam system; Pasternak layer; Moving mass; Struble’s technique; differential transform method; Dynamic amplification.

Nomenclature

Symbol	Description
EI	Flexural rigidity of each beam ($N\ m^2$)
μ	Mass per unit length of each beam ($kg\ m^{-1}$)
L	Span length (m)
k_1	Winkler stiffness of the layer ($N\ m^{-2}$)
R_0	Shear parameter of the layer (N)
μ_0	Viscous damping coefficient of the layer ($N\ s\ m^{-2}$)
M_L, v, g	Moving mass (kg), its speed ($m\ s^{-1}$), gravitational acceleration ($m\ s^{-2}$)

U_1, U_2	Transverse deflections of the upper and lower beams (m)
ϕ_m, β_m	Modal amplitudes of the upper and lower beams
ω_m	mth natural circular frequency of a single beam (rad s^{-1})
Ω_m	Modified frequency of the upper beam due to load inertia (rad s^{-1})
Θ_m	Excitation frequency $m\pi v/L$ (rad s^{-1})
ε, λ	Mass ratio $M_L/(\mu L)$ and Struble parameter $\varepsilon/(1 + \varepsilon)$
α, K, S, ζ	Speed, stiffness, shear and damping parameters, Eq. (9)
ξ	Dimensionless load position vt/L
δ_{st}	Static mid-span deflection of a single beam, $M_L g L^3/(48EI)$
$\bar{\phi}_m(k), \bar{\beta}_m(k)$	Differential transforms of ϕ_m and β_m
N, h	Number of transform terms per step and MsDTM step length

INTRODUCTION

Beams are among the most widely used structural elements in civil, mechanical, and aerospace engineering, and many practical systems consist of two beams joined by a continuous deformable layer. Examples include floating-slab railway tracks, sandwich and composite members, double-deck girders and auxiliary beams used for passive vibration control, in which a secondary beam suppresses the vibration of a primary beam that carries the external load. The free and forced vibration of such elastically connected double-beam systems has therefore attracted sustained attention. Oniszczyk (2000, 2003) derived the free and forced responses of simply supported double beams joined by a Winkler layer; obtained exact solutions for a damped double beam under harmonic excitation; and Li and Hua (2007) developed a spectral finite element formulation. Later studies included the effects of compressive axial load (Zhang et al., 2008; Stojanović & Kozić, 2012; Mao & Wattanasakulpong, 2015), viscoelastic interlayers (Palmeri & Adhikari, 2011; Li et al., 2016) and a Pasternak middle layer (Mohammadi & Nasirshoabi, 2015).

The dynamic response of structures to moving loads is a classical problem in bridge and railway engineering (Frýba, 1972). For single beams, closed-form and semi-analytical solutions are available for moving forces with general boundary conditions (Abu-Hilal & Zibdeh, 2000). Moving masses, whose inertia couples the load to the structure, have been treated analytically (Michaltsos et al., 1996; Gbadeyan & Oni, 1995) and numerically (Esen, 2011; Chen et al., 2014). The two-parameter Pasternak foundation, which adds a shear-interaction term to the Winkler model, has been shown to change both the natural frequencies and the moving-load response of beams considerably (Uzzal et al., 2012; Obara, 2014; Rahbar-Ranji & Shahbaztabar, 2017; Abdelghany et al., 2015).

For double-beam systems, Abu-Hilal (2006) obtained the response of a damped double Euler–Bernoulli beam to a moving constant load. Jiang et al. (2019) extended the analysis to successive moving loads, and Yang et al. (2019) considered several symmetric boundary conditions, while Lai et al. (2018) analyzed multiple supported beams. Nearly all of these studies idealize the load as a moving force, and neglect its inertia. Moving-mass problems for double beams are fewer, largely because the inertia terms introduce time-varying coefficients into the coupled equations. Gbadeyan and Hammed (2017) combined a generalized finite integral transform, a modified Struble technique, and the differential transform method (DTM) to study a moving mass on a viscoelastically connected double Rayleigh beam. Hammed et al. (2020) applied a similar approach to a double beam with elastically restrained ends, and Gbadeyan and Agboola (2012) treated a partially distributed moving load on a double Rayleigh beam. More recently, Hammed et al. (2025) analyzed a double beam with a linear Pasternak middle layer under a travelling distributed load, but again without load inertia.

The DTM, introduced by Zhou (1986), expresses the solution of a differential equation as a Taylor series whose coefficients are generated recursively. It has been applied widely to beam vibration problems (Ho &

Chen, 1998; Attarnejad et al., 2010; Agboola et al., 2017). However, a single truncated series is accurate only near its expansion point. For moving-load problems, where the response must be computed over the whole traverse time $0 \leq t \leq L/v$, and the product $\omega_m L/v$ is usually much greater than unity, a short series cannot represent the oscillatory response. The multi-step DTM (MsDTM) of Odibat et al. (2010) overcomes this limitation by applying the transform on a sequence of short sub-intervals.

This study addresses a double Euler–Bernoulli beam system connected by a viscoelastic Pasternak-type layer, comprising Winkler springs, a shear layer and viscous dampers, and traversed by a moving mass. Compared with Gbadeyan, and Hammed (2017), and Hammed et al. (2020, 2025).

Mathematical formulation

Problem description

The system, shown in Fig. 1, consists of two identical, parallel, uniform Euler–Bernoulli beams of length L , flexural rigidity EI and mass per unit length μ . Both beams are simply supported at $x = 0$ and $x = L$. They are joined by a continuous viscoelastic Pasternak-type layer, which is modelled as distributed linear springs of stiffness k_1 and viscous dampers of coefficient μ_0 acting in parallel, together with a shear layer of parameter R_0 that transmits transverse shear between adjacent springs. A concentrated mass M_L enters the upper beam at $x = 0$ when $t = 0$ and travels along it at constant speed v . The load is assumed to remain in contact with the beam, and the deflections are assumed to be small.

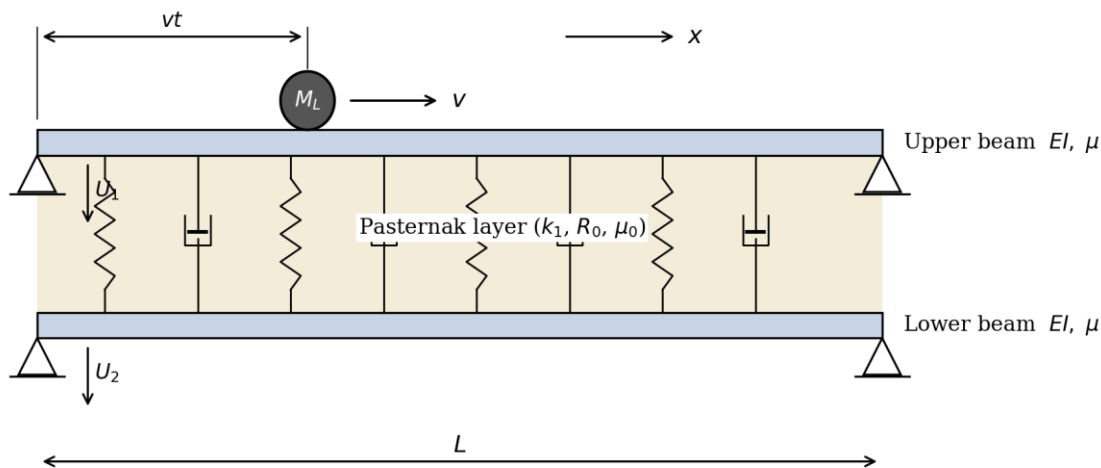


Fig. 1. Double Euler–Bernoulli beam system interconnected by a viscoelastic Pasternak-type layer and traversed by a moving mass.

Governing equations

The transverse deflections $U_1(x, t)$ and $U_2(x, t)$ of the upper and lower beams satisfy

$$EI \frac{\partial^4 U_1}{\partial x^4} + \mu \frac{\partial^2 U_1}{\partial t^2} + k_1(U_1 - U_2) - R_0 \frac{\partial^2}{\partial x^2}(U_1 - U_2) + \mu_0 \frac{\partial}{\partial t}(U_1 - U_2) = F(x, t), \quad (1)$$

$$EI \frac{\partial^4 U_2}{\partial x^4} + \mu \frac{\partial^2 U_2}{\partial t^2} - k_1(U_1 - U_2) + R_0 \frac{\partial^2}{\partial x^2}(U_1 - U_2) - \mu_0 \frac{\partial}{\partial t}(U_1 - U_2) = 0, \quad (2)$$

where the force exerted by the moving mass on the upper beam is

$$F(x, t) = M_L \left[g - \left(\frac{\partial^2 U_1}{\partial t^2} + 2v \frac{\partial^2 U_1}{\partial x \partial t} + v^2 \frac{\partial^2 U_1}{\partial x^2} \right) \right] \delta(x - vt). \quad (3)$$

In Eq. (3), $M_L g$ is the gravitational force of the load. The three bracketed terms are the local, Coriolis and centripetal components of the vertical acceleration of the mass, evaluated at the contact point $x = vt$, and $\delta(\cdot)$ is the Dirac delta distribution. Setting these three terms to zero gives the classical moving-force problem with constant load $P = M_L g$. For any function f that is continuous on $[0, L]$, the sifting property

$$\int_0^L f(x) \delta(x - vt) dx = f(vt), \quad 0 < vt < L, \quad (4)$$

is used below. This property is equivalent to expanding $\delta(x - vt)$ in the Fourier cosine series of its even extension on $[-L, L]$, as done by Frýba (1972), but it avoids the resulting double series.

2.3. Boundary and initial conditions

For simply supported ends, the boundary conditions are

$$U_i(0, t) = U_i(L, t) = 0, \quad \frac{\partial^2 U_i}{\partial x^2}(0, t) = \frac{\partial^2 U_i}{\partial x^2}(L, t) = 0, \quad i = 1, 2. \quad (5)$$

The system is assumed to be at rest when the load enters, so that

$$U_i(x, 0) = 0, \quad \frac{\partial U_i}{\partial t}(x, 0) = 0, \quad i = 1, 2. \quad (6)$$

Dimensionless parameters

The natural circular frequencies of a single simply supported beam and the classical critical speed are

$$\omega_m = \left(\frac{m\pi}{L} \right)^2 \sqrt{\frac{EI}{\mu}}, \quad m = 1, 2, \dots, \quad (7)$$

$$v_{cr} = \frac{\omega_1 L}{\pi}. \quad (8)$$

The results are presented in terms of the dimensionless speed, mass, stiffness, shear and damping parameters

$$\alpha = \frac{v}{v_{cr}}, \quad \varepsilon = \frac{M_L}{\mu L}, \quad K = \frac{k_1}{\mu \omega_1^2}, \quad (9)$$

$$S = \frac{R_0 \pi^2}{\mu L^2 \omega_1^2}, \quad \zeta = \frac{\mu_0}{2\mu \omega_1}.$$

The dimensionless load position is $\xi = vt/L \in [0, 1]$. Deflections are normalised by the static mid-span deflection of a single beam under $M_L g$,

$$\delta_{st} = \frac{M_L g L^3}{48EI}, \quad (10)$$

so that the ratio $U_i(L/2, t)/\delta_{st}$ is a dynamic amplification factor (DAF) referred to an isolated beam. Because the layer transfers part of the load to the lower beam, the upper-beam DAF of the connected system can be less than unity.

Method of solution

Modal reduction

The eigenfunctions $\sin(m\pi x/L)$ satisfy the boundary conditions (5) for both beams. The deflections are therefore expanded as

$$U_1(x, t) = \sum_{m=1}^{\infty} \phi_m(t) \sin \frac{m\pi x}{L}, \quad U_2(x, t) = \sum_{m=1}^{\infty} \beta_m(t) \sin \frac{m\pi x}{L}. \quad (11)$$

Eq. (11) is substituted into Eqs. (1)–(3), and each equation is multiplied by $\sin(j\pi x/L)$ and integrated over $0 \leq x \leq L$. Using the orthogonality relation

$$\int_0^L \sin \frac{m\pi x}{L} \sin \frac{j\pi x}{L} dx = \frac{L}{2} \delta_{mj}, \quad (12)$$

where δ_{mj} is the Kronecker delta, together with the sifting property (4), and dividing by $\mu L/2$, gives

$$\ddot{\phi}_j + \omega_j^2 \phi_j + \kappa_j(\phi_j - \beta_j) + \eta(\dot{\phi}_j - \dot{\beta}_j) + 2\varepsilon s_j(t) \sum_{m=1}^{\infty} [s_m(t) \ddot{\phi}_m + 2\theta_m c_m(t) \dot{\phi}_m - \theta_m^2 s_m(t) \phi_m] = 2\varepsilon g s_j(t), \quad (13)$$

$$\ddot{\beta}_j + \omega_j^2 \beta_j - \kappa_j(\phi_j - \beta_j) - \eta(\dot{\phi}_j - \dot{\beta}_j) = 0, \quad (14)$$

for $j = 1, 2, \dots$ and $0 \leq t \leq L/v$, where an overdot denotes differentiation with respect to t and

$$s_m(t) = \sin \theta_m t, \quad c_m(t) = \cos \theta_m t, \quad \theta_m = \frac{m\pi v}{L}, \quad (15)$$

$$\kappa_j = \frac{1}{\mu} \left(k_1 + R_0 \frac{j^2 \pi^2}{L^2} \right), \quad \eta = \frac{\mu_0}{\mu}.$$

The shear term contributes $+R_0(j\pi/L)^2(\phi_j - \beta_j)/\mu$ to the upper-beam equation and the equal and opposite term to the lower-beam equation, so the layer forces on the two beams are self-equilibrating. Eqs. (13) and (14) are coupled in two ways: through the layer terms, which couple each upper-beam mode to the same lower-beam mode, and through the inertia sum in Eq. (13), which couples all upper-beam modes through time-varying coefficients. When $\varepsilon \rightarrow 0$ with $M_L g$ held fixed, the inertia sum vanishes and Eqs. (13)–(14) reduce to the moving-force problem.

Modified frequency from Struble's technique

Following Gbadeyan and Oni (1995), only the diagonal term $m = j$ of the inertia sum is retained in the derivation of the modified frequency. The off-diagonal terms oscillate at the combination frequencies $\theta_m \pm \theta_j$ and do not produce secular terms at first order. With $2s_m^2 = 1 - \cos 2\theta_m t$ and $2s_m c_m = \sin 2\theta_m t$, the diagonal part of Eq. (13) becomes

$$[1 + \varepsilon(1 - \cos 2\theta_m t)] \ddot{\phi}_m + 2\varepsilon \theta_m \sin(2\theta_m t) \dot{\phi}_m + [\omega_m^2 - \varepsilon \theta_m^2(1 - \cos 2\theta_m t)] \phi_m + \mathcal{L}_m = 2\varepsilon g s_m(t), \quad (16)$$

where $\mathcal{L}_m = \kappa_m(\phi_m - \beta_m) + \eta(\dot{\phi}_m - \dot{\beta}_m)$ collects the layer terms. The Struble parameter is defined as

$$\lambda = \frac{\varepsilon}{1 + \varepsilon}, \quad 0 \leq \lambda < 1, \quad (17)$$

so that $\varepsilon = \lambda + O(\lambda^2)$ and

$$\frac{1}{1 + \varepsilon(1 - \cos 2\theta_m t)} = (1 - \lambda)[1 + \lambda \cos 2\theta_m t + O(\lambda^2)]. \quad (18)$$

To obtain the modified frequency, the layer terms and the load are temporarily removed. Dividing the homogeneous part of Eq. (16) by its inertia coefficient and using Eq. (18) gives

$$\ddot{\phi}_m + \omega_m^2 \phi_m + \lambda[2\theta_m \sin(2\theta_m t) \dot{\phi}_m - (\omega_m^2 + \theta_m^2)(1 - \cos 2\theta_m t) \phi_m] + O(\lambda^2) = 0. \quad (19)$$

For $\lambda = 0$, the solution of Eq. (19) is $\phi_m = A_m \cos(\omega_m t - a_m)$. Struble's technique (Struble, 1962) seeks the first-order solution in the form

$$\phi_m(t) = A_m(t) \cos[\omega_m t - a_m(t)] + \lambda \phi_m^{(1)}(t) + O(\lambda^2), \quad (20)$$

in which the amplitude A_m and phase a_m vary slowly, that is,

$$\dot{A}_m, \dot{a}_m = O(\lambda), \quad \ddot{A}_m, \ddot{a}_m = O(\lambda^2). \quad (21)$$

Writing $\psi_m = \omega_m t - a_m$ and retaining terms up to $O(\lambda)$, the derivatives of Eq. (20) are

$$\dot{\phi}_m = \dot{A}_m \cos \psi_m - A_m(\omega_m - \dot{a}_m) \sin \psi_m + \lambda \dot{\phi}_m^{(1)} + O(\lambda^2), \quad (22)$$

$$\ddot{\phi}_m = -(A_m \omega_m^2 - 2A_m \omega_m \dot{a}_m) \cos \psi_m - 2\omega_m \dot{A}_m \sin \psi_m + \lambda \ddot{\phi}_m^{(1)} + O(\lambda^2). \quad (23)$$

Eqs. (20), (22) and (23) are substituted into Eq. (19). The products of $\sin 2\theta_m t$ and $\cos 2\theta_m t$ with $\sin \psi_m$ and $\cos \psi_m$ oscillate at $\omega_m \pm 2\theta_m$. Provided that $2\theta_m$ is not close to $2\omega_m$, which for the fundamental mode requires α not close to unity, these products are non-secular and are absorbed into $\phi_m^{(1)}$. Equating the coefficients of the secular terms $\sin \psi_m$ and $\cos \psi_m$ to zero gives the variational equations

$$-2\omega_m \dot{A}_m = 0, \quad 2\omega_m A_m \dot{a}_m - \lambda(\omega_m^2 + \theta_m^2) A_m = 0. \quad (24)$$

Hence A_m is constant and

$$a_m(t) = \frac{\lambda(\omega_m^2 + \theta_m^2)}{2\omega_m} t + a_{m0}, \quad (25)$$

where a_{m0} is a constant. The first approximation to the free motion is therefore

$$\phi_m(t) = A_m \cos(\Omega_m t - a_{m0}), \quad (26)$$

with the modified frequency

$$\Omega_m = \omega_m \left[1 - \frac{\lambda}{2} \left(1 + \frac{\theta_m^2}{\omega_m^2} \right) \right] = \omega_m \left[1 - \frac{\lambda}{2} \left(1 + \frac{\alpha^2}{m^2} \right) \right]. \quad (27)$$

Eq. (27) shows that the inertia of the moving mass lowers every natural frequency of the upper beam, and that the reduction grows with both the mass ratio and the speed. When $\varepsilon \rightarrow 0$, $\Omega_m \rightarrow \omega_m$. Because the Coriolis term in Eq. (19) averages to zero over a cycle, the amplitude A_m remains constant at first order; the load inertia does not by itself introduce damping. To first order in λ , Ω_m^2 also equals $(\omega_m^2 - \varepsilon \theta_m^2)/(1 + \varepsilon)$, which is the stiffness-to-inertia ratio of Eq. (16) after averaging its coefficients over one period of $2\theta_m t$.

Reduced equations of motion

The homogeneous operator acting on ϕ_m is now replaced by the equivalent free-system operator with frequency Ω_m . The layer terms and the load, which in Eq. (16) are divided by the same inertia coefficient, are restored with its time average, $1 - \lambda = 1/(1 + \varepsilon)$. The reduced equations for the m th mode are

$$\ddot{\phi}_m + \Omega_m^2 \phi_m + (1 - \lambda) [\kappa_m (\phi_m - \beta_m) + \eta (\dot{\phi}_m - \dot{\beta}_m)] = (1 - \lambda) F_0 \sin \theta_m t, \quad (28)$$

$$\ddot{\beta}_m + \omega_m^2 \beta_m - \kappa_m (\phi_m - \beta_m) - \eta (\dot{\phi}_m - \dot{\beta}_m) = 0, \quad (29)$$

with zero initial conditions and

$$F_0 = 2\varepsilon g = \frac{2M_L g}{\mu L}. \quad (30)$$

The factor $1 - \lambda$ ensures that the reduced model reproduces the exact static deflection in the limit $\alpha \rightarrow 0$, since then $\Omega_m^2 \rightarrow (1 - \lambda)\omega_m^2$ to first order in λ . Eqs. (28) and (29) are uncoupled across modes, but each pair remains coupled through the layer. For $\varepsilon = 0$, one has $\lambda = 0$ and $\Omega_m = \omega_m$, and Eqs. (28)–(29) become the moving-force equations of the system.

3.4. Multi-step differential transform method

Let the interval $[0, T]$, where $T = L/v$ is the traverse time, be divided into n sub-intervals $[t_i, t_{i+1}]$ of equal length $h = T/n$. On each sub-interval, the differential transform of an analytic function $w(t)$ and its inverse are defined as (Zhou, 1986; Odibat et al., 2010)

$$\bar{W}(k) = \frac{1}{k!} \left[\frac{d^k w(t)}{dt^k} \right]_{t=t_i}, \quad w(t) = \sum_{k=0}^{\infty} \bar{W}(k) (t - t_i)^k, \quad t_i \leq t \leq t_{i+1}. \quad (31)$$

In practice, the series in Eq. (31) is truncated after $N + 1$ terms. The operational properties used in this work are listed in Table 1.

Table 1. Differential transforms of basic operations on the sub-interval $[t_i, t_{i+1}]$.

Original function $w(t)$	Transformed function $\bar{W}(k)$
$u(t) \pm y(t)$	$\bar{U}(k) \pm \bar{Y}(k)$
$c u(t)$	$c \bar{U}(k)$
du/dt	$(k + 1) \bar{U}(k + 1)$
d^2u/dt^2	$(k + 1)(k + 2) \bar{U}(k + 2)$
$u(t) y(t)$	$\sum_{r=0}^k \bar{U}(r) \bar{Y}(k - r)$
$\sin(\theta t)$	$\frac{\theta^k}{k!} \sin\left(\theta t_i + \frac{k\pi}{2}\right)$
$\cos(\theta t)$	$\frac{\theta^k}{k!} \cos\left(\theta t_i + \frac{k\pi}{2}\right)$

Applying Table 1 to Eqs. (28) and (29) gives the recurrence relations

$$\bar{\phi}_m(k + 2) = \frac{(1 - \lambda) [\bar{F}_m(k) - \kappa_m \Delta_m(k) - \eta(k + 1) \Delta_m(k + 1)] - \Omega_m^2 \bar{\phi}_m(k)}{(k + 1)(k + 2)}, \quad (32)$$

$$\bar{\beta}_m(k+2) = \frac{\kappa_m \Delta_m(k) + \eta(k+1) \Delta_m(k+1) - \omega_m^2 \bar{\beta}_m(k)}{(k+1)(k+2)}, \quad (33)$$

for $k = 0, 1, \dots, N-2$, where;

$$\Delta_m(k) = \bar{\phi}_m(k) - \bar{\beta}_m(k), \quad \bar{F}_m(k) = F_0 \frac{\theta_m^k}{k!} \sin\left(\theta_m t_i + \frac{k\pi}{2}\right). \quad (34)$$

The recurrences are started on each sub-interval from the state at its left end,

$$\begin{aligned} \bar{\phi}_m(0) &= \phi_m(t_i), & \bar{\phi}_m(1) &= \dot{\phi}_m(t_i), \\ \bar{\beta}_m(0) &= \beta_m(t_i), & \bar{\beta}_m(1) &= \dot{\beta}_m(t_i), \end{aligned} \quad (35)$$

with $\phi_m(0) = \dot{\phi}_m(0) = \beta_m(0) = \dot{\beta}_m(0) = 0$ from Eq. (6). The state is then advanced by

$$\phi_m(t_{i+1}) = \sum_{k=0}^N \bar{\phi}_m(k) h^k, \quad \dot{\phi}_m(t_{i+1}) = \sum_{k=1}^N k \bar{\phi}_m(k) h^{k-1}, \quad (36)$$

and similarly, for β_m . On the first sub-interval ($t_0 = 0$) the moving-force case gives $\bar{\phi}_m(2) = \bar{\beta}_m(2) = \bar{\beta}_m(3) = 0$ and $\bar{\phi}_m(3) = F_0 \theta_m / 6$. The first non-zero coefficients are listed in Appendix A.

If a single sub-interval covering the whole traverse is used ($n = 1$), the truncation error is of order $(\hat{\omega}T)^{N+1} / (N+1)!$, where $\hat{\omega}$ is the highest frequency present. Since $\omega_1 T = \pi / \alpha$, this error is large for all practical speeds, as Section 4.1 confirms. The step length is therefore chosen so that

$$\hat{h} = h \left(\sqrt{\max(\omega_M, \Omega_M)^2 + 2\kappa_M + \theta_M} \right) \leq 0.5, \quad (37)$$

where M is the highest retained mode. The mid-span deflections are finally obtained from

$$U_1\left(\frac{L}{2}, t\right) = \sum_{m=1,3,\dots}^M \phi_m(t) \sin \frac{m\pi}{2}, \quad U_2\left(\frac{L}{2}, t\right) = \sum_{m=1,3,\dots}^M \beta_m(t) \sin \frac{m\pi}{2}. \quad (38)$$

Even-numbered modes vanish at mid-span. With $M = 9$ (five odd modes), the peak mid-span DAF changes by less than 0.01% when M is increased to 19.

3.5. Exact solution of the moving-force problem

For $\varepsilon = 0$, the two beams have the same modal frequency and Eqs. (28)–(29) decouple exactly in the in-phase and out-of-phase coordinates

$$\sigma_m = \phi_m + \beta_m, \quad d_m = \phi_m - \beta_m. \quad (39)$$

Adding and subtracting Eqs. (28) and (29) gives

$$\begin{aligned} \ddot{\sigma}_m + \omega_m^2 \sigma_m &= F_0 \sin \theta_m t, \\ \ddot{d}_m + 2\eta \dot{d}_m + \Gamma_m^2 d_m &= F_0 \sin \theta_m t, \quad \Gamma_m^2 = \omega_m^2 + 2\kappa_m. \end{aligned} \quad (40)$$

The in-phase motion is therefore unaffected by the layer, which acts only on the relative motion of the two beams. With zero initial conditions and $\theta_m \neq \omega_m$, the solutions are

$$\sigma_m(t) = \frac{F_0}{\omega_m^2 - \theta_m^2} \left[\sin \theta_m t - \frac{\theta_m}{\omega_m} \sin \omega_m t \right], \quad (41)$$

$$\begin{aligned} d_m(t) = D_m \sin(\theta_m t - \vartheta_m) &+ e^{-\eta t} \left[D_m \sin \vartheta_m \cos \gamma_m t \right. \\ &\left. + \frac{D_m (\eta \sin \vartheta_m - \theta_m \cos \vartheta_m)}{\gamma_m} \sin \gamma_m t \right], \end{aligned} \quad (42)$$

where

$$D_m = \frac{F_0}{\sqrt{(\Gamma_m^2 - \theta_m^2)^2 + 4\eta^2 \theta_m^2}}, \quad (43)$$

$$\tan \vartheta_m = \frac{2\eta \theta_m}{\Gamma_m^2 - \theta_m^2}, \quad \gamma_m = \sqrt{\Gamma_m^2 - \eta^2}.$$

The modal amplitudes are recovered as $\phi_m = (\sigma_m + d_m)/2$ and $\beta_m = (\sigma_m - d_m)/2$. At resonance ($\theta_m = \omega_m$), Eq. (41) is replaced by its limit $\sigma_m = F_0(\sin \omega_m t - \omega_m t \cos \omega_m t)/(2\omega_m^2)$. Eqs. (39)–(43) provide an independent benchmark for the MsDTM.

RESULTS AND DISCUSSION

All computations were carried out in Python (NumPy and SciPy). The baseline parameters and the ranges considered are listed in Table 2. Because the results are expressed in the dimensionless groups of Eq. (9), they apply to any beam with the same parameter values. Unless stated otherwise, the MsDTM uses $N = 12$ terms per step and the step criterion (37).

Table 2. Dimensionless parameters used in the numerical study.

Parameter	Symbol	Baseline value	Values examined
Speed parameter	α	0.3	0.1, 0.3, 0.5 (0.05–0.95 in Fig. 4)
Mass ratio	ε	0.15	0.05, 0.15, 0.25
Winkler stiffness parameter	K	1.0	0, 1, 5
Shear parameter	S	0.5	0, 0.5, 1.0
Damping parameter	ζ	0.05	0.05, 0.25, 0.50
Retained modes	M	9	1–19 (convergence check)

Validation and convergence of the MsDTM

Table 3 lists the maximum absolute error of the MsDTM in the upper-beam mid-span DAF, relative to the exact solution of Section 3.5, for the baseline moving-force case. The error decreases rapidly as the number of terms N increases and as the normalised step $\hat{h}$ decreases. With $N = 12$ and $\hat{h} = 0.5$, the error is at the level of machine precision. Fig. 2(a) confirms that the MsDTM and exact solutions coincide for both beams over the whole traverse.

Table 3. Maximum absolute error of the MsDTM in $U_1(L/2, t)/\delta_{st}$ relative to the exact moving-force solution (baseline parameters).

N	$\hat{h} = 2.0$	$\hat{h} = 1.0$	$\hat{h} = 0.5$	$\hat{h} = 0.25$
4	1.7×10^{-2}	3.3×10^{-3}	4.4×10^{-4}	5.5×10^{-5}
6	1.9×10^{-3}	6.7×10^{-5}	2.1×10^{-6}	6.2×10^{-8}
8	1.0×10^{-4}	9.6×10^{-7}	7.8×10^{-9}	6.0×10^{-11}
10	3.4×10^{-6}	8.1×10^{-9}	1.7×10^{-11}	3.2×10^{-14}
12	7.6×10^{-8}	4.6×10^{-11}	2.4×10^{-14}	7.8×10^{-16}

By contrast, a single DTM series truncated at t^6 and applied over the whole traverse fails almost immediately (Table 4 and the dotted curve in Fig. 2(a)). By the time the load reaches $\xi = 0.1$, the truncated series already predicts a deflection more than three orders of magnitude larger than the exact value and of the wrong sign,

and at the far support its magnitude exceeds 10^6 , about seven orders of magnitude above the exact value. Deflections obtained from such a single short series, whatever the parameter values, cannot therefore be relied upon.

Table 4. Upper-beam mid-span DAF from the exact solution and from a single DTM series truncated at t^6 (baseline moving-force case).

Load position ξ	Exact	Single-step DTM (t^6)
0.1	0.0398	-1.36×10^2
0.2	0.2522	-4.19×10^3
0.3	0.5349	-3.06×10^4
0.5	0.8308	-3.63×10^5
1.0	0.1365	-9.15×10^6

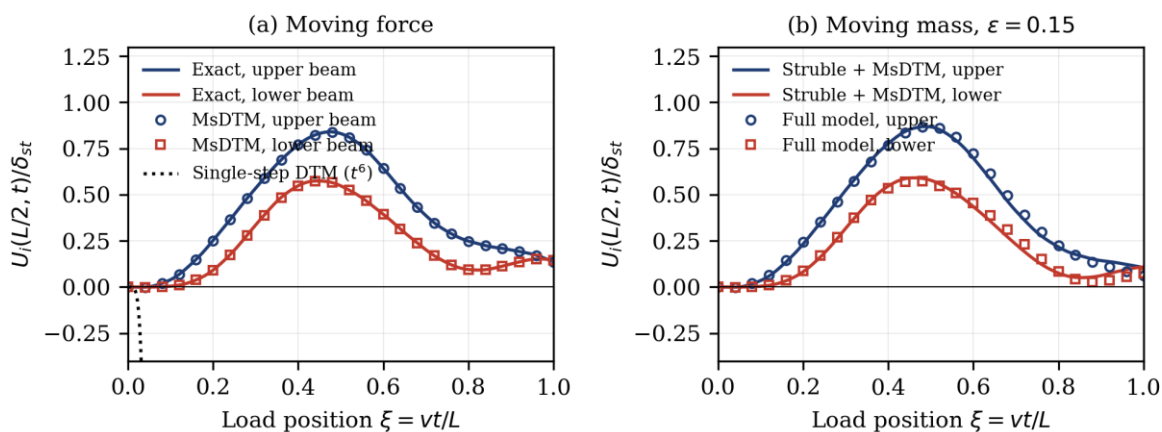


Fig. 2. Validation of the solution procedure (baseline parameters). (a) Moving force: exact solution, MsDTM and a single-step DTM series truncated at t^6 . (b) Moving mass with $\varepsilon = 0.15$: Struble-reduced model solved by MsDTM compared with the full time-varying-inertia model.

Accuracy of the Struble reduction

The Struble-reduced model, Eqs. (28)–(29), was compared with a full model that retains all ten modes of Eq. (13), including the complete time-varying inertia matrix. The full model was integrated with an eighth-order Runge–Kutta scheme (DOP853) at a relative tolerance of 10^{-9} ; doubling the number of modes changed its peak response by less than 2×10^{-4} . Fig. 2(b) shows close agreement between the two models over the entire traverse. Table 5 summarises the peak DAFs. For $\alpha \leq 0.5$ and $\varepsilon \leq 0.25$, the Struble-reduced model reproduces the peak DAF of the full model to within 3.5% for the upper beam and 4.6% for the lower beam. At higher speeds the reduction becomes less accurate: it underestimates the upper-beam peak by up to about 8% for $0.5 < \alpha < 1$ (Fig. 4). This is consistent with the first-order nature of Eq. (27) and with the approach of the parametric resonance condition $\theta_1 \rightarrow \omega_1$.

Table 5. Peak mid-span DAF of the moving-mass problem: Struble-reduced model (MsDTM) versus full model, with the moving-force value for reference.

α	ε	Upper: Struble	Upper: full	Upper: force	Lower: Struble	Lower: full	Lower: force
0.1	0.05	0.675	0.674	0.676	0.420	0.418	0.421
0.1	0.15	0.671	0.668	0.676	0.418	0.413	0.421

0.1	0.25	0.664	0.661	0.676	0.414	0.406	0.421
0.3	0.05	0.851	0.851	0.840	0.581	0.574	0.574
0.3	0.15	0.869	0.871	0.840	0.591	0.574	0.574
0.3	0.25	0.883	0.890	0.840	0.599	0.573	0.574
0.5	0.05	0.964	0.970	0.949	0.767	0.760	0.758
0.5	0.15	0.990	1.010	0.949	0.781	0.763	0.758
0.5	0.25	1.011	1.047	0.949	0.791	0.765	0.758

Limitations of the first-order reduction. The Struble reduction has three limitations that define its range of use. First, Eq. (27) is accurate only to $O(\lambda)$, and the neglected $O(\lambda^2)$ terms grow with the mass ratio. Second, the non-secular terms discarded between Eqs. (19) and (24) oscillate at $\omega_m \pm 2\Theta_m$; they become resonant as $\Theta_m \rightarrow \omega_m$, that is, as $\alpha \rightarrow 1$ for the fundamental mode, and the assumption in Eq. (21) that the amplitude varies slowly then fails. Third, the inertia of the load lowers the frequency, so the onset of this resonant region moves to lower speeds. Setting $\Theta_1 = \Omega_1$ in Eq. (27) gives the first-order estimate of the modified critical speed ratio, the root of $\alpha = 1 - \lambda^2 / (1 + \alpha^2)$, which is $\alpha \approx 0.95$, 0.88 and 0.83 for $\varepsilon = 0.05$, 0.15 and 0.25, respectively. The reduction can therefore be expected to degrade before the classical critical speed is reached, which is consistent with the error growth beyond $\alpha = 0.5$ in Fig. 4.

The following guidance is proposed. (i) For $\alpha \leq 0.5$ and $\varepsilon \leq 0.25$, the first-order model is adequate for design-level estimates; its peak-DAF error was at most 4.6% in Table 5. (ii) For $0.5 < \alpha \leq 0.8$ or $\varepsilon > 0.25$, it should be used only as a preliminary estimate and verified against the full model, Eq. (13). (iii) Near and above the modified critical speed ratio, higher-order treatment is required. Options are a second-order Struble or multiple-scales expansion that retains the $O(\lambda^2)$ terms and allows the amplitude to vary, a Floquet analysis of the parametrically excited system, Eq. (16), or direct numerical integration of the full model, which is inexpensive for the small number of modes needed here.

Effect of load speed

Fig. 3 shows the mid-span deflection histories for, $\alpha = 0.1$, 0.3 and 0.5, and Fig. 4 shows the peak DAF as a function of speed for $\varepsilon = 0.25$. At the lowest speed, the response is quasi-static: it follows the static influence line with small superimposed oscillations from the system's free vibration. The moving-mass and moving-force curves are then almost identical, and the load inertia in fact reduces the upper-beam peak slightly (0.671 against 0.676 in Table 5). As the speed increases, the peak DAF rises and shifts towards the far support. For the upper beam under a moving force, it reaches a maximum of about 0.95 near $\alpha = 0.5$ (Fig. 4(a)). The lower beam, which is excited only through the layer, has a peak DAF that continues to rise up to $\alpha \approx 0.8$ (Fig. 4(b)).

For $\alpha \gtrsim 0.2$, the moving-mass response of the upper beam exceeds the moving-force response throughout. In the full model with $\varepsilon = 0.25$, the upper-beam peak is about 10% higher at $\alpha = 0.5$ and about 21% higher at $\alpha = 0.95$. The excess arises from the reduction of the upper-beam frequencies by the load inertia, Eq. (27), which brings the system closer to resonance. The lower beam is much less sensitive to load inertia: in the full model its peak DAF stays within 1% of the moving-force value for $\alpha \leq 0.5$ and exceeds it by at most about 7% (near $\alpha = 0.8$). The moving-force idealisation is therefore non-conservative mainly for the loaded beam, at moderate and high speeds.

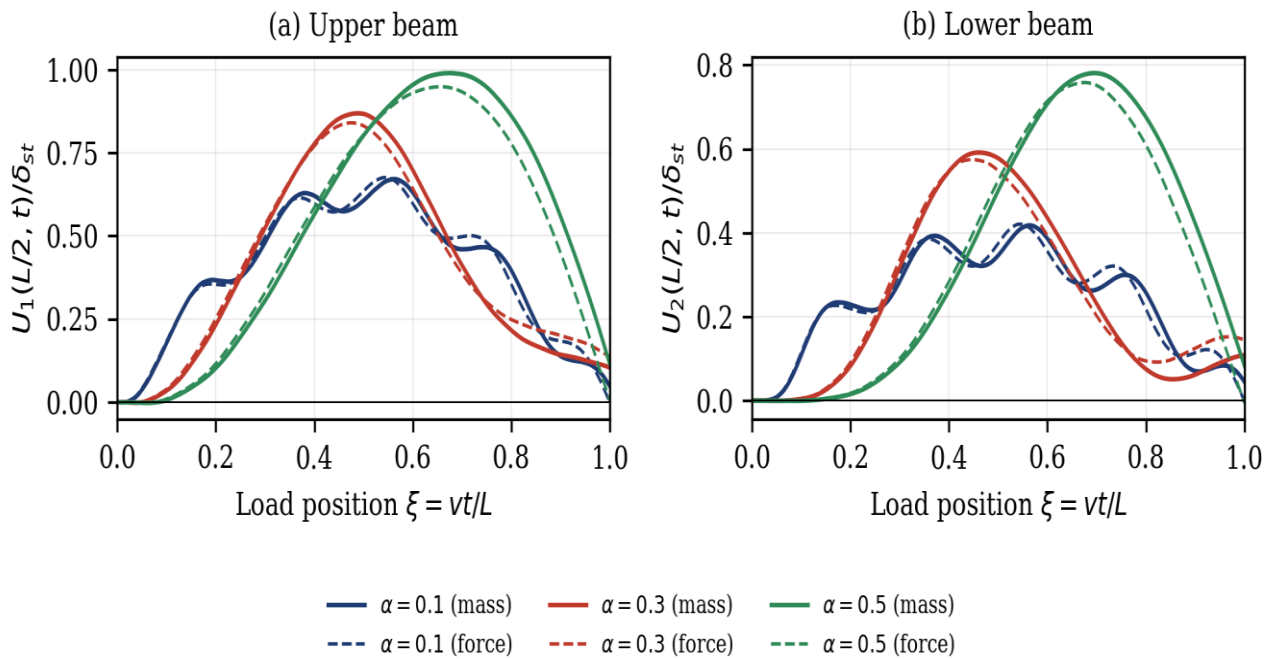


Fig. 3. Mid-span deflection histories for different speed parameters: (a) upper beam; (b) lower beam. Solid lines, moving mass ($\epsilon = 0.15$); dashed lines, moving force.

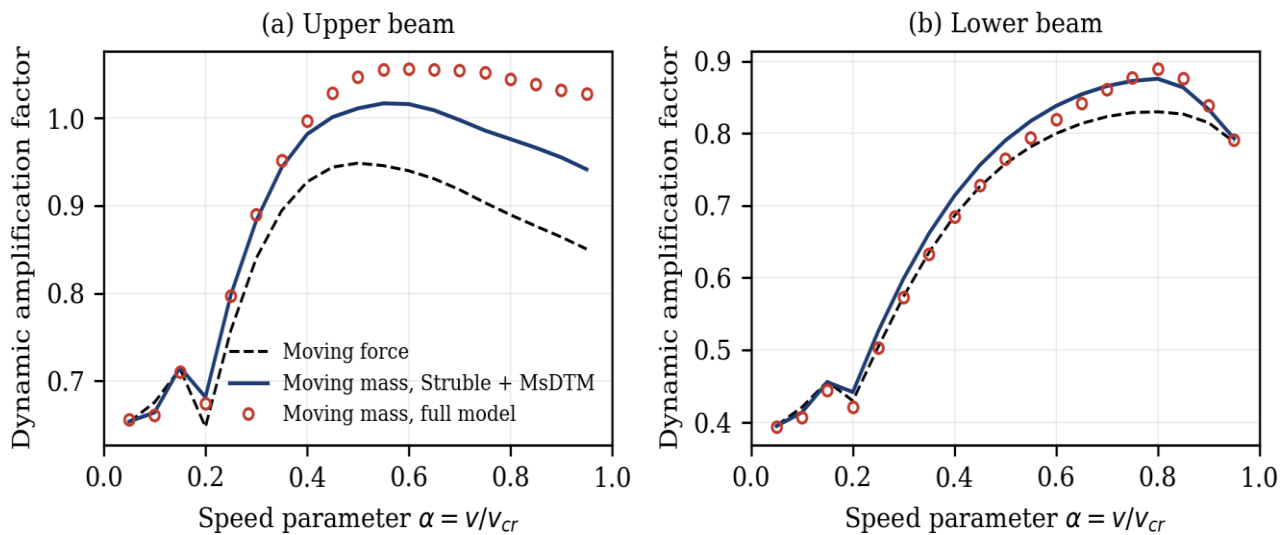


Fig. 4. Peak dynamic amplification factor versus speed parameter for $\epsilon = 0.25$: (a) upper beam; (b) lower beam.

Effect of mass ratio

Fig. 5 shows the effect of the mass ratio at $\alpha = 0.3$. Increasing ϵ from 0.05 to 0.25 raises the upper-beam peak DAF from 0.851 to 0.883, which is 1.3% and 5.1% above the moving-force value of 0.840. The Struble-reduced model also predicts a small rise in the lower-beam peak, from 0.581 to 0.599. However, the full model (Table 5) shows that the lower-beam peak is practically insensitive to ϵ at this speed, so this rise is mainly a consequence of the first-order reduction. The peak also occurs later, moving from $\xi = 0.480$ to $\xi = 0.495$, because the lower modified frequency lengthens the response period. After the peak, the heavier load produces a lower residual deflection as it approaches the far support, which reflects the phase shift of the free-vibration component. These trends agree with the frequency-reduction mechanism described by Eq. (27). They also show that, at practical speeds, neglecting load inertia underestimates the deflection of the loaded beam.

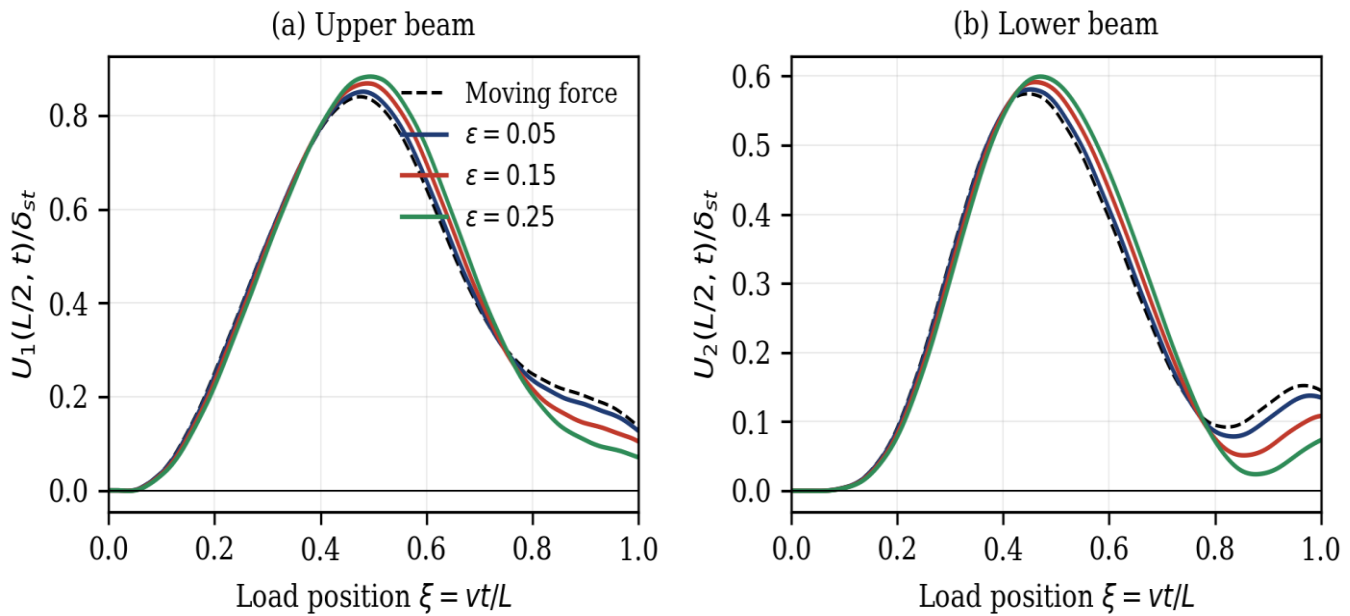


Fig. 5. Effect of the mass ratio on the mid-span deflection histories ($\alpha = 0.3$): (a) upper beam; (b) lower beam. The dashed curve is the moving-force solution.

Effect of the shear parameter

Fig. 6 shows the influence of the shear parameter S of the Pasternak layer. Increasing S from 0 to 1 reduces the upper-beam peak DAF under the moving mass from 0.902 to 0.844 (−6.4%) and increases the lower-beam peak from 0.565 to 0.609 (+7.8%). The moving-force results follow the same trend. According to Eq. (15), the shear layer adds the mode-dependent stiffness $R_0(m\pi/L)^2/\mu$ to the coupling coefficient κ_m . It therefore stiffens the relative (out-of-phase) motion, particularly in the higher modes, and transfers a larger share of the load to the lower beam. The in-phase motion is unaffected, as Eq. (40) shows. Consequently, the shear interaction redistributes the response between the two beams rather than simply suppressing it.

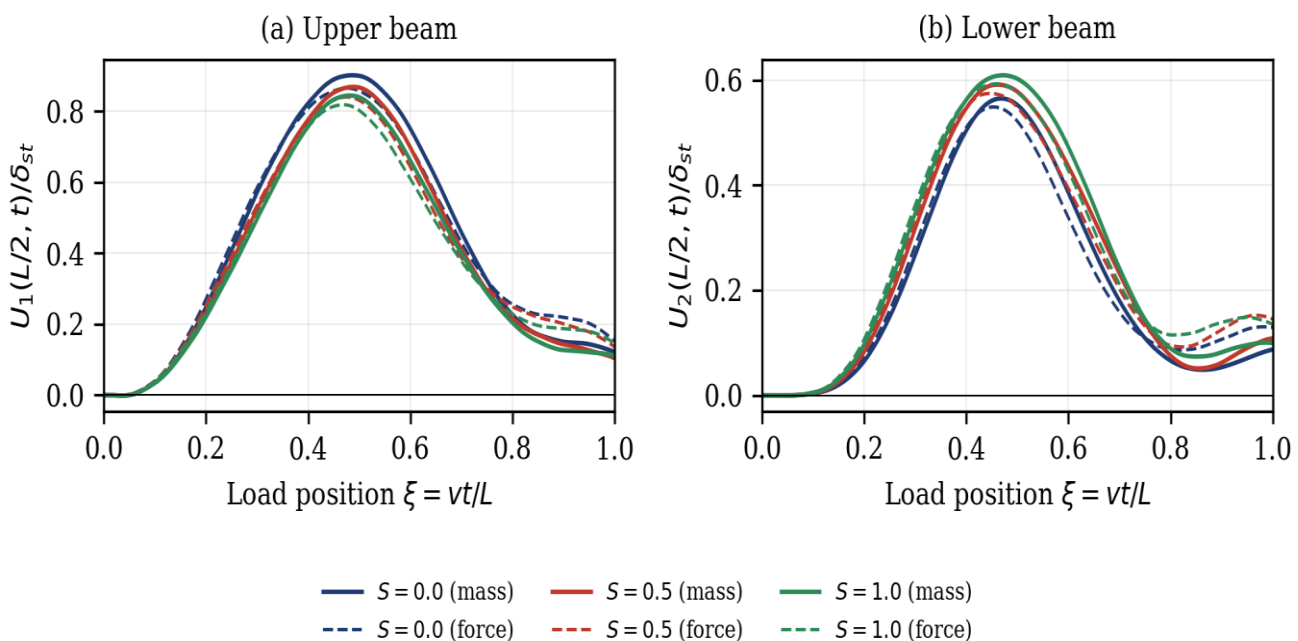


Fig. 6. Effect of the shear parameter S on the mid-span deflection histories: (a) upper beam; (b) lower beam. Solid lines, moving mass; dashed lines, moving force.

Effect of the Winkler stiffness

The Winkler stiffness has the strongest influence of all the layer parameters (Fig. 7). When K increases from 0 to 5, the upper-beam peak DAF under the moving mass falls from 1.006 to 0.775 (−23%), while the lower-beam peak rises from 0.473 to 0.674 (+42%). In the limit of a very stiff layer, the two beams move together, and each carry approximately half of the load, so that both DAFs approach one half of the isolated-beam value at low speed. In the opposite limit $K = S = 0$, the upper beam behaves as an isolated beam and the lower beam is excited only through the dampers. For every value of K considered, the moving-mass peak exceeds the moving-force peak by about 3–5%.

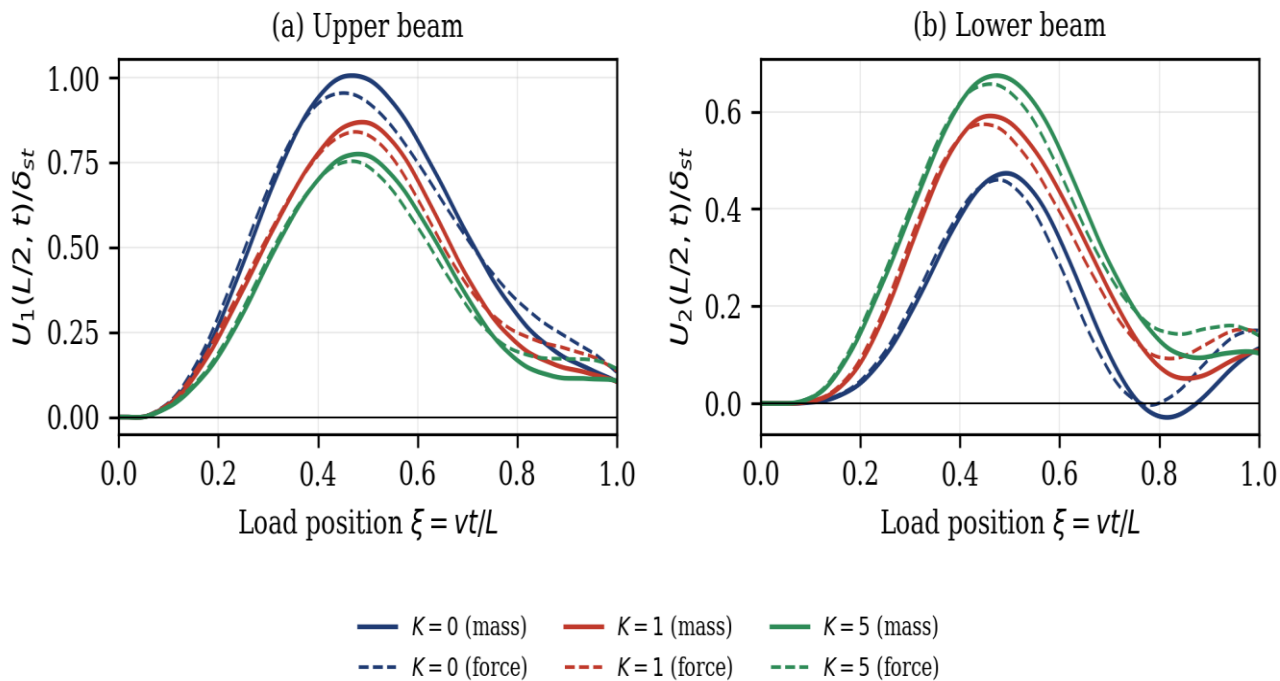


Fig. 7. Effect of the Winkler stiffness parameter K on the mid-span deflection histories: (a) upper beam; (b) lower beam. Solid lines, moving mass; dashed lines, moving force.

Effect of layer damping

Fig. 8 shows that the layer damping has only a weak influence on the peak response during a single passage. Increasing ζ tenfold, from 0.05 to 0.50, reduces the upper-beam peak DAF by about 1% (from 0.869 to 0.860) and increases the lower-beam peak by about 0.6%. The dampers act only on the relative motion, Eq. (40). Their main effect is to transfer energy from the upper to the lower beam and to attenuate the out-of-phase oscillation that follows the peak, which is visible near the far support. Over the short traverse time considered here, the in-phase component, which dominates the peak and is undamped in the present model, controls the maximum deflection. Damping therefore becomes important mainly for the free vibration after the load has left the span and for repeated loading, which are beyond the scope of this study. Repeated loading and fatigue. The present results concern a single passage and should not be extrapolated to cumulative effects. The layer dampers act only on the relative motion of the beams, Eq. (40), so the in-phase component is undamped in the present model, and material damping and support losses are not included. The residual vibration left after one passage therefore decays only through the out-of-phase component. For successive loads, such as trains or vehicle streams with headway T_h the residual vibration of one passage can add to the response to the next, and the largest build-up is expected when T_h is commensurate with the free-vibration period; this expectation is an inference and has not been evaluated here. Fatigue assessment additionally requires stress-range histories and cycle counting for the response during and after each passage. The step-by-step structure of the MsDTM accommodates repeated loading naturally, and this extension is recommended for future work together with the inclusion of beam damping.

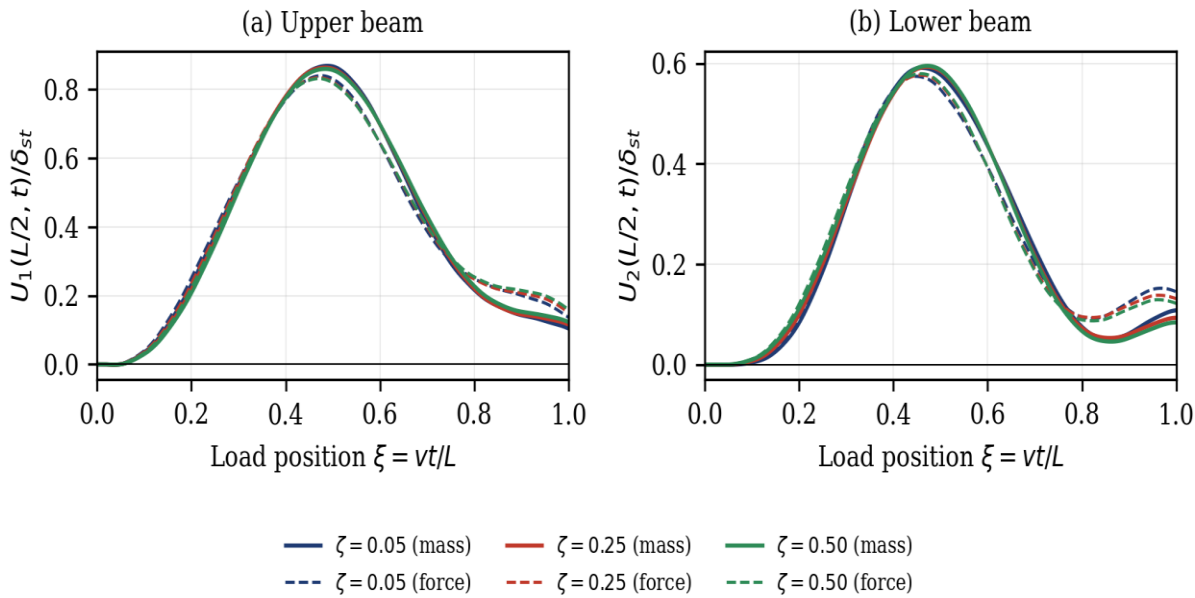


Fig. 8. Effect of the damping parameter ζ on the mid-span deflection histories: (a) upper beam; (b) lower beam. Solid lines, moving mass; dashed lines, moving force.

Table 6 summarizes the parametric effects on the peak DAF at the baseline speed $\alpha = 0.3$.

Table 6. Peak mid-span DAFs of the moving-mass (Struble–MsDTM) and moving-force (exact) problems for the parametric cases ($\alpha = 0.3$, $\varepsilon = 0.15$ unless varied).

Parameter varied	Value	Upper: mass	Upper: force	Lower: mass	Lower: force
S	0	0.902	0.865	0.565	0.549
	0.5	0.869	0.840	0.591	0.574
	1.0	0.844	0.818	0.609	0.593
K	0	1.006	0.955	0.473	0.460
	1	0.869	0.840	0.591	0.574
	5	0.775	0.754	0.674	0.657
ζ	0.05	0.869	0.840	0.591	0.574
	0.25	0.864	0.834	0.593	0.577
	0.50	0.860	0.832	0.595	0.580

Implications for Design

The results give the following screening guidance for simply supported, identical double-beam systems such as floating-slab tracks and double-deck girders, in which the load is idealized as a moving mass M_L and $\varepsilon = M_L/(\mu L)$ is the ratio of load mass to the mass of one beam.

- Low speed**, $\alpha < 0.2$. The moving-force model is adequate; the two load models differ by about 2% or less in Table 5.
- Moderate speed**, $0.2 \leq \alpha \leq 0.5$. The moving-force model under-predicts the peak response of the loaded (upper) beam. At $\alpha = 0.5$, the ratio of the full-model peak to the moving-force peak is 1.02, 1.06 and 1.10 for $\varepsilon = 0.05$, 0.15 and 0.25 (Table 5), which is reproduced to within 0.5% by the amplification factor $1 + 0.4\varepsilon$. At $\alpha = 0.3$ the same factor is conservative, since the ratio is 1.01–1.06, and it also covers

the layer-parameter cases in Table 6 (ratios 1.03–1.05 at $\varepsilon=0.15$). Moving-force results for the upper beam should therefore be multiplied by at least $1+0.4\varepsilon$, about 1.10 for a load mass equal to a quarter of the beam mass. In the tabulated cases the lower-beam peak changes by less than 1% in this speed range, so no additional margin on load inertia is needed for that member.

3. **High speed**, $\alpha>0.5$. The under-prediction by the moving-force model reaches about 10% at $\alpha=0.5$ and 21% at $\alpha=0.95$ for $\varepsilon=0.25$, and the lower-beam peak can exceed its moving-force value by up to about 7%. The full model should be used, and the design speed should be kept well below the modified critical speed ratio estimated in Section 4.2.
4. **Layer stiffness**. Stiffening the layer relieves the loaded beam but loads the other one: raising K from 0 to 5 reduces the upper-beam peak by 23% and raises the lower-beam peak by 42%. Both members must therefore be checked when the layer stiffness is selected, and the shear parameter S acts in the same direction.

These figures apply to the parameter ranges studied (Table 2) and to a single passage. They are screening values for preliminary design and do not replace the dynamic factors and load models of the applicable design codes. The dimensionless groups of Eq. (9) allow a project to be mapped onto Table 2, but the margins should be confirmed with a project-specific model before they are adopted.

CONCLUSIONS

The dynamic response of a simply supported double Euler–Bernoulli beam system interconnected by a viscoelastic Pasternak-type layer and traversed by a moving mass has been analyzed using a Galerkin modal reduction, Struble’s asymptotic technique and a multi-step differential transform method. The main conclusions are as follows.

1. A consistent first-order Struble reduction yields the explicit modified frequency $\Omega_m = \omega_m[1 - \lambda(1 + \alpha^2/m^2)/2]$ and a load term scaled by $1/(1 + \varepsilon)$. The resulting model reproduces the peak DAFs of a full time-varying-inertia model to within 3.5% (upper beam) and 4.6% (lower beam) for $\alpha \leq 0.5$ and $\varepsilon \leq 0.25$; its accuracy for the upper beam decreases to about 8% as α approaches unity.
2. The MsDTM converges rapidly and agrees with the exact moving-force solution, derived here in in-phase and out-of-phase coordinates, to machine precision. A single truncated DTM series diverges before the load has travelled one-tenth of the span, so a multi-step or equivalent approach is essential for moving-load problems.
3. Load inertia increases the peak response of the upper beam relative to the moving-force idealization, by up to about 10% at $\alpha = 0.5$ and 21% at $\alpha = 0.95$ for $\varepsilon = 0.25$, and it delays the time of peak response. At very low speeds ($\alpha \approx 0.1$), the two load models differ by less than 2%.
4. Increasing the Winkler stiffness or the shear parameter of the layer reduces the upper-beam response and increases the lower-beam response by redistributing the load between the beams. The Winkler stiffness has the stronger effect: upper -23% and lower $+42\%$ as K increases from 0 to 5.
5. The layer damping has little influence on the peak deflection during a single passage (about 1% for a tenfold increase in ζ), because it acts only on the relative motion of the beams.
6. The moving-force model is non-conservative for the loaded beam once α exceeds about 0.2. For $\alpha \leq 0.5$, multiplying the moving-force upper-beam response by about $1 + 0.4\varepsilon$ reproduces the full-model peak; at higher speeds the load inertia should be included explicitly. These are screening values for simply supported identical beams, not code-calibrated factors.

7. The first-order Struble reduction loses accuracy as α approaches the modified critical speed ratio, estimated at 0.95, 0.88 and 0.83 for $\varepsilon = 0.05, 0.15$ and 0.25 . Beyond $\alpha \approx 0.5$ its results should be verified against a full model, and near the critical region a higher-order or numerical treatment is needed.

Extensions of this work to other boundary conditions, non-uniform or non-identical beams, accelerating loads, higher-order asymptotic treatment near the critical speed, and repeated or multi-pass loading with fatigue assessment are recommended.

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